Applied Optics
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Lecture 44
Rectangular Aperture

Hello everyone, welcome to the class. Today we will talk about diffraction from rectangular aperture.

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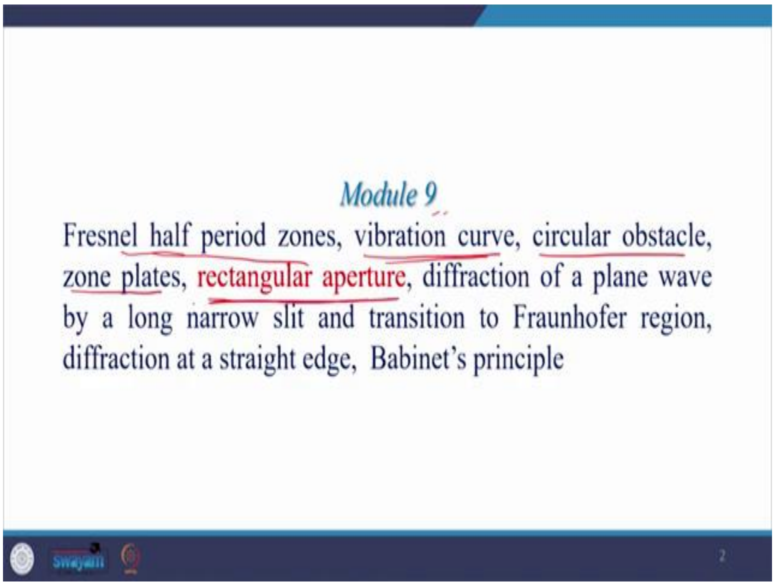

Now in module 9 we talked about Fresnel half-period zone and then I introduced you a very nice graphical method, which is called vibration curve. And then we implemented this knowledge of vibration curve in circular obstacle. And then we discussed about zone plates. Now proceeding ahead today we will deal rectangular aperture from this graphical perspective.

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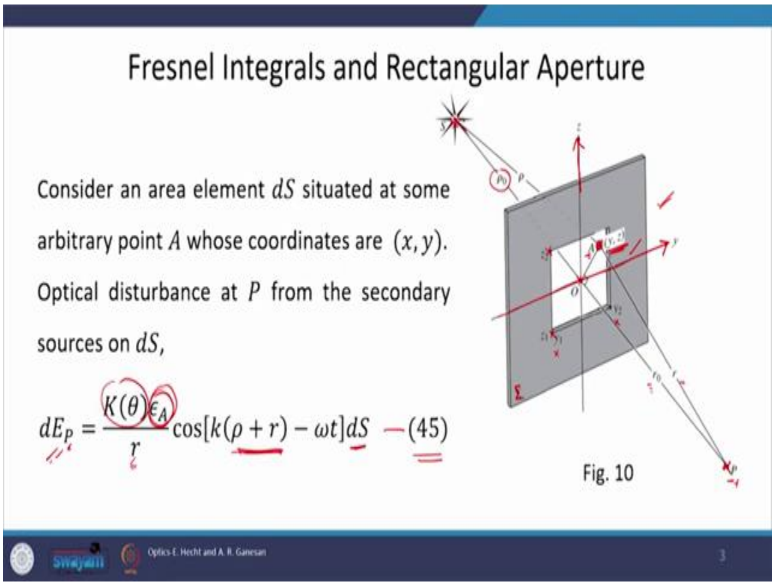

Now this rectangular aperture as you already know, since we are in the Fresnel diffraction regime we will deal this rectangular aperture through vibration curve. And we will also talk about Fresnel integrals. Now the rectangular aperture is shown here in this right figure schematically and associated with this Σ plane of the aperture is coordinate system and the y axis of the coordinate system is pointing in this direction, while the z axis is pointing in the vertical direction.

The origin is situated right at the center of the rectangular aperture. Now the rectangular aperture in horizontal direction is extending from point y_1 to point y_2 while in the vertical direction its extent is from point z_1 to point z_2. A source which is at a distance ρ_0 is illuminating this aperture and the point of observation P is situated at a distance r_0 from point O, the origin, which is at the center of the aperture.

Now we here we also consider area element dS which is situated at some arbitrary point A and the coordinate of this arbitrary point A is (y, z) as shown here in this figure. Now the disturbance which would be regarded at point P, it will come from the point sources, the secondary sources, which are situated at the area element dS. And we know how to calculate this disturbance, we have done it many times in our previous lecture.

This disturbance will be given by equation number 45, where dE_p represents the disturbance observed at point of observation P due to the area element dS. K(θ) is obliquity factor, ϵ_A is the strength per unit area of aperture source, r is the distance which is shown here. The r is the separation between the point A and the point of observation P. And $\rho + r$ is the separation between the source and the point of observation P.

The light is coming from S to A and then it is going to P, therefore with K we have $\rho + r$. And ϵ_A is the source strength per unit area, therefore we will have to multiply it with dS to get the complete strength, the right strength of the source. And thus, equation number 45 represents the optical disturbance at P due to the area element dS or due to the secondary sources situated at area element dS.

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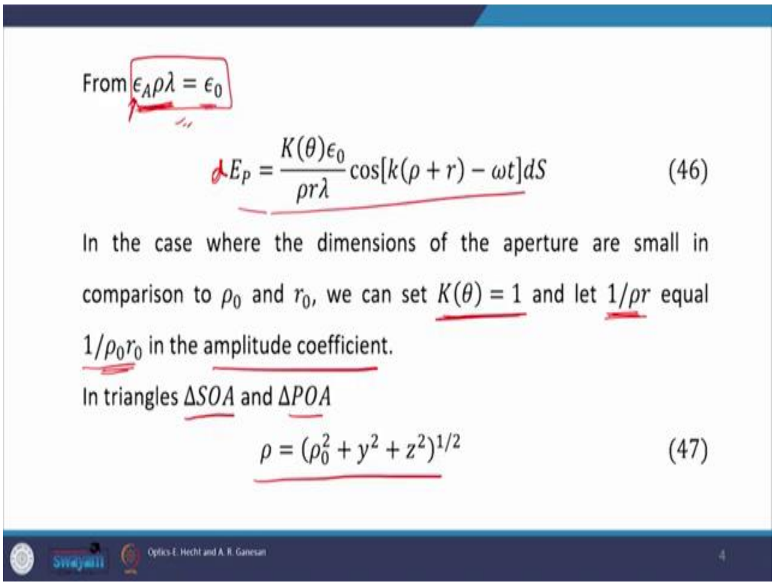

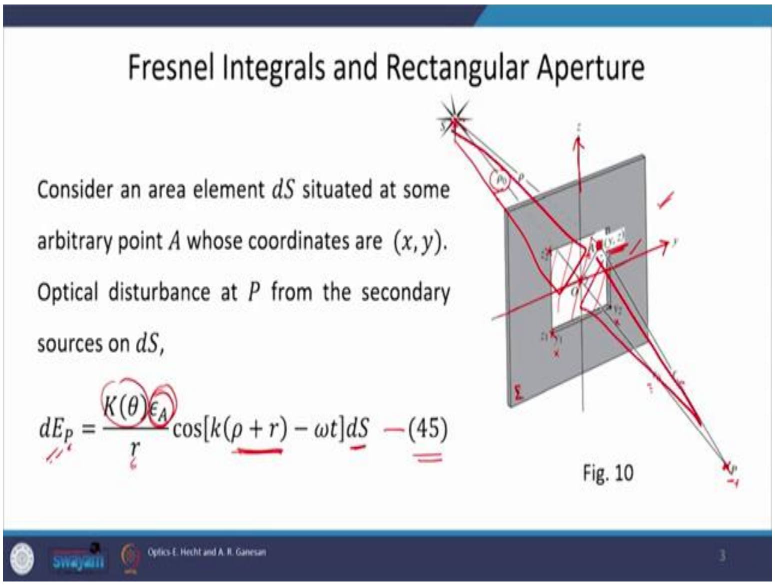

Now from the last lecture we know that $\epsilon_A \rho \lambda = \epsilon_0$. This we studied while studying vibration curve. This is the relation which we found from there. Now if you substitute the ϵ_A in previous expression using this equation, then the field get modified like this. Sorry, its dE_p.

Now in the case where the dimensions of the aperture are small in comparison to ρ_0 and r_0 then we can safely substitute K(θ) is equal to n. Why? Because if this aperture dimension is very small as compared to the aperture observation point distance, then the angle subtended by the point sources situated at the aperture on the point of observation P it would be almost equal to 0.

And under this assumptions K(θ), the obliquity factor would be 1. And we also assume that $1/\rho r = 1/\rho_0 r_0$, which is the correct assumption. But make it a point that this assumption is only true as long as this replacement is happening only in the amplitude part, not in the phase part. And thus, if we do this replacement in the amplitude coefficient the field would be almost unaffected.

Now consider two triangles, triangles SOA and POA. This is the SOA triangle, this is the first triangle and this is the second triangle POA. And these two triangle, we just apply the Pythagoras theorem and we can write this relation where y and z is the coordinate of point A situated on the aperture.

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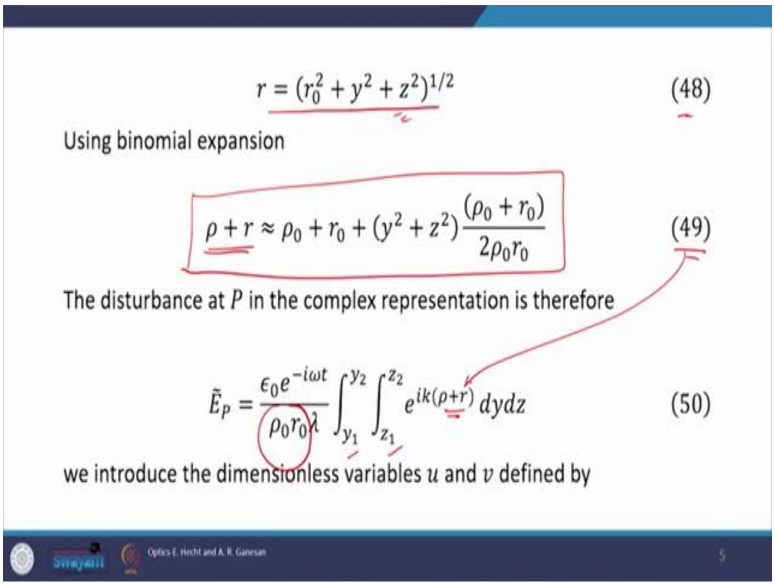

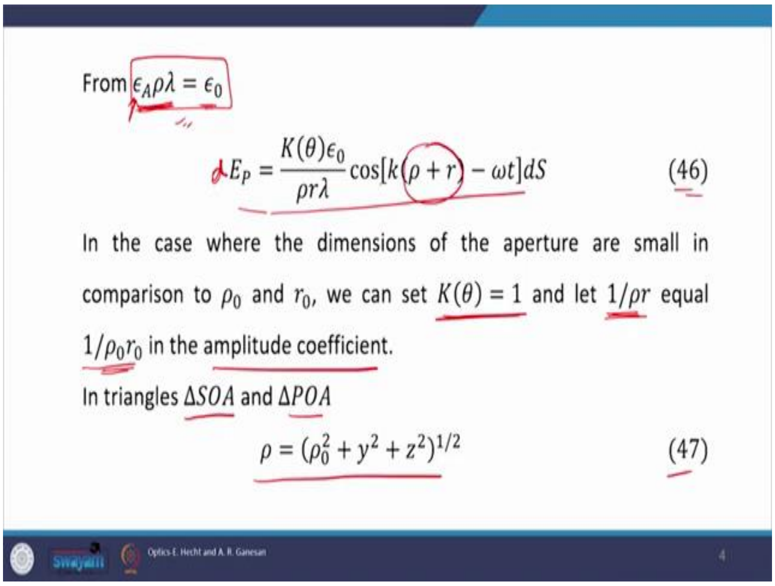

And $r = \sqrt{r_0^2 + y^2 + z^2}$. Now we exercise binomial expansion with these 2 expressions, the expression 47 and 48, and then add them up. And the addition gives equation number 49. Now you see that in equation 46 we have $\rho + r$ in the phase part. Now with this $\rho + r$ we can replace with equation number 49.

Now apart from this, in the amplitude part we have ρr which can be replaced by $\rho_0 r_0$ which is a constant term. And this is what we did here. Now if we want to calculate the field due to whole area of the aperture we have to perform integration. Since its the aperture has some area, therefore we will perform double integration, one along y and the other long z. And $\rho + r$ would be replaced by equation number 49.

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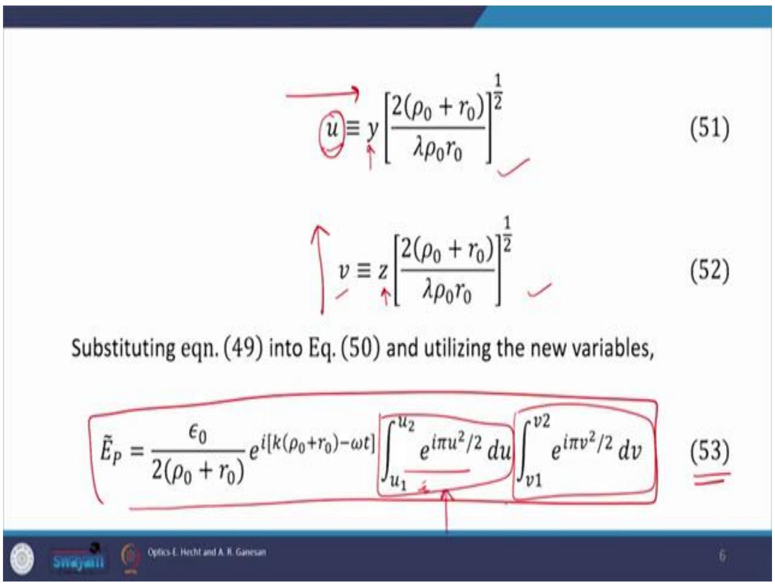

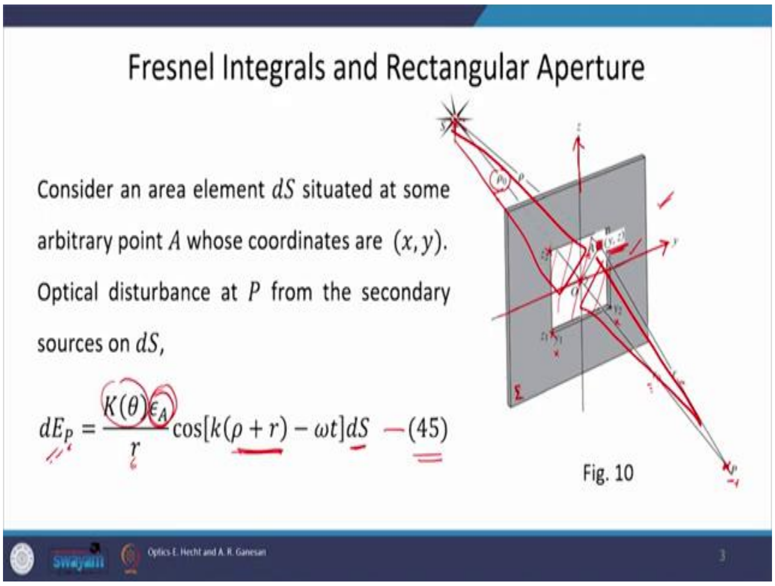

Now after the replacement we introduce 2 new variables u and v, and these are defined by equation number 51 and 52. Now after substituting 51 and 52 back into equation 50, we get this big expression. Now here u dependent term r in this integration and the v dependent on r terms are there in this integration and make it a point u is related to y while v is related to z.

And if you go in the first figure you see you see that z is the vertical axis and y is the horizontal axis. Therefore, u is in the horizontal direction while v would be in the vertical direction. Now with this we will have to evaluate equation number 53 and the major difficulty in evaluating equation number 53 is solving these 2 integrals.

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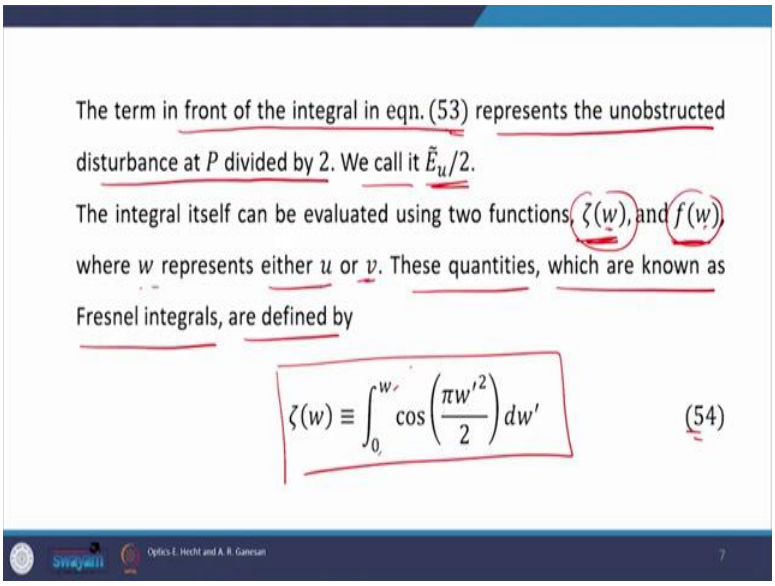

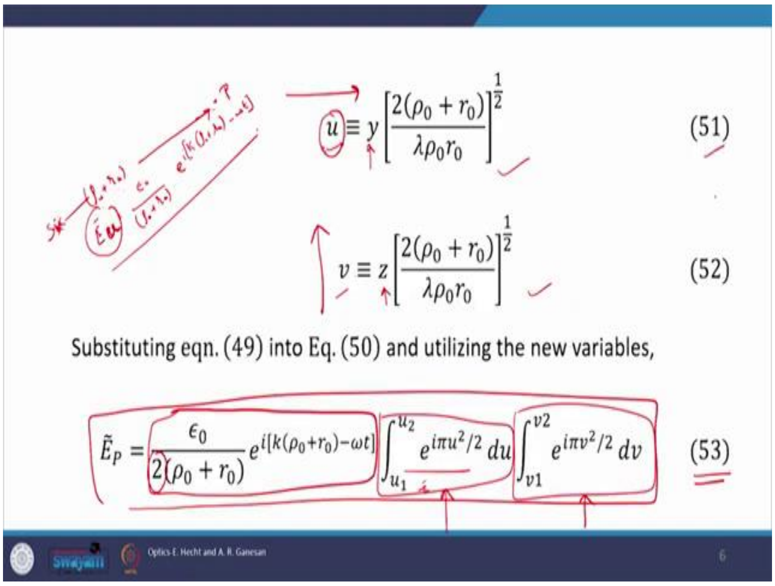

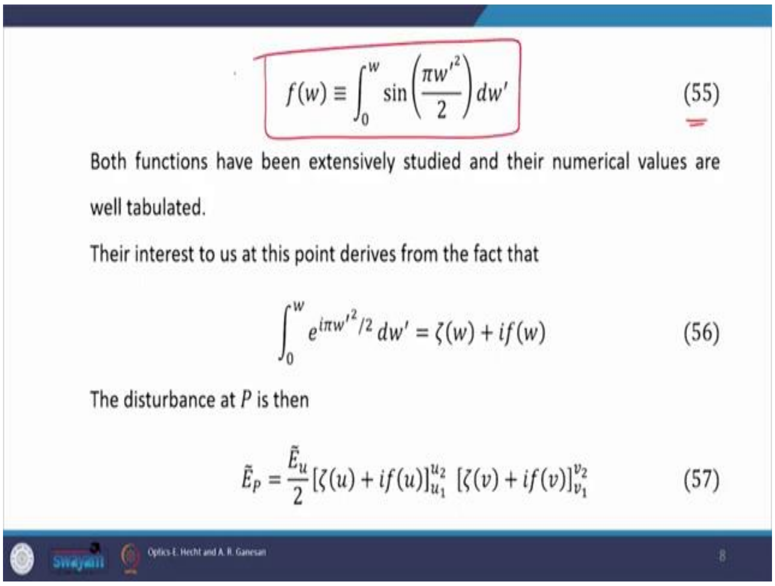

Now the terms in front of the integral in equation 53 represents the unobstructed disturbance at P divided by 2, here this term if you look it closely then you see that it is nothing but the field due to an unobstructed disturbance, which is situated at a distance $\rho_0 + r_0$ from the point of observation.

We have a point source S here and we have observer here and if this distance is $\rho_0 + r_0$ the field at P would be given by $e^{i(k(\rho_0+r_0)-\omega t)}$, this is the phase part and here it is the field strength by $\rho_0 + r_0$. This is the field due to unobstructed source, which is at a distance $\rho_0 + r_0$ from point of observation P.

And this field is halved here. You are seeing that we have, let us say that it is E_u and we have an extra term in the denominator which is 2. Therefore, we call it $\tilde{E}_u/2$. The amplitude part in the previous expression which is a field due to unobstructed point source. Now the difficulty solving the integrals. The integrals itself can be evaluated using 2 functions.

Now we introduce 2 functions, first is ζ which is function of w and second function is fwhich is function of again w, where w represents either u or v, u and v are introduced in equation number 51 and 52. Now here we have 2 integrals one is u dependent other is v dependent and to solve this integral we are introducing 2 new functions ζ and f, which are functions of w and w is nothing but it is either u or v.

Now these quantities which are known as Fresnel integrals are defined by this relation, equation number 54 and equation number 55. These are called Fresnel integrals. Now you see that in this expression we have integration of $cos(\pi w'^2/2)dw'$ where the limit of integration varies from 0 to w. And if you compare it with this expression then you see that these integrals are nothing but complex representation of equation number 54 or equation number 55.

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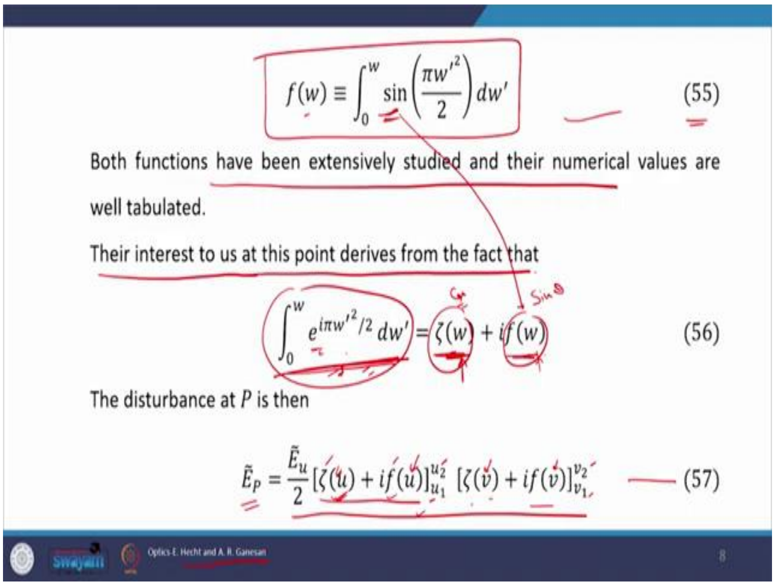

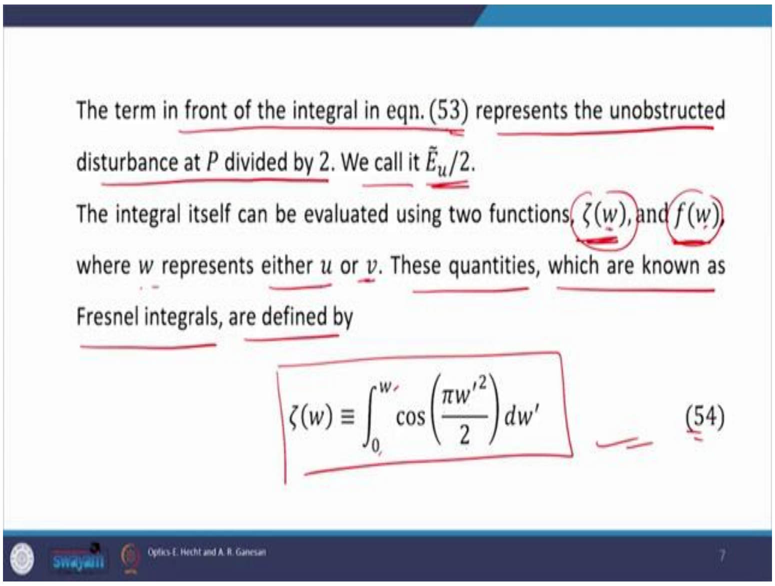

In 54 we have cosine term while in 55 we have sin term and this is the only difference between ζ and f. Now both functions have been extensively studied and their numerical values are well tabulated. And you can find the numerical values of these 2 functions in table 10.3 in a book by E. Hecht and A. R. Ganesan and the title of the book is Optics.

Now their interest to us at this point derives from the fact that this function can be or this integral can be written as $\zeta + \text{i}f$. The integral which is here in equation number 53, they can be expressed, since you see that here in the exponent you have i, it is a complex function. Therefore, the solution will be complex. If you solve this integral you will have a complex number.

And the complex number will have a real part and imaginary part, and let us assume that we will have a real part and imaginary part as ζ and f function of this integral. And this is how this ζ and f are defined. ζ is real part of this integral, therefore in ζ you have cos term, while f is imaginary part, therefore you have sin term here in f, in the expression of f, sin comes here and cosine comes here.

Therefore, $cos(\pi w'^2/2)$ will be embedded here in ζ function and simply the sin will be here in f function, which is quite obvious and therefore, the equation 53 can be rewritten in terms of these 2 new functions, ζ and f. And if you remove this integral which are there in equation number 53 with ζ and f, you get new expression for the disturbance at P, which is written here, where u_1 and u_2 and v_1 and v_2 are limits of the 2 integrals respectively.

In the first bracket we have ζ and f fwhich are functions of u, while in the second bracket this are functions of v. We will have to just you evaluate ζ and f in equation 57 to get the expression for field E_p.

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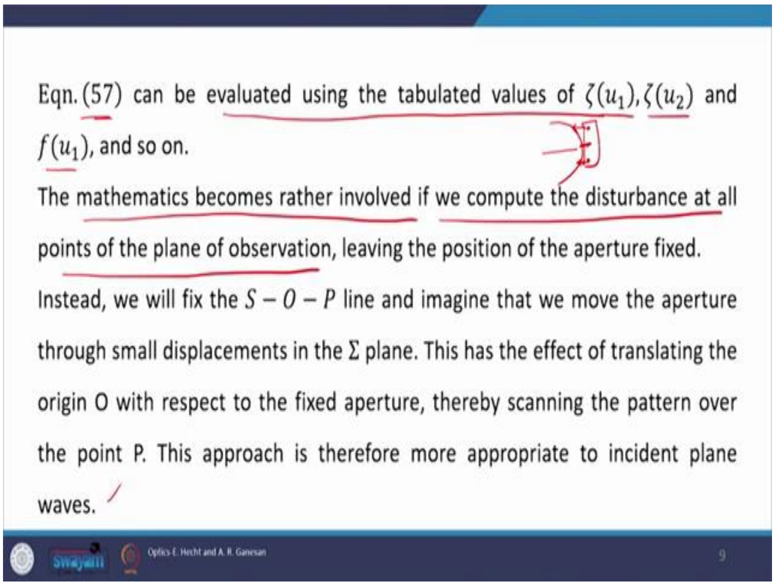

Now the equation 57, the previous equation can be evaluated using tabulated value of $\zeta(u_1)$ $\zeta(u_2)$ and $f(u_1)$ $f(u_2)$ and so on. And this tabulated values are given in, as I said before in the book Optics by E. Hecht and A. R. Ganesan and the table number is 10.3. And these are very standard integrals and ζ and f are very standard integral and you can see them in any standard book.

The mathematics becomes rather involved if we compute the disturbance at all points of the plane of observation. Because till now we have calculated, we have an aperture plane and we have a screen plane and we are calculating the field at some point in the screen plane, but if you want to calculate the field at all the points in the screen plane, then the mathematics would be very much involved.

It will impose too much difficulty. Now instead what people do is that they fix they axis, the source is here, the origin O is situated at the center of the aperture and the P is a point on the observation plane and what people do is that they fix this SOP line and then they imagine that instead of moving point P in the screen plane they move the aperture through small displacements in aperture plane.

They only move the aperture and therefore the value of the field or irradiance at the point of observation which is fixed in the screen plane, it will change. It is like scanning the aperture and once you scan the aperture you will come to know the off axis field and this has the effect of translating the origin O with respect to the fixed aperture and therefore, by scanning the pattern over the point P.

And this approach therefore is more appropriate to the incident plane waves because the plane wave illuminates the irradiance uniformly. Well, if you launch a spherical wave then there would be a phased difference on illumination, the center part of the aperture would be illuminated earlier because if this is the aperture and this is the wave, which is illuminating the aperture, then this part will be illuminated earlier.

While this part would be later, because there is a difference, there is a time difference between the illumination of the central part and this part because of the curvature of the wave front. Therefore, this approach is more suited for plane wave illumination.

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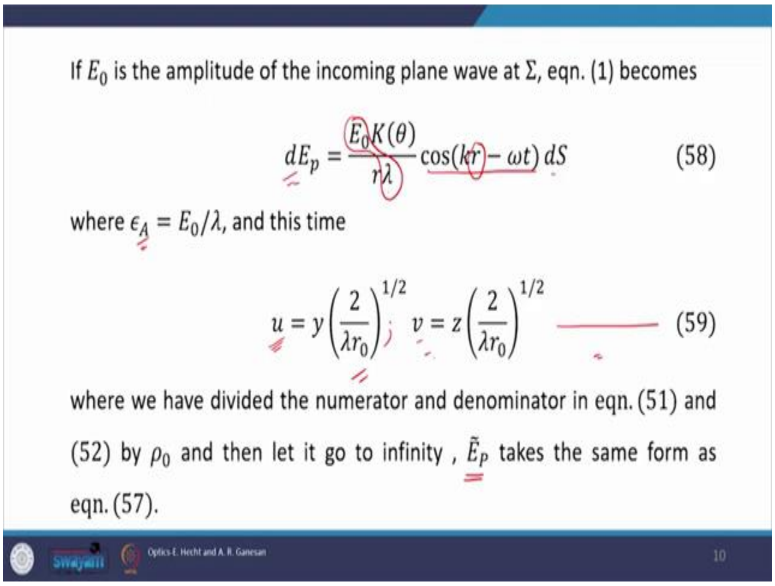

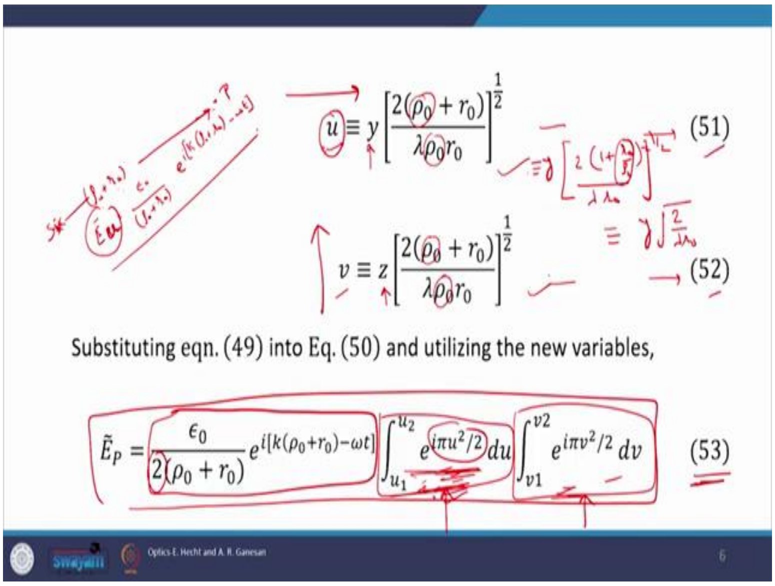

Now assume that E_0 is the amplitude of the incoming plane. We have now assumed that we are launching the plane wave with amplitude E_0 on the aperture and therefore, the equation 1 which we studied in our previous lectures, which evaluates the field contribution due to the elementary area dS now becomes this $dE_p = E_0 K(\theta)/r\lambda(cos(kr - \omega t))dS$.

Since the illumination is by plane wave you see that here with k we are getting only r, no ρ, because for a plane wave we know ρ is infinity. And E_0/λ is source strength, ϵ_A. Here too, we get a new expression for the variable u and v which are given by equation number 59, and this new expression we got from this these two expression, equation number 51 and 52. In equation 51 and 52, the illumination was due to a wave coming from a point source.

Now in this second case there are plane waves which are illuminating the aperture, therefore what we will do is that we will divide equation number 51 as well as 52 by ρ_0 and after division we will equate ρ_0 to infinity. Now you can see here if you divide 51 with, both numerator and denominator of 51 by ρ_0, then you get this $y(2(1 + r_0/\rho_0)/\lambda r_0)^{1/2}$ here in the numerator.

And here you will get this. And if you now say that ρ_0 is infinity that is for plane wave then this term would be equal to 0. And what you are left with is $y\sqrt{2/\lambda r_0}$. And this is what you see here in equation number 59, both u and v reduces to these new expressions. Now we will calculate the total field at point of observation P due to the whole aperture.

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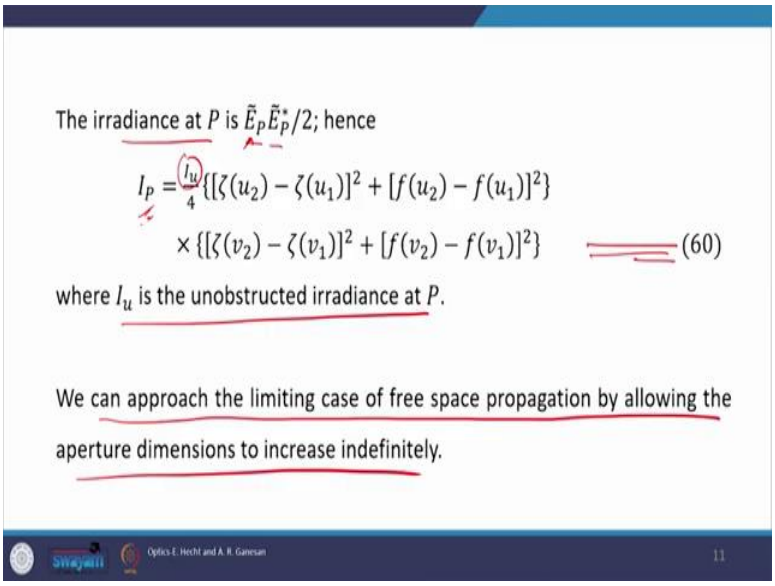

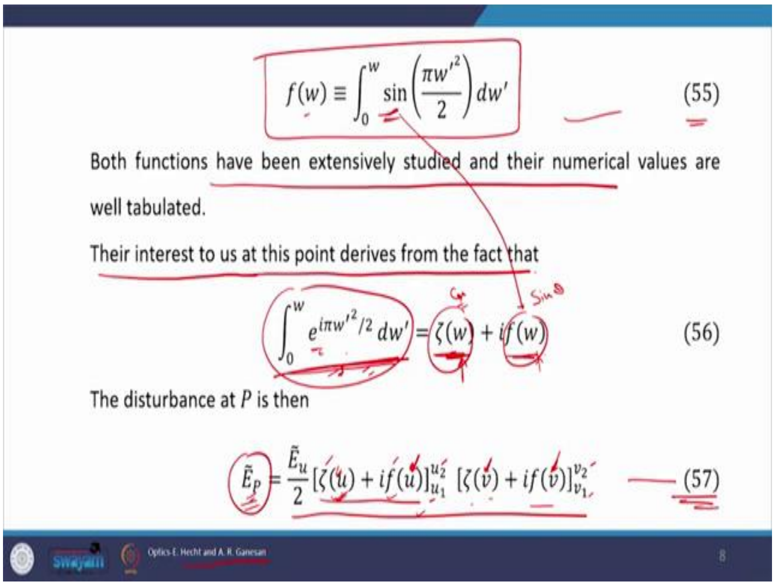

And once this is done then we calculate irradiance at P. The irradiance would be given by $\tilde{E}_p\tilde{E}_p^*/2$. And once the expression for $\tilde{E}_p$ is known, we can say clearly easily write the expression for I_p that is a radiance at P which is given by equation number 60. Now if you go back then here it is the expression for $\tilde{E}_p$. Now in this expression 57 only the expression for u and v are modified because the illumination is now plane wave.

If you take the complex conjugate of $\tilde{E}_p$ and then multiply the $\tilde{E}_p$ with its complex conjugate you get this, you get expression 60. And here I_u is the unobstructed irradiance at point of observation P, which is nothing but in equation 57 $\tilde{E}_u$, is $|\tilde{E}_u|^2$. Now we can approach the limiting case of free space propagation by allowing the aperture dimensions to increase indefinitely.

If we keep increasing the aperture dimension then it would be like unobstructed illumination. For this you will have to increase the values of u and v or you have to extend u_2 to $+\infty$ and u_1 to $-\infty$. Similarly, you have to extend v_2 to $+\infty$ and v_1 to $-\infty$.

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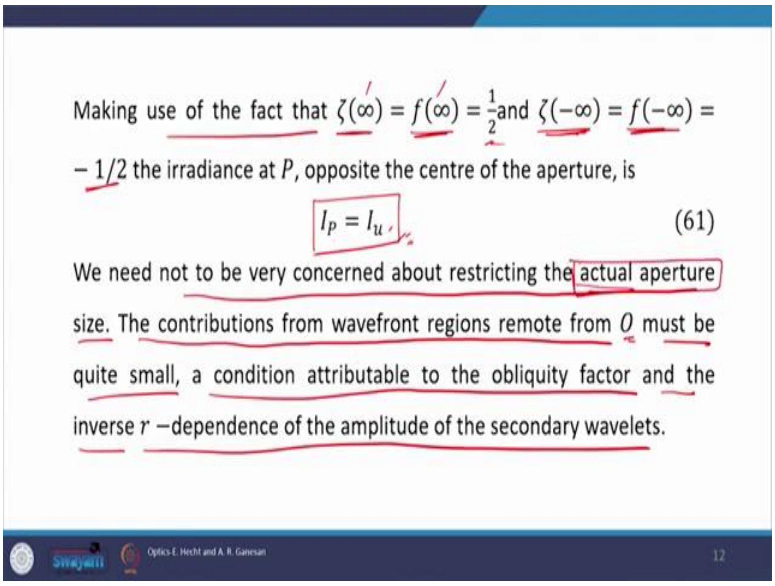

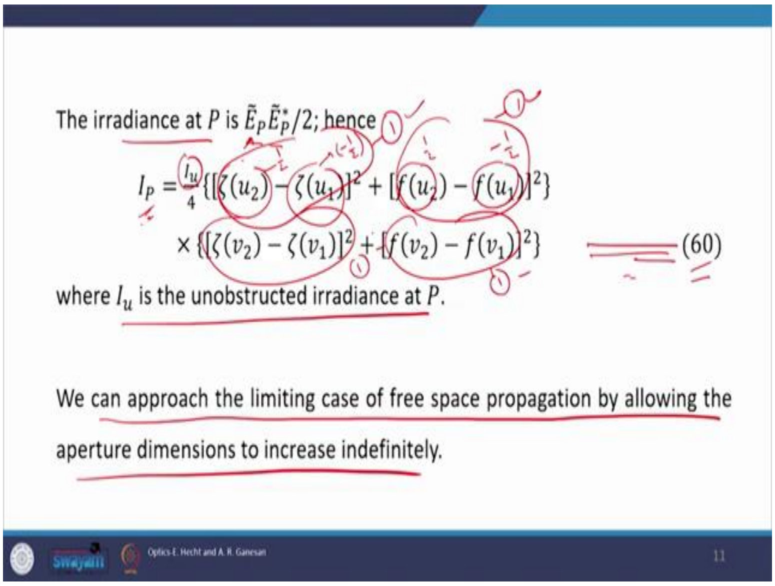

Now making use of the fact that $\zeta((\infty) = f(\infty) = 1/2$ while $\zeta(-\infty) = f(-\infty) = -1/2$. These are the tabulated value as I said earlier. At $+\infty$, both $\zeta = f = 1/2$ while at $-\infty$ both $\zeta = f = -1/2$. With this we can calculate the irradiance at P using equation 60.

And this will give us the irradiance due to unobstructed source and if you substitute these value back here in equation 60, then you see that you will get this value would be replaced by half, this value will be replaced by -1/2, similarly this +1/2, this -1/2 and this -1/2, minus of minus half will give you 1, similarly this bracket will also give you 1, similarly this bracket will give you 1 and this bracket will also give you 1.

And if you add them up you will get 1, 2, 3, 4, you will get $4I_u/4$, which is equal to I_u and this is what is written here $I_p = I_u$. It means things are calculated very well, things are moving in a proper direction, in a correct direction because once we extended the boundaries of the aperture to infinity then we get irradiance at point P due to unobstructed point source and since these two expressions are matching we are in a correct line.

Now we need not to be very concerned about restricting the actual aperture size. Why? The contributions from wavefront regions remote from O must be quite small, a condition attributable to the obliquity factor and the inverse r dependence of the amplitude of the secondary wavelets. This statement says that the aperture size does not play a very important role in calculating the irradiance value at the point of observation P.

Because if the aperture is a little big then as you increase the aperture size the θ values goes up, and therefore, the irradiance goes down, therefore the contributions from portions which are very far from the origin in aperture plane that contribution is very small, very little and in addition as the wave propagate r is there in the denominator if we consider the wave to be spherical.

And therefore, the amplitude of the secondary wave decays down very rapidly. And therefore, the shape of the wave, the actual aperture shape or actual aperture size does not play a major role. It should not be of too much of concern.

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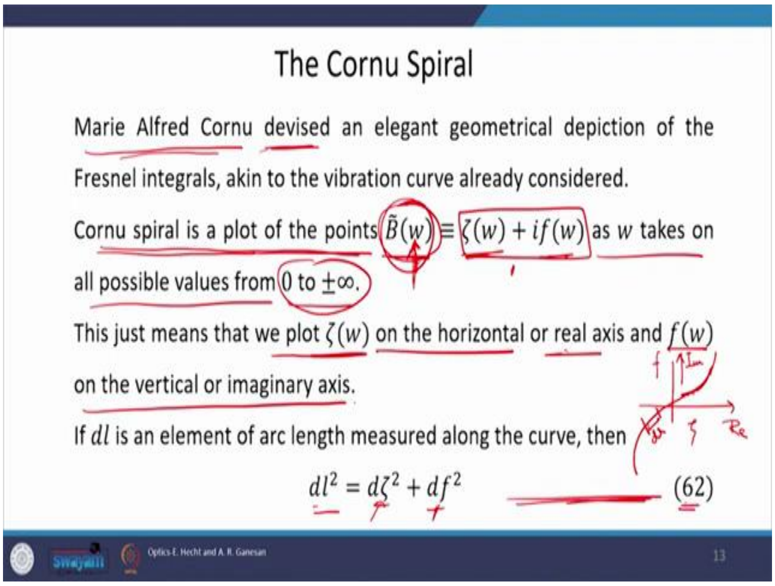

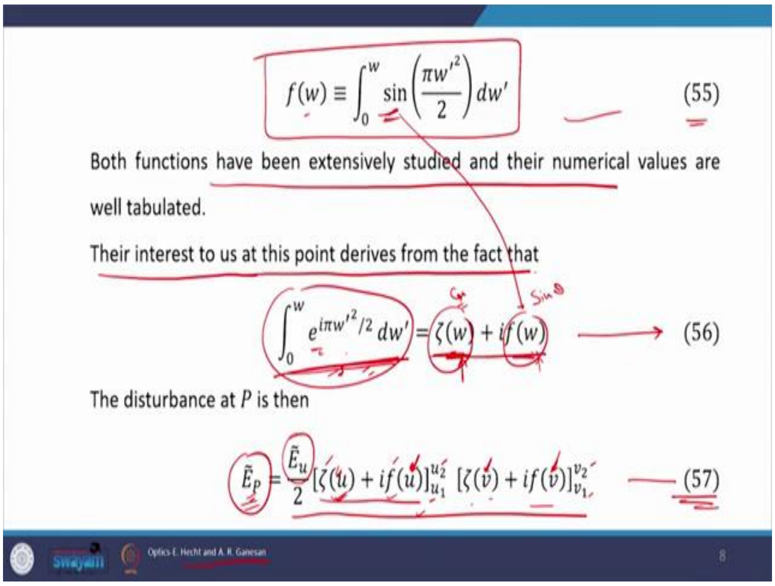

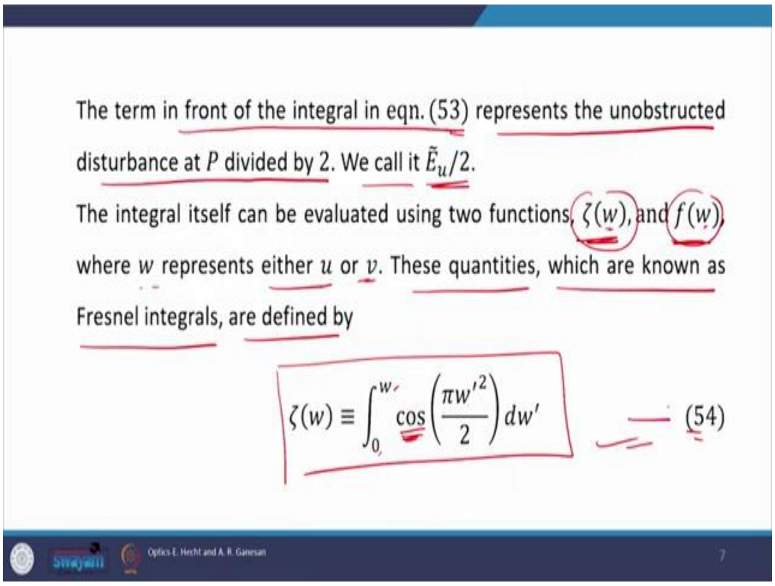

Now this is all about the rectangular aperture. Although we solve equation 16 in a limiting case but what if u_1, u_2 or v_1, v_2 are not equal to $+\infty$ or $-\infty$, then how to solve it. For this we will have to go to some tabulated values of these functions. But there is a very nice graphical method to solve these integrals or to have these values and what is this graphical method?

This graphical method is called Cornu spiral. Marie Alfred Cornu devised elegant geometrical depiction of the Fresnel integral, and this geometrical interpretation is almost similar to that of vibration curve. Cornu spiral is a plot of points $\tilde{B}(w)$ which is equal to $\zeta(w) + if(w)$. As w takes on all possible values from 0 to $\pm\infty$. And this function we have already seen in our previous slides.

This is nothing but this integral, equation number 56, you see $\zeta(w) + if(w)$ and the only our motto of introducing ζ and f was to solve this integration. Now this integration here in Cornu spiral is represented by this function $\tilde{B}$. And here the independent variable w it is allowed to take any possible values between 0 to $\pm\infty$. This just means that we plot $\zeta(w)$ on horizontal our real axis and $f(w)$ on vertical or imaginary axis.

In Cornu spiral as is said here that it is a plot of $\tilde{B}$ and B is a complex number. So how to plot a complex number? In horizontal direction plot real part of this complex number while in the vertical direction plot imaginary part of this complex number or in other words on horizontal axis plot ζ, while on the vertical axis plot f and with this you get a set of points.

Just with this set of points you can draw a curve which would be nothing but Cornu spiral. Now say that you have some spiral here, and say that there is a length element dl on this spiral. Now this length element $dl = \sqrt{d\zeta^2 + df^2}$, dl is a length element which is measured along the curve and this length element would be equal to $\sqrt{d\zeta^2 + df^2}$ which is very much obvious.

Now let us substitute the values of $d\zeta$ and df because we know what is the expression of ζ and f. ζ and f are given here in our previous slide by equation number 54 and equation number 55. Therefore, once ζ and f are given we will have to just differentiate them to get $d\zeta$ and df.

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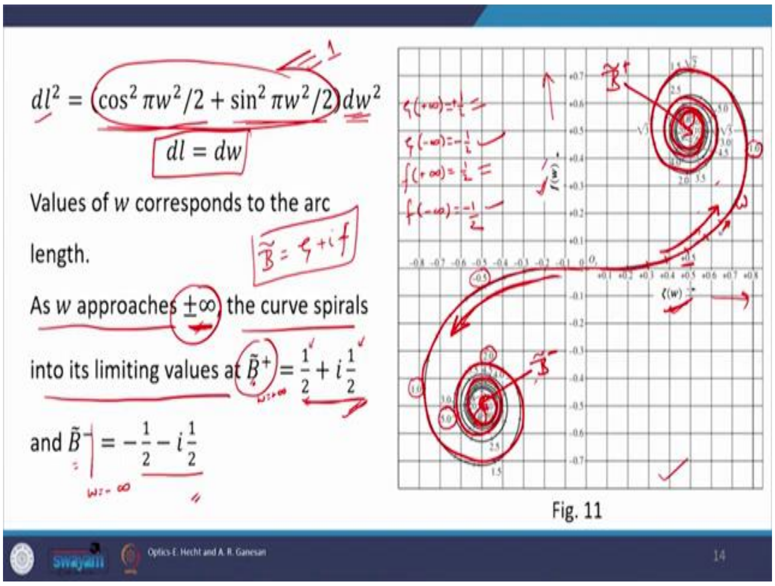

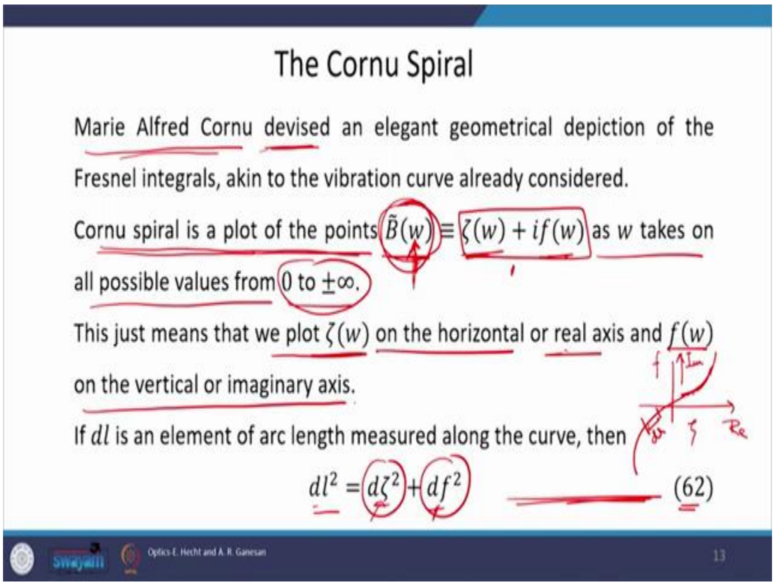

Now with differentiation the integral sign will go away and we will have this expression $(cos^2\pi dw^2/2 + cos^2\pi dw^2/2)dw^2$, this is what we will get on the right hand side of this after substitution in equation number 62. What we did is that we substituted for $d\zeta^2$ and df^2. We know ζ is equal to integration of some parameter since we are difference integral integration the integral sign will go away and we will be left with these 2 terms.

Now we know that $sin^2\theta + cos^2\theta = 1$, then therefore, this term would be equal to 1 and therefore $dl = dw$. which means the values of w on the Cornu spiral will correspond to the arc length. Now this is a representative figure of Cornu spiral. How Cornu spiral look like? On the horizontal axis ζ is plotted while in the vertical axis f is plotted.

ζ is the real part of $\tilde{B}$ while f is imaginary part of $\tilde{B}$, here on the vertical axis its imaginary part, it is an imaginary axis, while the horizontal one is the real axis. Now the values which you are seeing on the spiral it is w, make it a point, on the spiral we are plotting w. And the w values are marked by this dash line, you can see here the value of w is 0.5, the value of w here is 1, here it is $\sqrt{2}$, here it is 1.5, similarly on the other side.

These are the values of w and the spiral is going in this direction in the first quadrant, while it is going in this direction in the third quadrant. Now as the value approaches $\pm\infty$ you see that values of w is increasing in positive direction along this axis, along this spiral, while the values of w are increasing in negative direction in along this axis. Now when w approaches $\pm\infty$ the curve is spirals into its limiting value.

And what is its limiting value? We know the limiting value because when w is equal to either $+\infty$ or $-\infty$ we know both ζ and f, they become equal, when $w = +\infty,\ \zeta(\infty) = 1/2$ as well as $f(\infty) = 1/2$. Similarly, when $w = -\infty,\ \zeta(-\infty) = -1/2$ and $w = -1/2$.

Let me write it here, $\zeta(+\infty) = 1/2$, $\zeta(-\infty) = -1/2$, whenever I am saying half it is plus half. Similarly, $f(+\infty) = 1/2$ and $f(-\infty) = -1/2$. Sorry, it should be plus half here. Sorry minus half here, these 2 terms are minus while these 2 terms are plus.

And we know that $\tilde{B} = \zeta + if$, therefore at $w(+\infty)$ or at $w(-\infty)$ we can write the expression of $\tilde{B}$. Now at $w(+\infty)\tilde{B}$ value is given by $\tilde{B}^{+}$, this is at $w(+\infty)$. And since this is equal to $\zeta + if$ and ζ at ∞ is 1/2 as well as f at $+\infty$ is +1/2, therefore this would be the expression for $\tilde{B}^{+}$ at $w = +\infty$.

Similarly, at $w = -\infty$ $\tilde{B}$ is given by $\tilde{B}^{-}$ and its value is given by $-1/2 - i/2$. Since these are the maximum or minimum possible values of function $\tilde{B}$, therefore the upper spiral when it rounds, the ultimate value is B^{+} which is here, and the ultimate value is B^{-} which is shown here. All these spirals ends here. This value is $\tilde{B}^{+}$ plus here and this value is $\tilde{B}^{-}$.

These are the limiting values $\tilde{B}$ function, it is a $\tilde{B}$ is the limiting value and the limiting values are given by these 2 expressions. Now if w increases in the positive direction this is how the B evolves, this is how B increases and after a while it spirals around and ultimately it reaches to $\tilde{B}^{+}$. Similarly, if w increases in minus direction, then this is how the B evolves and it spirals and it ultimately reaches to $\tilde{B}^{-}$.

The values of ζ and f are expressed on the horizontal and vertical axes respectively.

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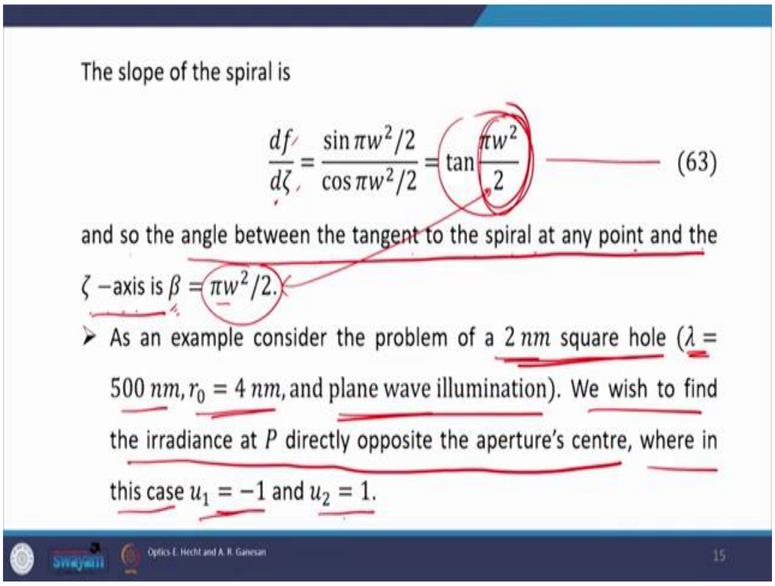

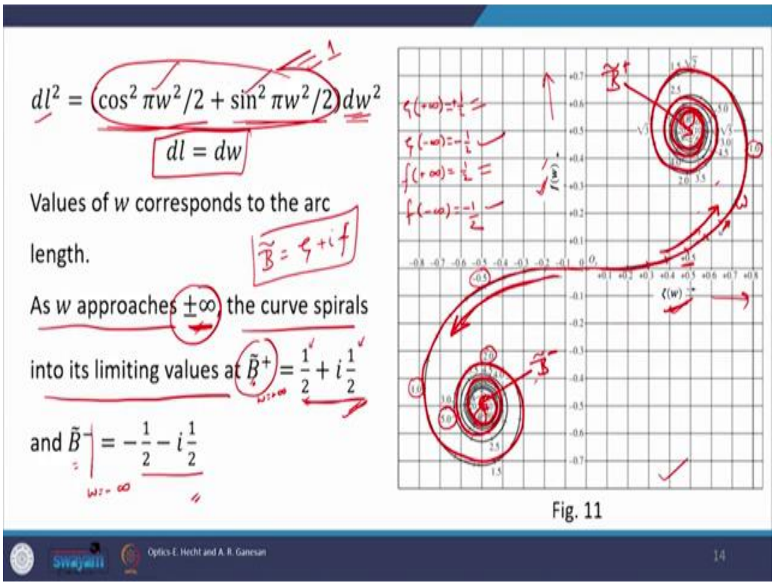

Now, let us now calculate the slope of this spiral. The slope would be given by $df/d\zeta$. We know the expression for df and $d\zeta$ which are used here, substitute them here and this gives the slope, $tan(\pi w^2/2)$. This is the slope. Once the slope is given then the angle between the tangent to the spiral at any point and the ζ axis would be given by $\pi w^2/2$. This is shown here in this figure.

If you draw a tangent here then this angle would be β which is the slope, which is coming here while calculating the slope $\pi w^2/2$. This is why it is said that the angle between the tangent to the spiral at any point and the ζ axis, the horizontal axis, is β which is given by $\pi w^2/2$, which we calculate here, calculated here in expression 63. Now let us consider some realistic example.

Let us consider the problem of a 2 nanometer square hole, where this hole is illuminated by a light of wavelength 500 nanometer, r_0 is 4 nanometer and the illumination is done with a plane wave. r_0 means the point of observation P is at a distance 4 nanometer from the square hole. Now what is the motto? We wish to find the irradiance at P directly opposite the aperture center where in this case $u_1 = -1$ and $u_2 = +1$. Why $u_1 = -1$ and $u_2 = +1$?

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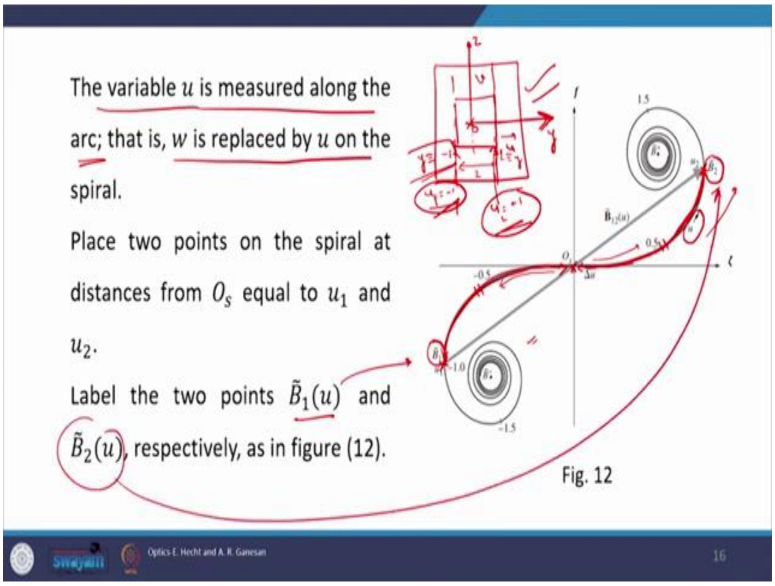

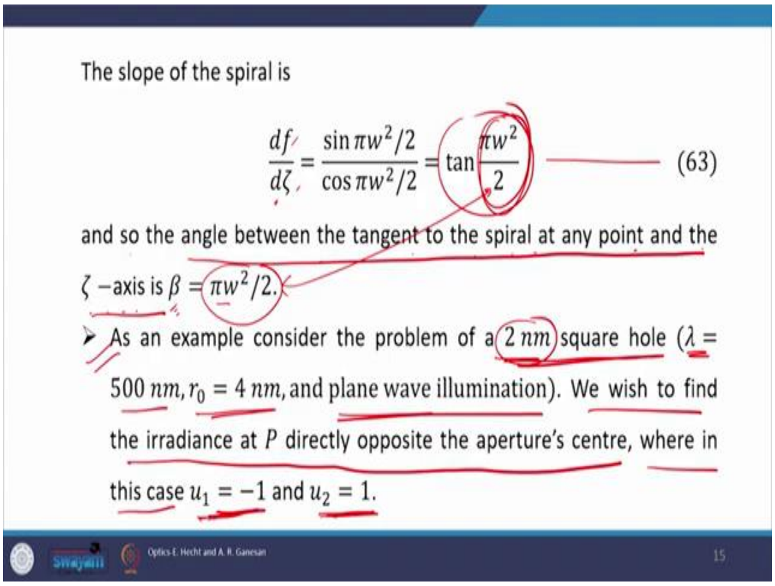

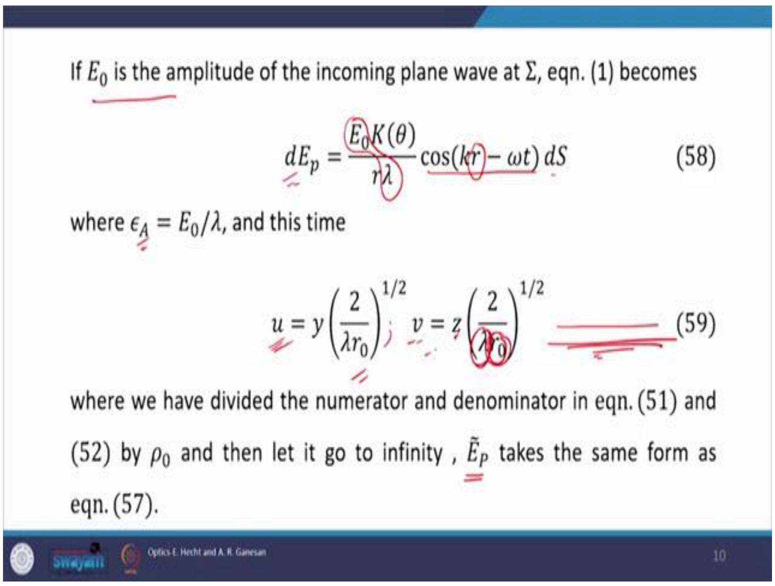

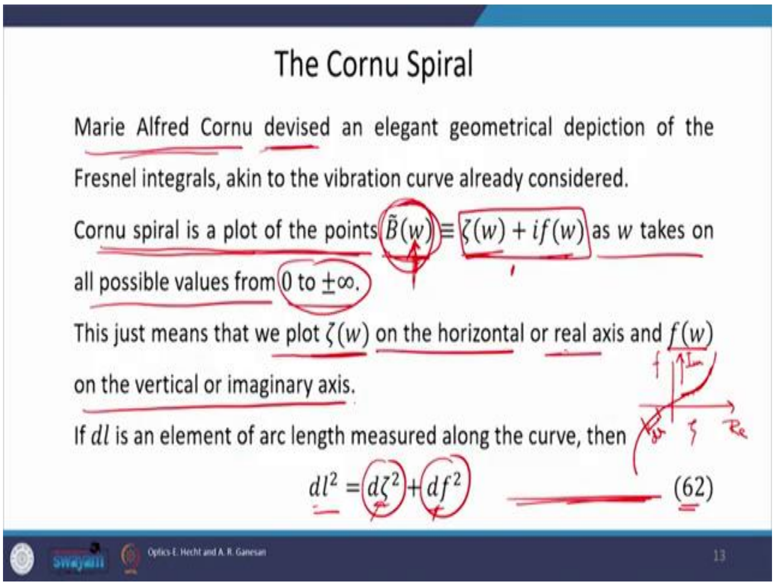

Now if this is our aperture, this is our square aperture, this is our z axis and this is our y axis. Now if you go back and see the definition of y and u_1 and u_2, then if you remember I said u is in the horizontal direction and v is in the vertical direction. Why? Because v is related to z and u is related to y. Therefore, you would be pointing in this direction, this would be the u direction and this would be the v direction.

Now its origin is at the center of this square aperture and we know that this square aperture is having sides which are equal to 2 nanometer, therefore, this whole length would be 2, but if we want to write the coordinate then it would be -1, it would be +1 here in y direction. The y value here would be -1 and the y value here would correspond to equivalent to be +1.

Now let us again go back and see that if it is a plane of illumination then this is how the u and v are calculated and with this calculation we found that in this particular example $u_1 = -1$ and $u_2 = +1$. If you put the value of λ, if you substitute the value of r_0 in the expression which we just talked about, I mean this expression, expression number 59, if you substitute the value of r_0, substitute the value of λ.

And the extremities of the square aperture, then you will get a values of u_1, u_2 and v_1, v_2 which would be given by -1 and +1. This is why I say that y correspond to -1 here and y correspond to +1 here or you can write $u_1 = -1$ for this point and for this point, it is equal to +1. And we call these two extremities as u_1 and u_2.

Now as you saw before in the Cornu spiral, the variable u are the variable w was being measured along the curve, now w was a generalized variable. Now since we are dealing with u here only we will say that u is varying along the curve. The variable u is measured along the arc. And here what we have done we have replaced w by u on the spiral. Right now we are just talking about the horizontal axis, y axis.

And therefore we are just talking about the extremities which are here, extremities of the aperture which are in the horizontal direction. The left most extremity is at point $u_1 = -1$ while the right most extremity is at a point where $u_2 = +1$. Now once the values of u_1 and u_2 are known we can put these values, we can mark these values on the Cornu spiral.

Therefore, we will place two points on the spiral at distances from the center of this spiral, which is OS and which distances would be equal to u_1 and u_2. Now you see that in this aperture O which is origin is center of the aperture and therefore u_1 and u_2 are at a same distance, they are situated at a same distance from the origin. Therefore, u_1 we can mark here and this distance represents the length of u_1.

And similarly, we can mark u_2 and this distance represents the length of u_2 and since $|u_1|$ is $|u_2|$ therefore these two distances would be equal. Do make it a point that u is varying along the curve, along this arc. Once these points are placed, having known the values of u_1, we can mark u_1 and this arc length would be equal to the length of u_1. Similarly, this arc length would be equal to length of u_2.

u_1 was on the left hand side or on the negative side of, since u_1 was on the negative side of the origin O, therefore we are marking u_1 in third coordinate. It is going down. And the positive value of u which is u_2 is marked in first quadrant, because positive values of u are here, measured in the first quadrant. The u is 0 at the center and it is increasing in this direction and it is increasing in negatively in the other direction.

This we have already talked about. Now we label these two points as $\tilde{B}_1$ and $\tilde{B}_2$, because once u_1 and u_2 values are known we can calculate the values value of $\tilde{B}_1$ and $\tilde{B}_2$ which was introduced here in this slide because Cornu spiral is a plot of these points $\tilde{B}$. Now once w is known, we can mark this two points, w here is replaced by u, we already said. And these marking are shown in this figure $\tilde{B}_1(u)$ is marked here while $\tilde{B}_2(u)$ is marked here.

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The phasor $\tilde{B}_{12}(u)$ drawn from $\tilde{B}_1(u)$ to $\tilde{B}_2(u)$ is just the complex number $\tilde{B}_2(u) - \tilde{B}_1(u)$

$$\tilde{B}_{12}(u) = [\zeta(u) + i f(u)]_{u_1}^{u_2} \quad (64)$$

and is the first term in eqn.(53) for $\tilde{E}_P$.

Similarly, for $v_1 = -1$ and $v_2 = 1$, $\tilde{B}_2(v) - \tilde{B}_1(v)$ is

$$\tilde{B}_{12}(v) = [\zeta(v) + i f(v)]_{v_1}^{v_2} \quad (65)$$

which is the latter portion of $\tilde{E}_P$. The magnitudes of these two complex numbers are just the length of the appropriate $\tilde{B}_{12}$ phasors.

The variable u is measured along the arc; that is, w is replaced by u on the spiral.

Place two points on the spiral at distances from O_s equal to u_1 and u_2.

Label the two points $\tilde{B}_1(u)$ and $\tilde{B}_2(u)$, respectively, as in figure (12).

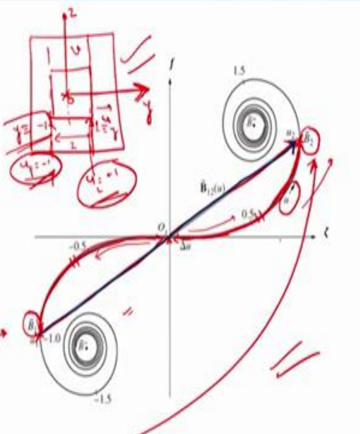

Fig. 12

$$u \equiv y\left[\frac{2(\rho_0 + r_0)}{\lambda \rho_0 r_0}\right]^{\frac{1}{2}} \quad (51)$$

$$v \equiv z\left[\frac{2(\rho_0 + r_0)}{\lambda \rho_0 r_0}\right]^{\frac{1}{2}} \quad (52)$$

Substituting eqn. (49) into Eq. (50) and utilizing the new variables,

$$\tilde{E}_P = \frac{\epsilon_0}{2(\rho_0 + r_0)} e^{i[k(\rho_0 + r_0) - \omega t]} \int_{u_1}^{u_2} e^{i\pi u^2/2} du \int_{v1}^{v2} e^{i\pi v^2/2} dv \quad (53)$$

Having done this, we will now have to calculate the phasor $\tilde{B}_{12}$. How to calculate phasor $\tilde{B}_{12}$? To calculate phasor $\tilde{B}_{12}$ we draw a line starting from $\tilde{B}_1$ and then extending till $\tilde{B}_2$. Let us pick a different color for better clarity. This phasor will represent $\tilde{B}_{12}$ and it is drawn from $\tilde{B}_1$ to $\tilde{B}_2$ and it is nothing but a complex number and this complex number is $\tilde{B}_2 - \tilde{B}_1$ and $\tilde{B}_{12} = \zeta(u + if)$.

Where the limits is varying from u_1 to u_2. This is the compressed expression of $\tilde{B}_2 - \tilde{B}_1$. And this exactly equation number 64 is first term in equation 53 where we were calculating the total field. Let us go to equation 53. Now you see here in equation 53 this term is nothing but it is $\tilde{B}_{12}$ which is function of u. And through Cornu spiral we just saw that just by drawing a phasor $\tilde{B}_{12}$ we calculated this integral.

The first part in equation 53. We will repeat the same thing for the second integral also and there we will replace u by v. And this is what is done next. Similarly till now this was our aperture and this was the origin, u was extending in this direction and v was extending in this direction u was varying from -1 to +1, while v, on the other hand, it also varies from -1 to +1.

v is varying from -1 to +1 also, therefore the values of v_1 and v_2 would be -1 and + 1. Similarly for these two values, we can calculate $\tilde{v}_2 - \tilde{v}_1$ on Cornu spiral we will draw the same Cornu spiral but here the variable u would be replaced by variable v. With that Cornu spiral we will again draw phasor $\tilde{B}_{12}$ which now will be function of v and that phasor will give you the second integral in equation number 53.

Once the length of the 2 phasors are measured the 2 integrals are solved and therefore, from 53 we can calculate the resultant field at the point of observation P. The magnitude of these 2 complex numbers $\tilde{B}_{12}(u)$ and $\tilde{B}_{12}(v)$ they are just the length of appropriate $\tilde{B}_{12}$ phasors in the Cornu spiral.

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The irradiance is then simply

$$I_P = \frac{I_u}{4}\left|\tilde{B}_{12}(u)\right|^2\left|\tilde{B}_{12}(v)\right|^2 \qquad (66)$$

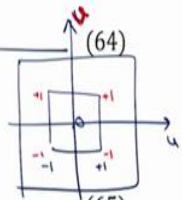
Notice that the arc lengths along the spiral are proportional to the aperture's overall dimensions in the y and z direction, respectively.

The arc lengths are therefore constant, regardless of the position of P in the plane of observation.

On the other hand, the phasors $\tilde{B}_{12}(u)$ and $\tilde{B}_{12}(v)$, which span the arc lengths are not constant, and they do depend on the location of P.

The phasor $\tilde{B}_{12}(u)$ drawn from $\tilde{B}_1(u)$ to $\tilde{B}_2(u)$ is just the complex number $\tilde{B}_2(u) - \tilde{B}_1(u)$

$$\tilde{B}_{12}(u) = [\zeta(u) + if(u)]_{u_1}^{u_2} \qquad (64)$$



and is the first term in eqn.(53) for $\tilde{E}_P$.

Similarly, for $v_1 = -1$ and $v_2 = 1$, $\tilde{B}_2(v) - \tilde{B}_1(v)$ is

$$\tilde{B}_{12}(v) = [\zeta(v) + if(v)]_{v_1}^{v_2} \qquad (65)$$

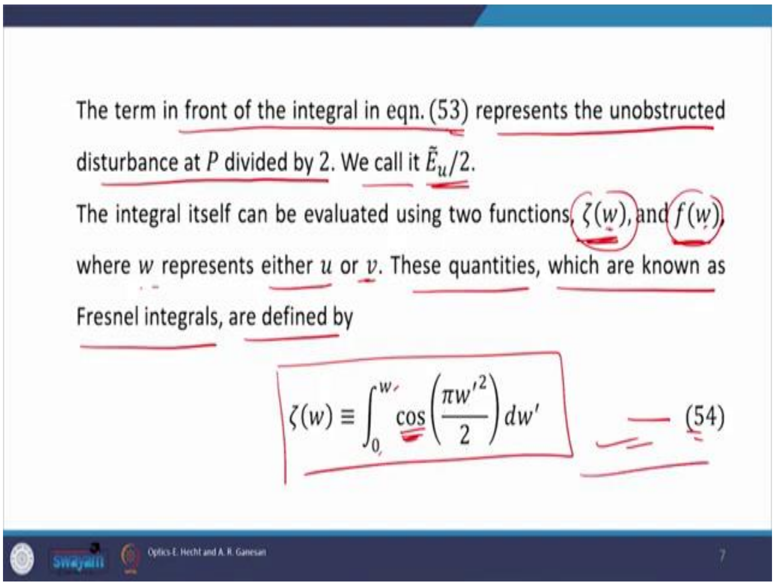
which is the latter portion of $\tilde{E}_P$. The magnitudes of these two complex numbers are just the length of the appropriate $\tilde{B}_{12}$ phasors.

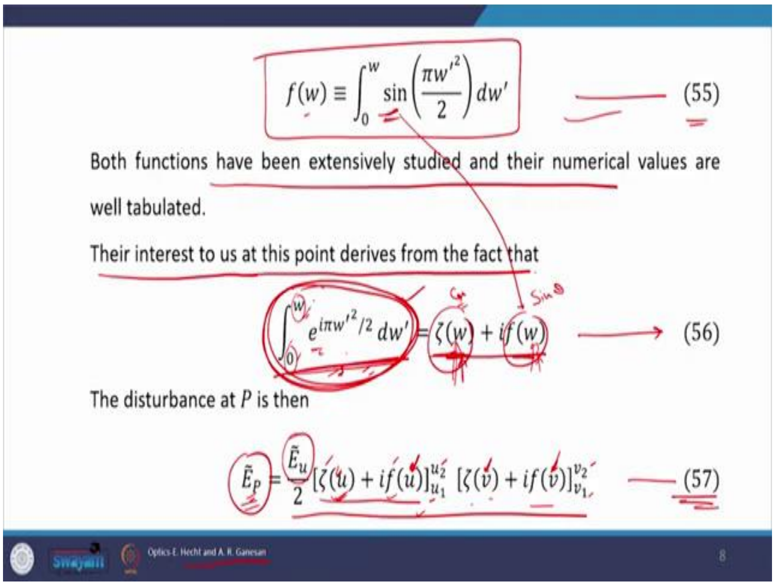
The slope of the spiral is

$$\frac{df}{d\zeta} = \frac{\sin \pi w^2/2}{\cos \pi w^2/2} = \tan\frac{\pi w^2}{2} \qquad (63)$$

and so the angle between the tangent to the spiral at any point and the ζ −axis is $\beta = \pi w^2/2$.

- As an example consider the problem of a $2\ nm$ square hole ($\lambda = 500\ nm, r_0 = 4\ nm$, and plane wave illumination). We wish to find the irradiance at P directly opposite the aperture's centre, where in this case $u_1 = -1$ and $u_2 = 1$.





Therefore, from equation 53, the irradiance can simply be written as $I_p = (I_u/4)|\tilde{B}_{12}(u)|^2|\tilde{B}_{12}(v)|^2$(, . And here I_u is the irradiance due to unobstructed source. Notice that the arc length along the spiral are proportional to the apertures overall dimension in yz direction.

Now the arc lengths are given translated in u and v terms and u and v are given by +1 and -1. Because these u and v were calculated from these values. The square of 2 nanometer size hole wavelength r_0. Now if you change the size of the square the u and v values will change and therefore arc lengths which we drew on the Cornu spiral that will also change.

And that will effectively change the irradiance observed at the point of observation P or irradiance to be observed at the point of observation P. The arc lengths are therefore constant, if the square size is fixed, if the aperture size is fixed then the arc lengths are constant regardless of the position of P in the plane of observation. You see that this arc lengths are function of size of the aperture only.

They are not the function of position of point P, on the other hand the phasor $\tilde{B}_{12}$ which is a function of u or phasor $\tilde{B}_{12}$ which is a function of v which spans the arc length are not constant. And they do depend upon the location of P. $\tilde{B}_{12}$ the phasor do depend upon the location of P. Why do they depend upon the location of P? Because in $\tilde{B}_{12}$ definition you see here we have ζ and f.

$\tilde{B}_{12} = \zeta + if$. And what are ζ and f? Let us go back again just to remind you ζ and f are given by equation 54 and 55 and here you see that there are variables w which are involved here. And this w varies between the limits of this integrals. And therefore, if you change the position of P, the value of phasor will change. The $\tilde{B}_{12}$, because I will reiterate it.

This u and v, $\tilde{B}_{12}$ is function of u and v and in u and v we have y and z, these are 2 variables. Now if we shift the aperture in aperture plane under plane wave evolution or if we shift the origin in the aperture plane the values of u and v will shift and why do we shift the u and v values or why do we shift the origin because to measure the off axis irradiance the location of P is varied.

And therefore u and v will vary, and therefore the phasor will vary. But with the size of the aperture the arc length which is the difference between u values and v values are constant. Therefore, Δu and Δv, which are the difference between u_1, u_2 and difference between v_1, v_2 respectively, which represents the arc length in the Cornu spiral they are independent of the position of P.

While this phasors, they do depend on the location of P. And we will see it in the next slide how do they depend.

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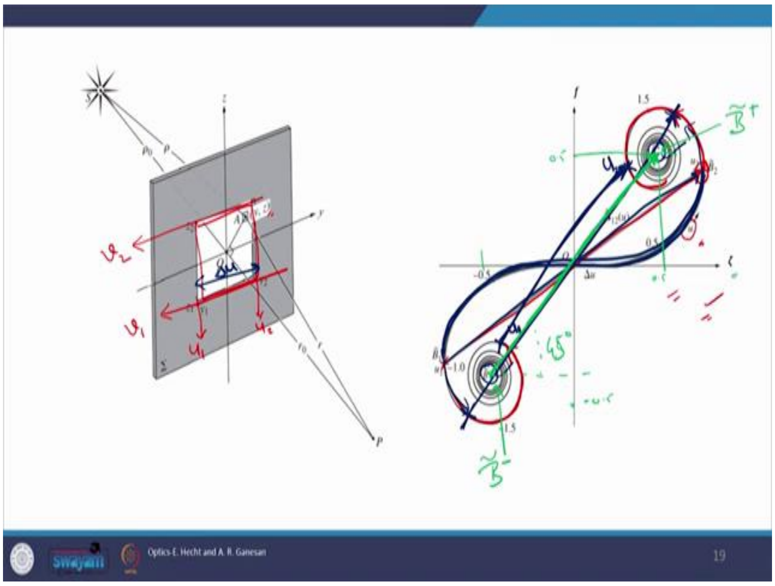

Now this is the Cornu spiral for this aperture. Now initially suppose that the aperture is very small. Now slowly what we do is that we slowly increase the size of the aperture, it is a square aperture and we slowly increasing the size of the aperture. Now once suppose we started with this size, with this size we know what are the value of u_1 here, what is the value of u_2 here, similarly we know what is the value of v_1 here and what is the value of v_2 here.

And depending upon these values of u_1, u_2 and $v_1,\ v_2$, we can draw 2 such spirals. This spiral is drawn for u, a similar spiral can be drawn for v because we are calculating 2 integrals and 1 spiral is drawn for one integral and the second spiral is done for the second integral. Now for u we represent u_1 here and u_2 here and then we draw this phasor and this phasor is nothing but the value of the integral.

Now if you increase the size of this square aperture slowly, then what will happen is that this point u_2, this will move in anticlockwise direction along this spiral. Similarly, the point u_1 will also move in anticlockwise direction along this spiral. They both will move, they move in anticlockwise direction with the increasing size of the aperture. Why they will move in such a way?

Because with increasing size of the aperture the arc length will increase and what is arc length, let me pick different color, this is our arc length, the blue color in this figure represents the arc length. And this arc length is proportional to the size of the aperture. Now in the horizontal direction where we are use measuring u this is Δu, Δu is difference between u_1 and u_2.

Now if you increase the size, this Δu will increase and therefore, this arc length will increase, the blue one, it will slowly go move towards the $\tilde{B}^{+}$ and $\tilde{B}^{-}$ values which are shown here. $\tilde{B}^{-}$ is here $\tilde{B}^{+}$ is here, which are the limiting values of Cornu spiral. Now you see that the length of the phasor is this much here, but if the arc length has increased, then the length of the phasor has now increased.

What will happen if the arc is increased such that the u_1 has reached here and u_2 has reached here, in this case the arc length will be this. Now this arc length is much smaller than the other 2 arc lengths. Now with this anti-clockwise rotation of u_1 and u_2 or with increasing arc length the u_1 and u_2 will spiral around this Cornu and the length of the phasor therefore will vary.

And this length of phasor will go through a series of maxima and minima, and therefore, at the point of observation P with increasing size of the aperture we will see sometimes less intensity and sometimes a more, a series of maxima and minima will appear at the point of observation. And if the aperture is opened so widely that it reaches to infinity then in that particular case we will get a phasor which will be given by this line.

It will start from point $\tilde{B}^{-}$ and it will reach to the limiting point $\tilde{B}^{+}$. This green line represents the phasor for unobstructed source where the aperture has open till infinity. And the green line it starts here at point $\tilde{B}^{-}$ and it ends here at point $\tilde{B}^{+}$. And we know at infinity this value of $\tilde{B}^{+}$ is 0.5 here and 0.5 here and here it is -0.5, again it is -0.5.

And this is a line which is making an angle of 45 degree with the horizontal axis, with the ζ axis. This was all about the widening of the aperture. The point of observation P was directly behind the center O, the origin O, it was directly behind this origin O and from there if you open up the aperture, then you see that a series of maxima and minima appear at P. But what will happen if we scan the aperture of access points?

Question is we know the intensity fluctuations on axis points but what will happen if P is shifted to some off axis points.

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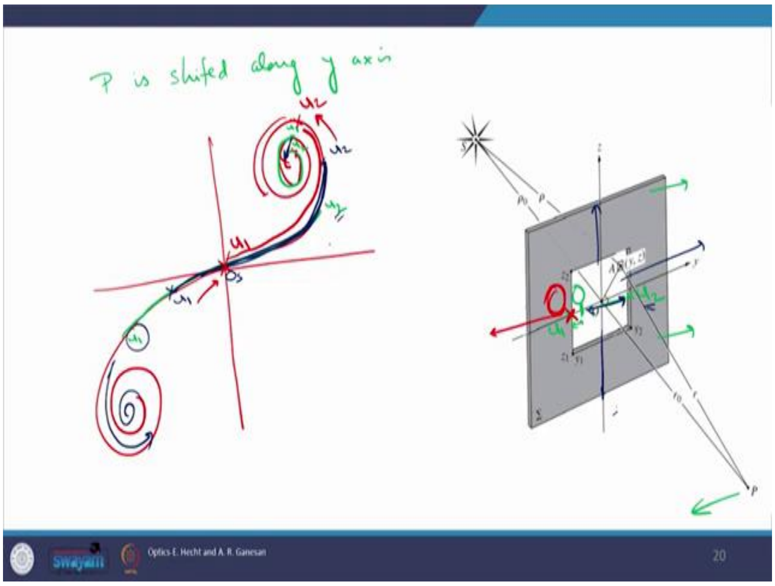

Let us also think about this, now say that we have a Cornu spiral and P is shifted on some off axis point and say that shift is along y axis. P is along y axis and this shift is in this direction. As we said before, shifting P is very complex mathematically, therefore, instead of shifting P on the left hand side people prefer to shift the whole aperture on the right hand side and therefore the new origin would be here.

This would be new origin, with the shift of the aperture the new origin would be shifted on the left hand side. Therefore, this point would be closer to the origin while this point would be farther to the origin. This u_1 would be closer to the origin and u_2 would be farther to the origin. With this understood initially this was our Cornu's spiral. And this was the arc length which we were getting with symmetric O.

Now when O is shifted towards left, along y axis, then u_1 has reduced, you see the value of new u_1, it got reduced, therefore, since this value of u_1 got reduced, the new u_1 will appear in a shorter distance from the origin OS here, this is the new origin new u_1, while new u_2 will appear here because the distance of u_2 got increased. This u_2 is now larger, therefore u_2 would be here.

And therefore this arc with the shift of origin towards left, this arc will shift up. If you shift the origin even more, say the new origin now is here, this is the new origin, with this what will happen is that u_2 will be shifted more towards the higher side and your u_1 would be shifted towards this direction. And therefore, when origin O is exactly at the edge of the aperture the u_1 would be here and u_2 would be here.

And then the new arc would be like this. Now if you shift the origin in the shadow region then this arc will keep shifting along this Cornu's spiral and a situation will come when everything will be very close to the $\tilde{B}^{+}$ point and if you go very far in the shadow you will not get any intensity, the arc length, it will keep rotating and suppose ultimately when we are far in the shadow region, then this would be the arc length.

u_1 would be this and u_2 would be this point and then if you draw the phasor, the phasor will look something like this, this would be the phasor. And this size of the phasor is very small. Then therefore, if you keep shifting in the shadow the phasor size will keep reducing and ultimately you will get negligible radiance at the point of observation P. Similarly if you go move the origin in this direction too, you will get the similar pattern.

And this arc will shift here in this direction. Now the arc will spiral around this spiral, the lower one and the same case will also happen if you move up or down in this aperture and there instead of plotting u, we will plot v.

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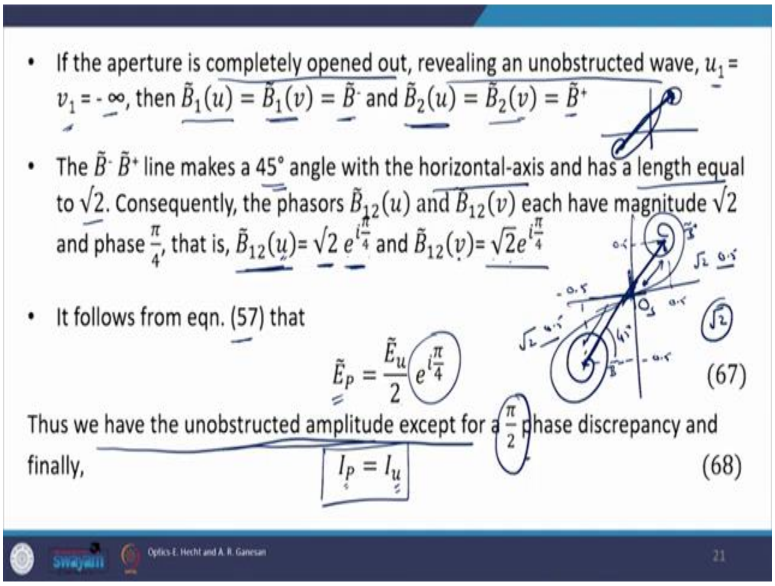

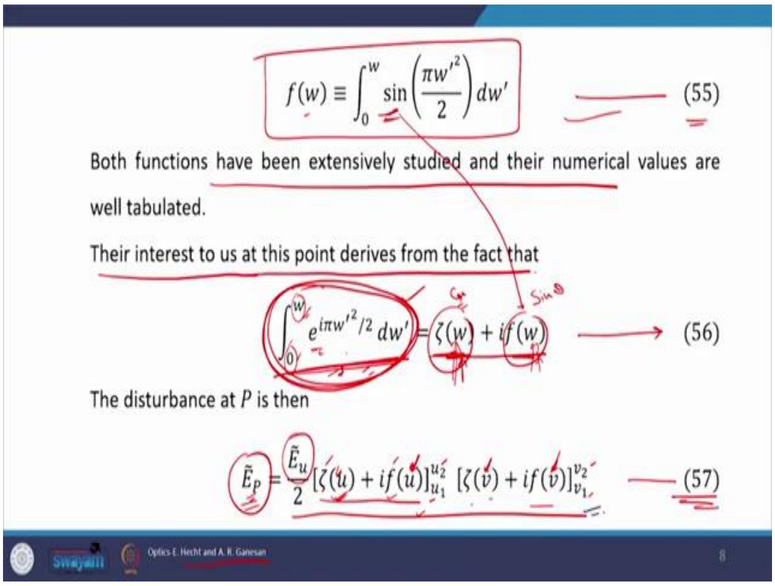

Now if the aperture is completely opened out revealing an unobstructed wave then we already know $u_1 = v_1$ and that would be equal to $-\infty$ and then $\tilde{B}_1(u)$ would be $\tilde{B}_1(v)$ which would be equal to $\tilde{B}^-$. And similarly, $\tilde{B}_2(u) = \tilde{B}_2(v) = \tilde{B}^+$, it means in this case this would be our points when the aperture is wide opened and this would be the final phasor.

Now this phasor, let me write again. Now this point is our $\tilde{B}^+$ and this point is $\tilde{B}^-$. Now this phasor for an unobstructed source will start from $\tilde{B}^-$ point and it will end at $\tilde{B}^+$ point. And it will pass through the center of course here, it will pass through the center OS. Now what would be the length of this phasor and what would be the orientation of this phasor?

We know that this point and this point they are at 0.5 unit away from the origin, similarly this point is at -0.5 and this point is -0.5. And from the geometry we can see that the orientation is 45 degree with the horizontal axis, and you can also calculate the length and length is equal to $\sqrt{2}$. Because this length would be equal to $\sqrt{0.5}$, this length, the lower part length, will again be equal to $\sqrt{0.5}$.

If you add them up then you will get $\sqrt{2}$. This would be the overall length of this phasor. Therefore, the expression for the phasor would be $\sqrt{2}$, which is the amplitude and the phase part $e^{i\pi/4}$. This is foru dependent phasor, for v dependent phasor that is in other direction, this would again be equal to $\sqrt{2}e^{i\pi/4}$.

These phasors are for aperture which is extended till ∞ that is equivalent to an unobstructed source. And from here if you substitute them back to equation number 57 the resultant field would be $(u/2)\sqrt{2}e^{i\pi/4}\sqrt{2}e^{i\pi/4}$, after multiplication you get $e^{i\pi/2}$. Now if you want to see equation number 57, let us go back and see this is the equation number 57.

Now with this if you want to calculate the irradiance, then irradiance would be equal to $\tilde{E}_p\tilde{E}_p^*/2$. And we will have the unobstructed amplitude except for a $\pi/2$ phase discrepancy. Because we know in the Fresnel formulism we always get this $\pi/2$ phase discrepancy because everything here is being calculated from secondary sources.

And we know that the in Fresnel formulism there is a $\pi/2$ phase difference between primary wave and secondary wave and apart from this discrepancy everything is on line and the intensity at point of observation P would be equal to the intensity which we usually calculate from the unobstructed source. With this I ends today's lecture. Thank you for bearing with me. Hope you are getting the concepts and see you in the next class.