Applied Optics
Professor Akhilesh Kumar Mishra
Department of Physics
Indian Institute of Technology, Roorkee
Lecture 40
Fresnel Diffraction

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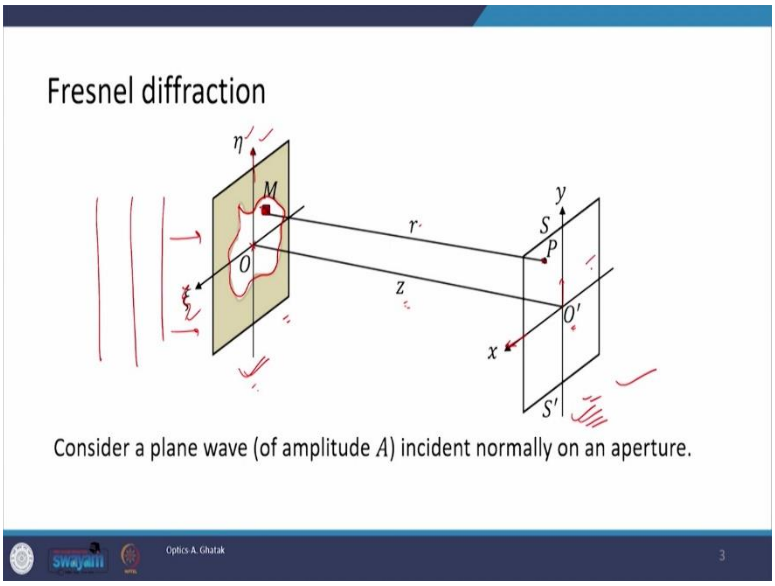

Hello everyone, welcome to my class. Today we will do a mathematical treatment of Fresnel diffraction and this is the last lecture in module 8. Now, this is a schematic arrangement where this is our aperture plane and this is our screen plane. Now in aperture plane you see a random aperture here and axis is associated with this aperture plane and in horizontal direction, we have η axis in the vertical direction while in the horizontal direction we have this ξ axis.

Now, in these two axes are crossing at O which is the origin in this aperture plane while in the screen plane, we have horizontal x and vertical y axis and the center is O', the distance between O and O' is z. And here in this aperture plane, we pick some differential area element which is designated here by M and the point of observation on the screen plane is P, the distance between the area element M and the point of observation P is r.

Now, this aperture plane is illuminated by a plane wavefront, this plane wavefront from the left illuminates this aperture plane and after the illumination, the pattern is being recorded on this screen.

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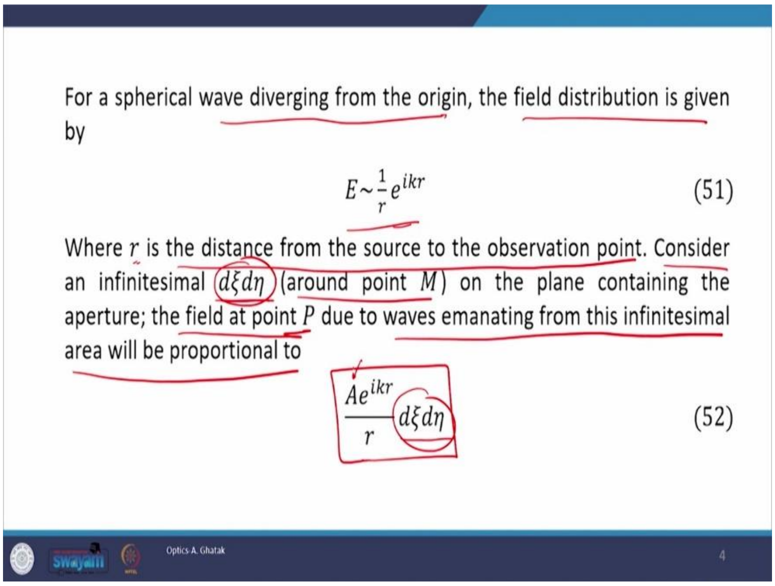

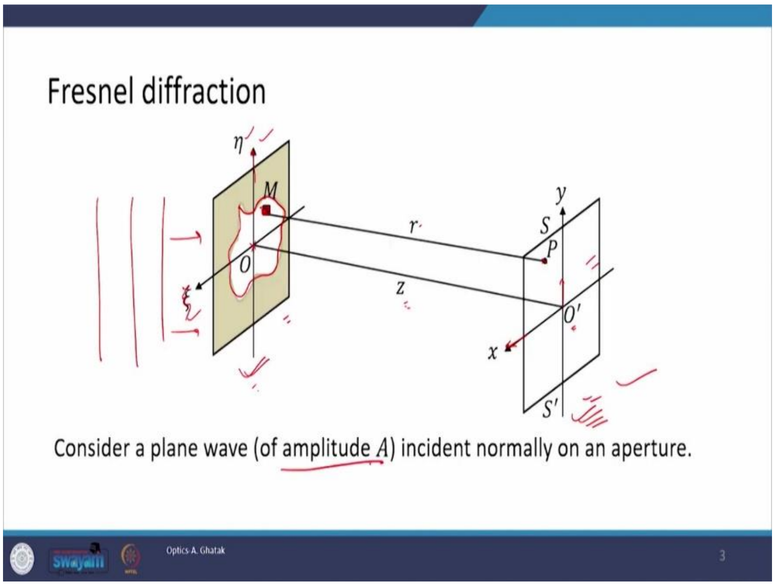

Now, in the aperture plane, we have a small area element which is M and out of this area element is spherical wave will emerge out. For a spherical wave, diverging from origin the field distribution is given by this relation where $E \sim (1/r)e^{ikr}$. Now, where r is the distance from the source to the observation point.

In the aperture plane as I discussed around point M, we have considered an infinitesimal area element. And the area is given by $d\xi d\eta$, the field at the point of observation P due to the wave emanating from this infinitesimal area will be proportional to this quantity which is $(Ae^{ikr}/r)d\xi d\eta$ which is the area element. Now, A is the amplitude of the incident plane wave which is given here in this slide.

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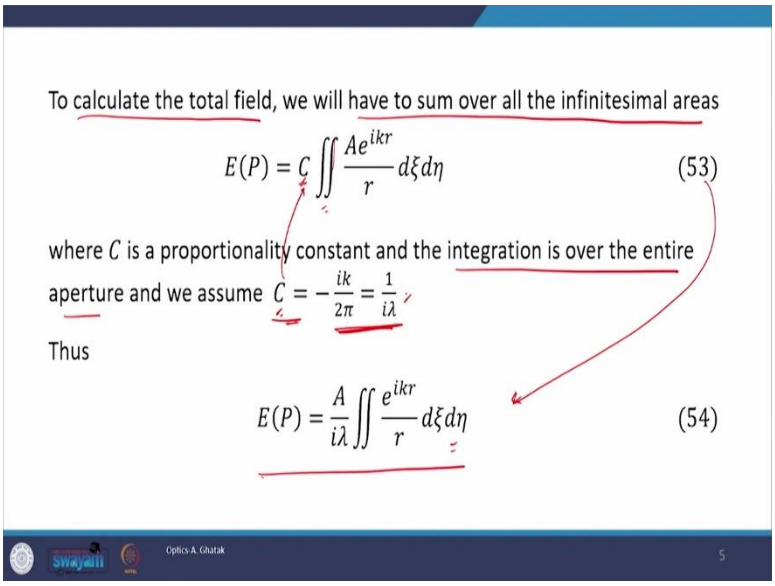

Now, to calculate the total field received in the screen plane, we will have to sum over all the infinitesimal areas and therefore, we just perform two-dimensional integration, two dimensional because we are integrating over area and here C is a proportionality constant and the integration is over the entire aperture.

Now, in this expression 53 we assume $C = -ik/2\pi = 1/i\lambda$, this comes from the Fourier transform, which is not in the purview of this course therefore, we assume that this constant $C = 1/i\lambda$. After substituting the value of C in equation number 53, the 53 reduces to this expression and this is the expression at point of observation P due to the entire aperture.

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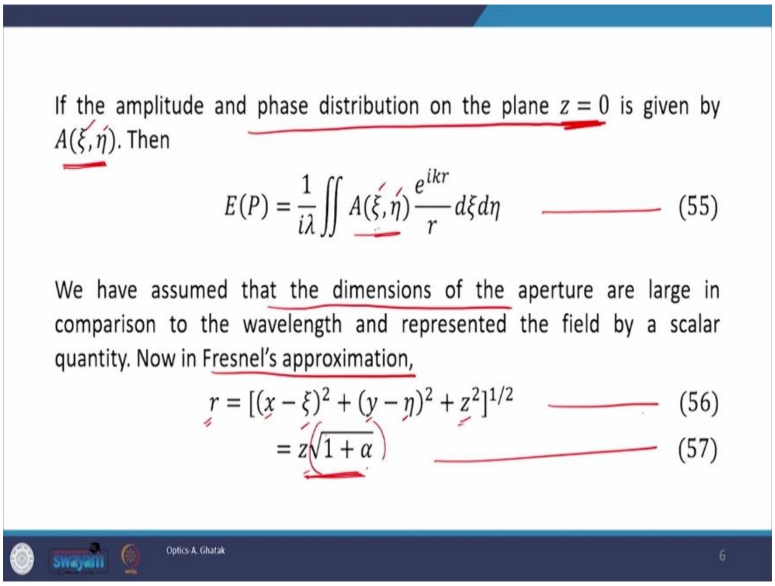

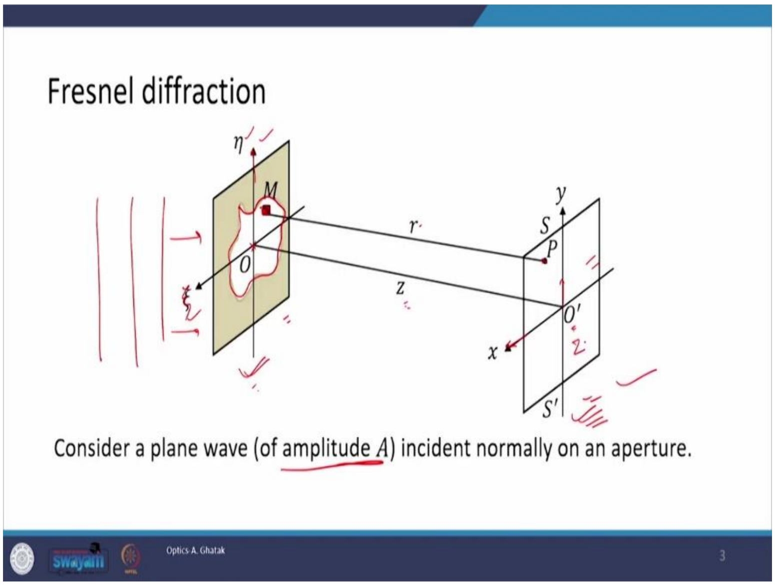

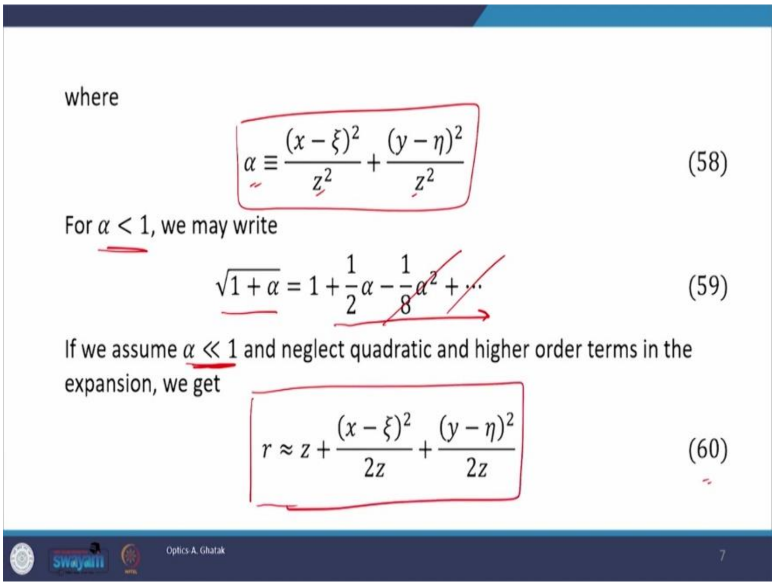

The amplitude and the phase distribution on the screen plane, the screen plane as you see in this figure here, at the screen plane, we have z is equal to 0 then at screen plane A would be function of the two variables and therefore, we replaced A with these two functions which gave equation number 55. Now, here we have assumed that the dimension of the aperture are large in comparison to the wavelength and represented the field by a scalar quantity. We did not talk about polarization of the field therefore the field is represented by a scalar quantity and the dimension of the aperture is large in comparison to the wavelength.

Now, we introduce Fresnel's approximation, which is as follows. Now, the r can be expressed by equation number 56, where x, y and z are the coordinates associated with the screen plane, while ξ, η are the coordinate associated by the aperture plane. Now, this equation number 56 can be expressed as given in equation number 57 where we have taken z out of the bracket and the rest of the quantities are expressed like this $\sqrt{1+\alpha}$ where α is given by this expression.

Now, since aperture is assumed to be small and the source to screen distances is large, $\alpha < 1$ because z is large here. Now, if α is a smaller quantity, then $\sqrt{1+\alpha}$ can be expressed like this and under these assumptions under this approximation of α much-much smaller than 1 we will neglect the quadrature and other higher order terms, with this the r can be expressed like this. And this expression of r can be now substituted in equation number 55.

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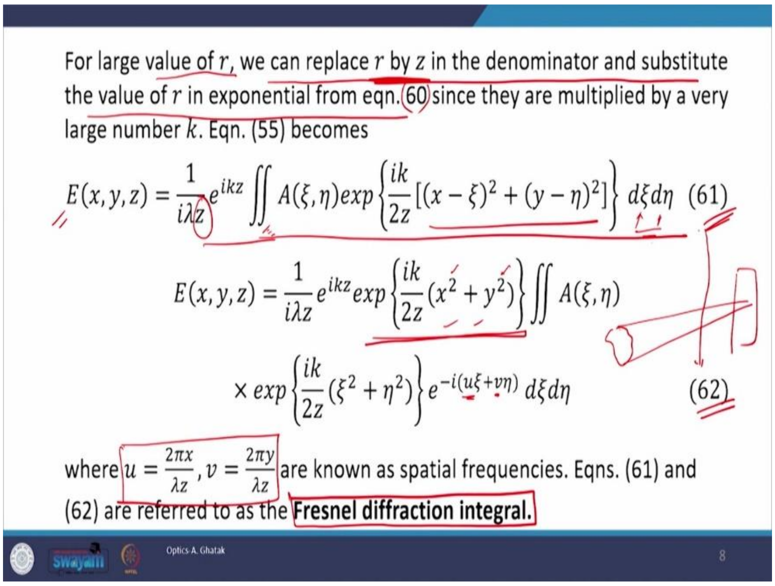

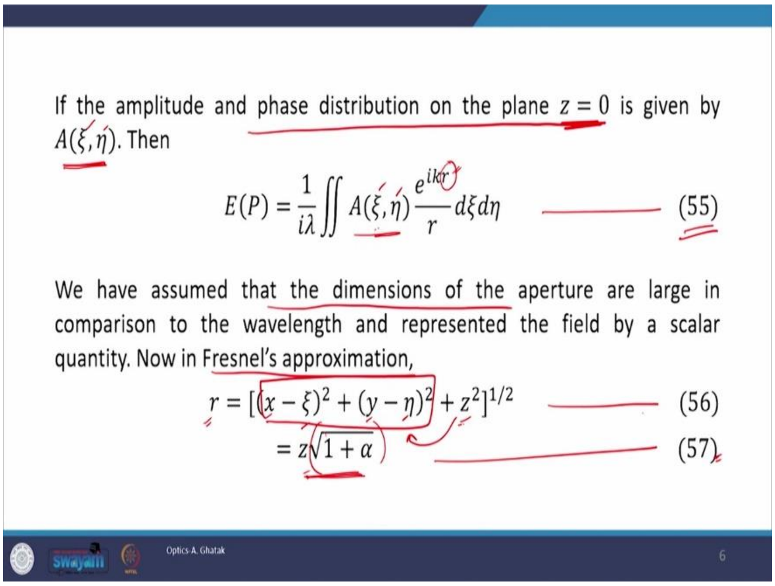

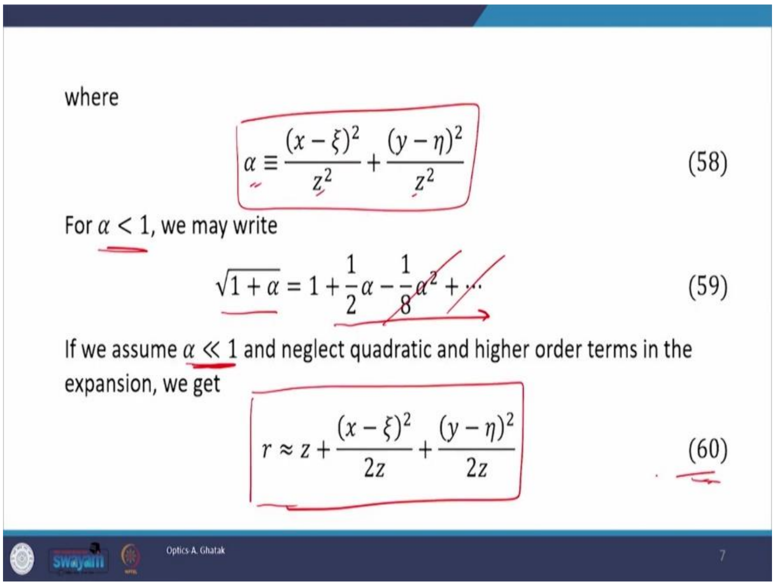

But note that for large value of r we can replace r by z in the denominator and substitute the value of r in the exponential from equation number 60. Why do we do so? We have already discussed that as long as we are in amplitude, we talk about the amplitude of a wave, the fluctuation in the amplitude would be very small, because aperture is very small the screen is very far therefore, this distance would almost be equal to this distance.

Therefore, in the denominator we have replaced r by z while in the phase part, which is here the phase part cannot be replaced by z because it is a sensitive parameter it comes with k, k is multiplied with r and in k we have $2\pi/\lambda$ and since λ is in the denominator and λ is very small quantity even the small fluctuations in r mix or incurs large change in the phase.

Therefore, we cannot apply this approximation in phase and then we will write or we will substitute the expression of r which is given in equation number 60 into the phase and therefore, the equation number 55 modifies like this. This is the newest expression of equation number 55 where in amplitude part we have replaced r by z while in the phase part r is replaced by equation number 60, this is the phase. On further simplification we can write equation number 61 as equation number 62 wherein we have taken this term out of this integral because it is constant for this integration.

We see that x and y are not variables here. Therefore, we can take the x and y dependent in out of the bracket. Now, here we have introduced two new terms u and v which are given by these expressions $u=2\pi x/\lambda z$ and $v= u=2\pi y/\lambda z$ and these terms are known as spatial frequencies, u is associated with x and v is associated with y. Now, equation number 61 which is given here and equation number 62, they are referred to as Fresnel diffraction integral, these two equations called Fresnel diffraction integrals.

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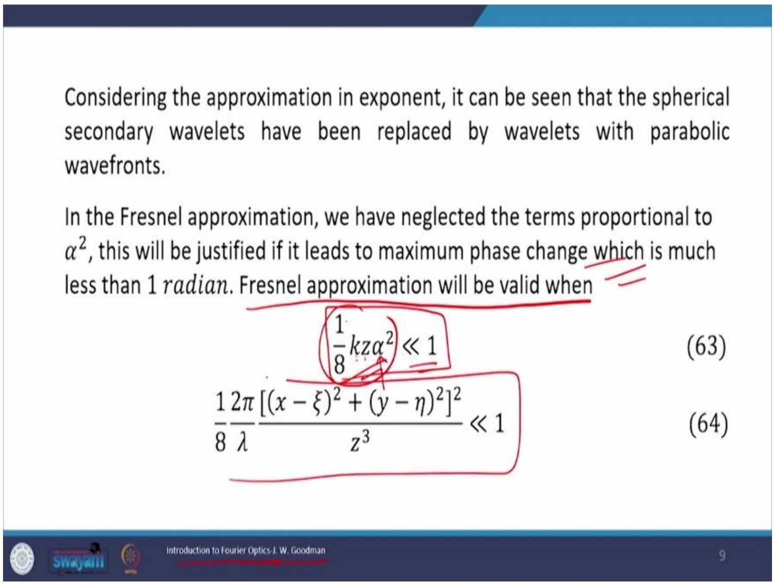

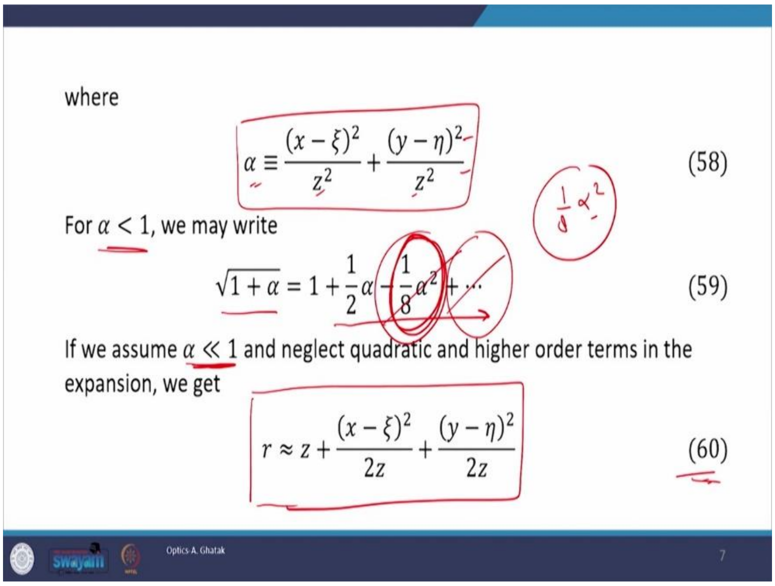

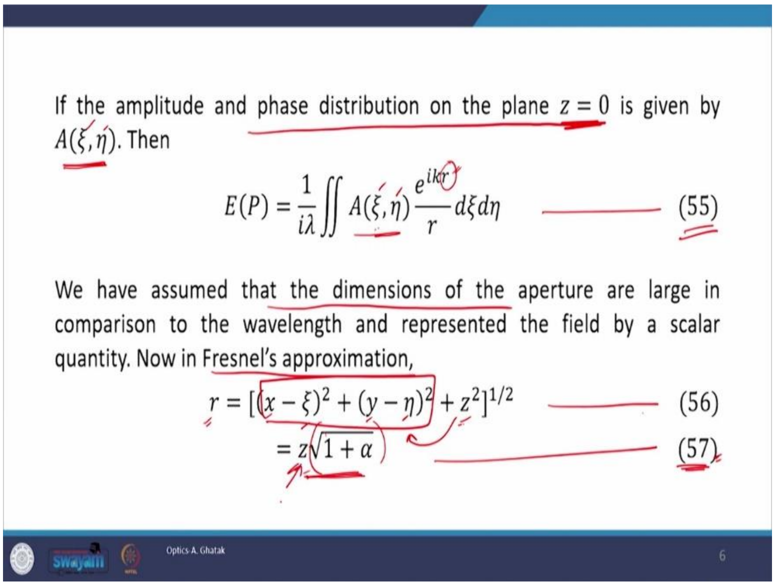

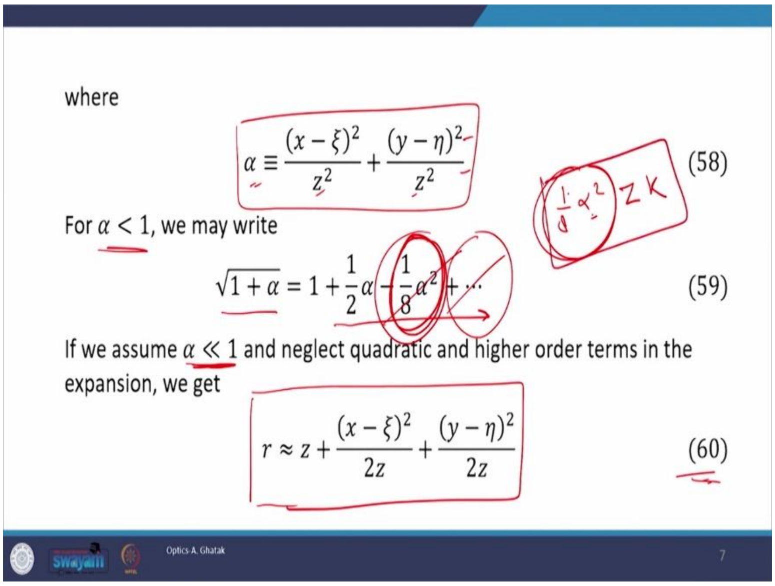

Now, considering the approximation in the exponent it can be seen that the spherical secondary wavelets have been replaced by wavelets with parabolic wavefront. You are seeing here the quadrature term; wavefront is now parabolic. Now, in Fresnel approximation, the only approximation till now which we use this, we have neglected the higher order term in this expansion.

Now, in the Fresnel approximation, we have neglected the terms proportional to α^2 and this will be justified if it leads to maximum phase change which is much smaller than 1 radian. Some author says that it should be smaller than 1 radian while some says it should be smaller than π, but let us stick with the reference, standard reference which is introduction to Fourier optics and this book is written by JW Goodman.

And as per this reference, the phase contribution from the α^2 term if it is less than 1 radian then its negligence is okay, it is acceptable to neglect this α^2 term, phase contribution from the α^2 term. Now Fresnel approximation therefore, will be valid when this quantity is much, much less than 1 radian and what is (1/8)kz α^2?

Let us go back now, here you see the term, the third term, on the right-hand side is (1/8) α^2, α is from equation 58 is unitless term. Now, this term gives some phase, how to calculate the phase? Now, go to equation number 57, you see that z is multiplied here, z is here in the multiplication factor.

Therefore, the distance related to 1/8 α^2 would be (1/8) α^2 z. How to calculate the phase? Then multiplied with k the phase therefore, would be equal to (1/8) α^2zk. And which is what written here (1/8) kzα^2 this phase as per the this above statement this must be much-much less than 1. Now, let us substitute the expression for α^2 with a substitution we get this modified expression.

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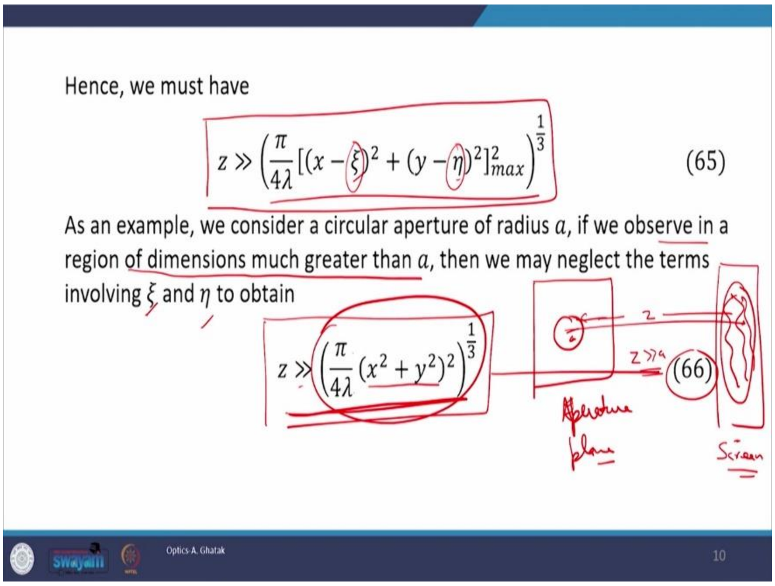

On further simplification we get for Fresnel approximation to be valid z must be much-much larger than this quantity and which looks very complex. But, to simplify it, let us consider an example, consider a circular aperture of radius a and now, if we observe in a region of dimensions much greater than a then we may neglect the terms involving ξ and η, why to neglect these two terms.

Suppose, we have a circular aperture and the radius of this circular aperture as per the example is smaller. Now, if the screen which is placed very far from this point say this is the distance between the circular aperture plane and the screen plane. Now, if we observe in the region of dimensions much greater than a.

And what we are observing? We are observing the diffraction pattern which is supposed to be on the screen here and this is your aperture plane, if this z is much-much larger than a then we may neglect these terms. Then with this, the above approximation reduces to equation number 66 which says z must be much-much larger than $(\pi/4\lambda(x^2+y^2)^2)^{1/3}$.

This is the reason where we will observe Fresnel diffraction pattern. If we put our screen at a distance z which is larger than this right-hand side quantity, then we observe Fresnel diffraction pattern, what about Fraunhofer?

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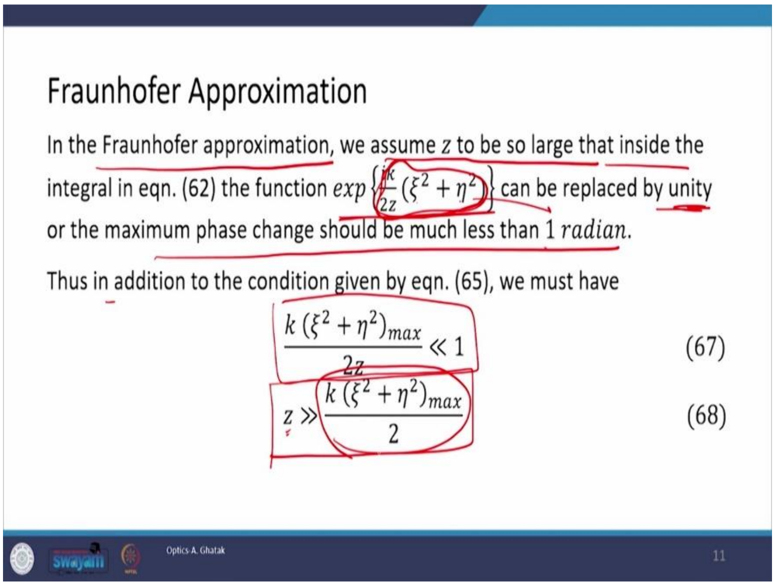

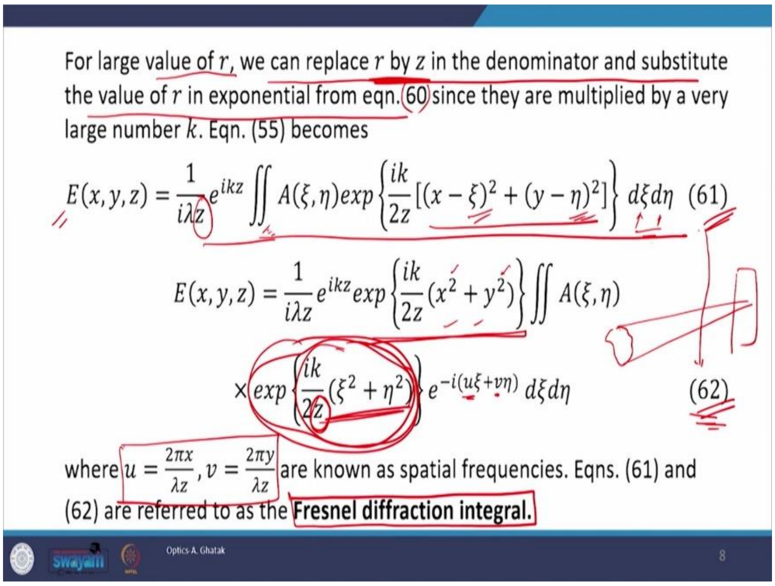

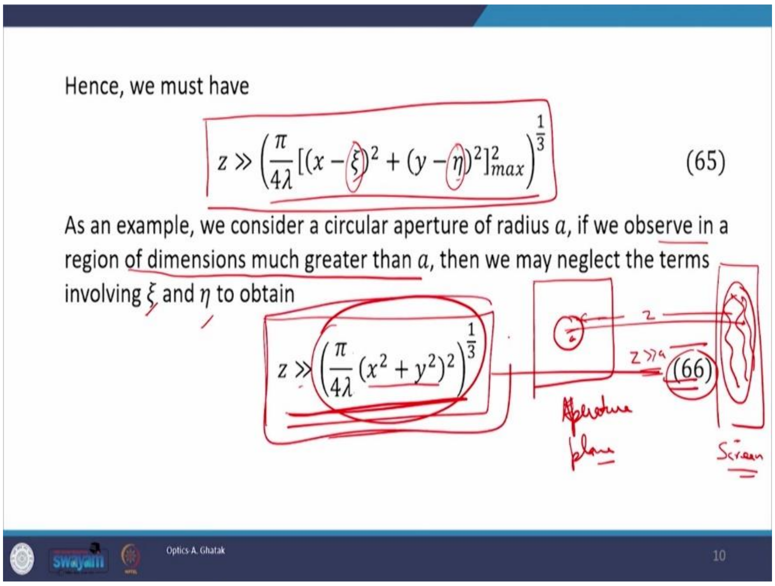

Now, in the Fraunhofer approximation we assume z to be so large even larger than the previous approximations, here z is assumed to be so large that in this equation, in equation 62, we neglect this term because z is so large that the z which is sitting in the denominator makes this term almost equal to 0. Therefore, whole of this exponential term reduces to 1.

With this approximation, we can replace this exponential term with unity or we can alternatively say that maximum phase change will be much less than 1 radian. Because of this phase, the phase introduced by this term if this is less than 1 radian then we can replace this exponential term with the unity. If this condition holds, then we are in the Fraunhofer regime of diffraction.

Therefore, in addition to the Fresnel condition, which is given by equation number 66 here, this condition in Fraunhofer region, we have one more condition and this condition is that this term, this phase term $k(\xi^2+\eta^2)/2z$, this term must be less, less than 1 and with this we get an expression for z and when z is much much larger than this quantity, then we are in the Fraunhofer region.

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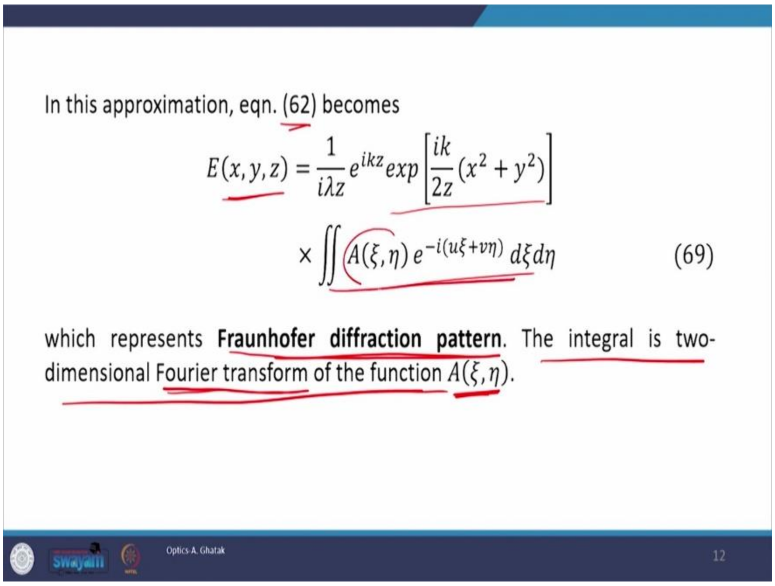

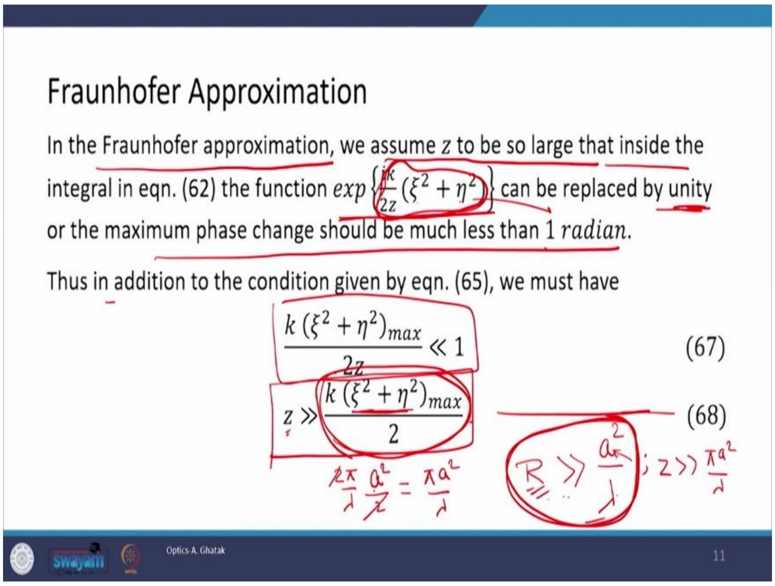

Now, if you rewrite equation number 62 in Fraunhofer region, then it looks like this where we have removed the second exponential term which has $\xi^2 + \eta^2$ term. And this represents Fraunhofer diffraction pattern, if you plot the radiance with this field and we get Fraunhofer diffraction, the integral is two-dimensional Fourier transform of this A function A (ξ,η). Since, the Fourier transform is not in the purview of this course, therefore, I am not commenting on this, but just for your information equation 69. In equation 69, the integral is Fourier transform of this function.

Now, if you see this term, equation number 68. And if you remember the Fraunhofer conditions, which we discussed while introducing the difference between Fraunhofer and Fresnel, the introduction of interference, then we talked about R and we said that R must be larger, much-much larger than a^2/λ, where a is the biggest dimension of the aperture. Now, if you look here, in equation number 68, this right-hand side of equation number 68 can be written as k which is 2π/λ, $\xi^2 + \eta^2$ nothing but it is a radius if we assume that the aperture is circular.

Then $\xi^2 + \eta^2$ is radius of the circle square, a is the radius of the circle and then we have 2 in the denominator this 2 and 2 will go away and this $\pi a^2/\lambda$, z is much-much larger than $\pi a^2/\lambda$, which is roughly equal to this equivalence, this expression that says R is much-much larger than $\pi a^2/\lambda$ and what is R? R is the distance from source to aperture and distance from aperture to the screen. Out of these two distances, the larger one is represented by R, this larger distance must be much-much larger than $\pi a^2/\lambda$. And equation number 68 also says the same thing.

Now, this is how the Fresnel diffraction is treated mathematically. And with this treatment, we also learned approximations, which are introduced by Fresnel as well as Fraunhofer diffraction. And these treatments are done very nicely in a book by Professor A. Ghatak and the title of the book is optics. Now with this I end my lecture, thank you for joining me. See you in the next class.