Applied Optics
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Lecture 36
Diffraction at a Rectangular Aperture

Hello everyone, welcome to the class. Today, we start new module which is module 8. In this module, we will talk about diffraction at rectangular and circular apertures then diffraction grating, then we will talk about the grating spectrum and its resolving power and ultimately we will start talking about Fresnel diffraction, we will slowly switch to Fresnel regime of diffraction.

Now, in the last module, we talked about single, double and multiple slit diffraction pattern, these are the traditional experiment which are easy to analyze and people perform the lab experiment over this. But instead of having slit which is a basically one dimensional entity, what will happen if we give some finite width to the slit. Although in the single slit class, we said that there is a some width there, but what if that width is at appreciable what if the slit width is such that we can call this slit a rectangular aperture or we start calling this slit a rectangular aperture okay.

Now, we will study this type of slit or slit with a wider width which we call now rectangular aperture, this would be covered today okay. Now, before starting rectangular aperture, let us do a very generalized study on some random shape aperture.

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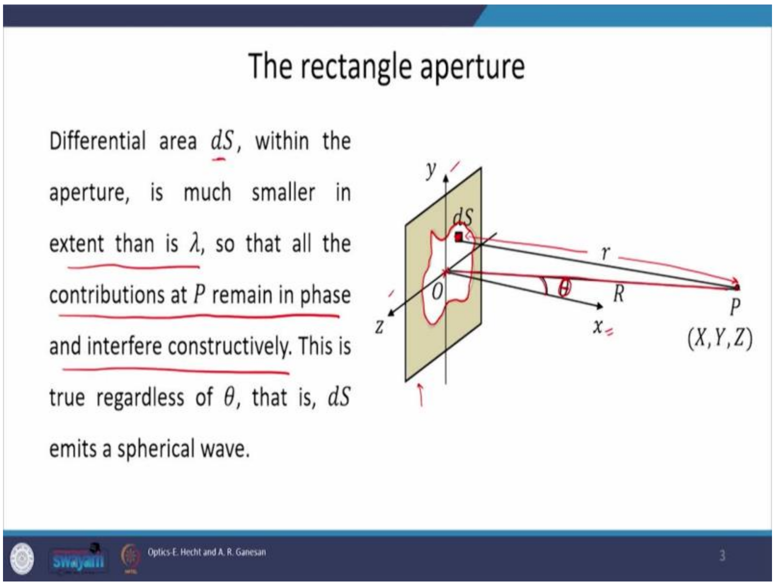

Suppose we have an aperture which is some random shape as shown here in this figure, it is some random shape aperture and we are signing plane wave on this aperture and the pattern is being observed, the diffraction pattern is being observed at a point P which is very far from the aperture plane and we are in the Fraunhofer regime therefore. Now, to study this case we pick a differential area of area dS from this aperture, we pick a very small area element in this aperture and we assume that area of this element is dS okay and this element, this area element is at a distance is small r from the point of observation P.

The origin is placed at the center of the aperture plane and from the origin the point of observation is at a distance R and this R is making or this line which joins origin to point P is making an angle θ with respect to the x axis. The directions of y and z axis's are given in the figure.

Now, the differential area which is inside this aperture is assumed to be much smaller in extent than λ. And if this is so, then the contributions at P remains in phase and all the rays which is starting from this differential area they will interfere constructively. Since this area element is assumed to be very small even smaller than the wavelength of the light, all the light which is emanating from this area element dS therefore will interfere constructively at a point of observation P and this is true regardless of θ.

Since it is an area element we will have to define the source strength per unit area. While dealing with the line source, we also defined the source strength but there it was defined per unit length, but now it is an area limit therefore the source strength definition is changed a bit.

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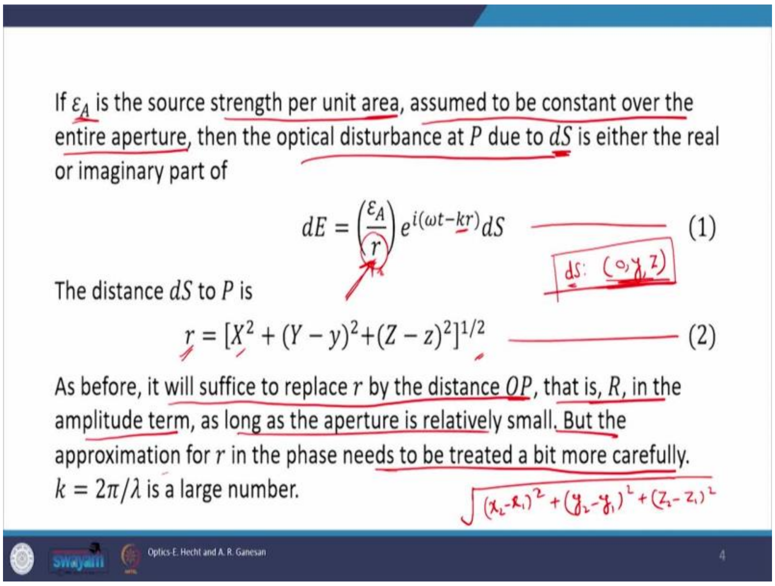

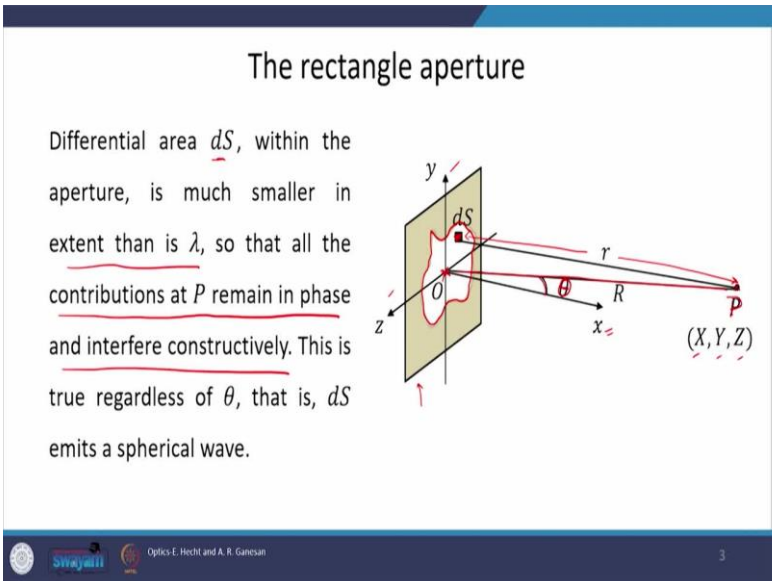

And here it is defined as a strength per unit area and is designated by ϵ_A and this ϵ_A is assumed to be constant over the entire aperture then the optical disturbance at P due to area element dS is given by equation number 1 in complex representation. And of course, its real or imaginary part will give you the real field at point of observation P.

Since dS is very small, we assume that it is emitting a spherical wave front and therefore, you are seeing r sitting in the denominator of the amplitude. Now, we will look for the expression for r.

Let us go back to the figure. Now, you see here O is the origin and our coordinate system x, y, z passes through this origin of course, and the coordinate of point P is X, Y, Z. Now, with this let us assume that coordinates of area element dS is given by, since area element dS is randomly picked, therefore, it may take any values and therefore, the coordinate of point dS this would be given by (0, y, z), why 0? because the aperture plane is in y z plane, aperture plane is in y z plane therefore, x axis value for any point in the aperture would be 0 and it is y and z which is varying in the aperture plane.

Therefore, coordinate of any point in the aperture plane can safely be given by (0, y, z). Once we know the coordinate of dS and the coordinate of point P then the separation between dS and P which is given by r can be expressed by equation number 2 which is nothing but the formula of separation between two points whose coordinates are known. You know that if we have two points whose coordinates are given by (x_1, y_1, z_1) and (x_2, y_2, z_2) then the distance between the two points would be equal to $[(x_2-x_1)^2 +(y_2 -y_1)^2+ (z_2 - z_1)^2]^{1/2}$ and this is what exactly is done here to in equation number 2.

Equation number 2 is nothing but this $[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2]^{1/2}$ is the distance between two points whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2). Now, as before it is sufficient to replace r by distance OP that is r in the amplitude term here, we are talking about the denominator in equation number 1 amplitude.

The denominator of amplitude in equation number 1, why we are saying so, because the point of observation is very far, we know that it is a spherical wave therefore, at the wave propagate it decays the field decays by 1/r and if the aperture size is very small, then we can replace this 1/ r with 1/R, once R is very huge as compared to the size of the aperture.

But we cannot do in the phase because the phase needs to be treated very carefully, it is a very sensitive term and 1/λ is there in multiplication with r. λ is coming in the denominator which is very small quantity therefore, any small fluctuations in r lead to appreciable change in the phase. Therefore, this approximation we cannot exercise in the phase part while in the amplitude we can do it very easily, very safely.

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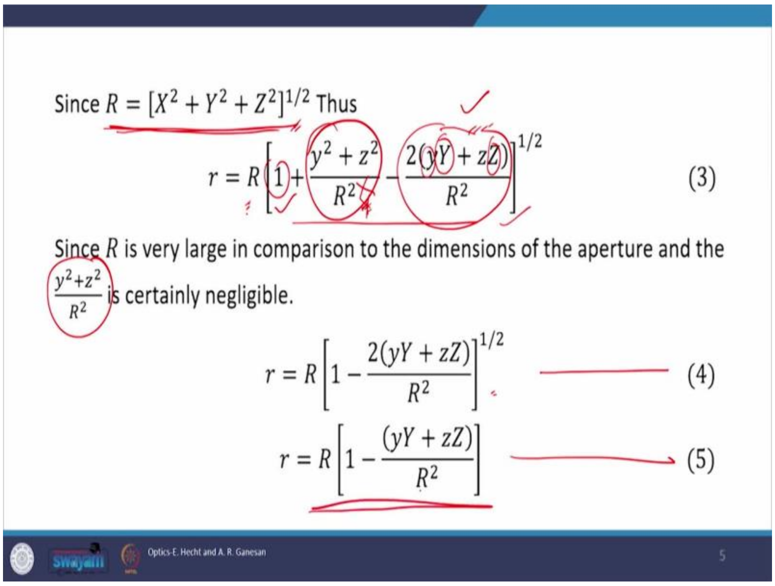

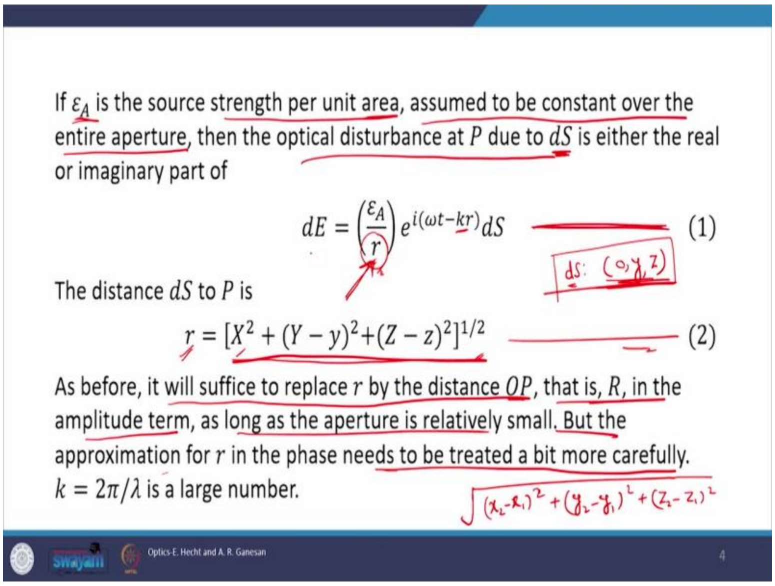

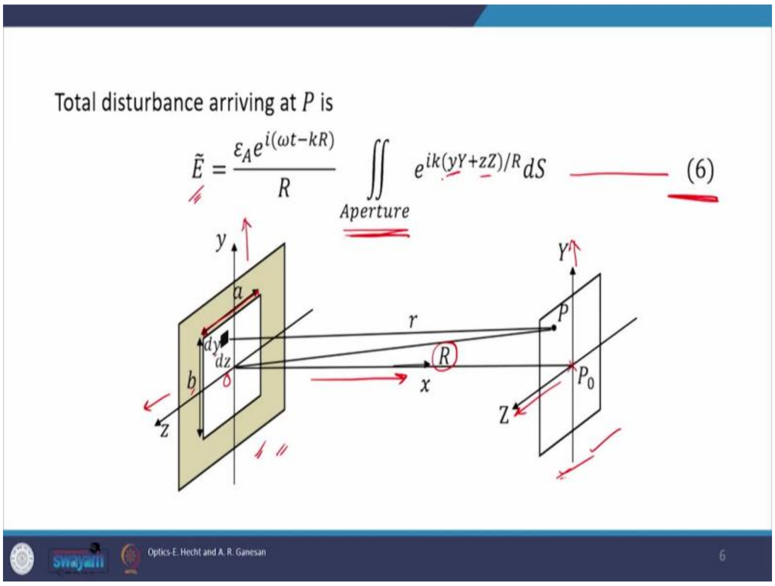

Now, what would be the expression for R, R is the distance from origin O to the point of observation P whose coordinate is (X,Y,Z) therefore, using the same formula which we discussed on the last slide, $R=[X^2+Y^2+Z^2]^{1/2}$.

And you see here r has this expression. Now, with this expression of R we can modify the previous equation, the equation number 2 and if you take R that is the $[X^2+Y^2+Z^2]^{1/2}$ out of the bracket then we are left with this term and we know that R is much-much larger than any other length expression here in equation number 3.

Now, since R is very large in comparison to the dimension of the aperture, therefore, this term, the second term in this expression 3 on the right hand side, this is negligible and we can neglect it because y and z they are the coordinates of points which are in the aperture plane and aperture size is very small as compared to R. Therefore, y^2 and z^2 they will be even smaller and since

R^2 is also there in the denominator, which is very huge term. Therefore, this term will go away, we will neglect it.

We would left with the third term and the first term only, these two terms will be there while this would be neglected, why are we not neglecting the third term because in the numerator here we have a y multiplied by Y, Y is a big number, it is a bigger quantity this saves the third time. Similarly, here it is Z, this also saves this third term from getting neglected.

With this we get a modified expression for equation number 3 that is for R and then slight similar mathematics gives equation number 5, the binomial expression. Now, once the expression of R is known, then we will substitute it into the expression of disturbance, what is the expression for disturbance, this equation number 1 is the expression for disturbance which is contributed by lights which are getting emanated from area element dS and they are reaching at point of observation P.

We will substitute here the expression of R which we just calculated and then integrate over the entire area of the aperture, this random side aperture. Let us do this, after this integration over aperture which is an area integration, the total field at the point of observation P would be given by equation number 6. Once we have this now, let us go to our problem which is to calculate the irradiance at screen due to the rectangular aperture.

Now, the rectangular aperture is kept here and this rectangular aperture has dimensions a and b and the center is here at point O that is the origin, x is pointing in this direction, y is vertically up and z is coming out of the paper. Now, the screen is placed here which is at a distance R from the origin O.

And we also associate a coordinate system with a screen plane. In this coordinate system Y is pointing up, Z is coming out of the plane of the paper while the X associated with the observation screen plane is coinciding with the x axis of the aperture plane, with this, we will solve equation number 6 to get the irradiance produced by rectangular aperture. Now, you see that in equation number 6 we have y and z dependent term which are coming in the exponential.

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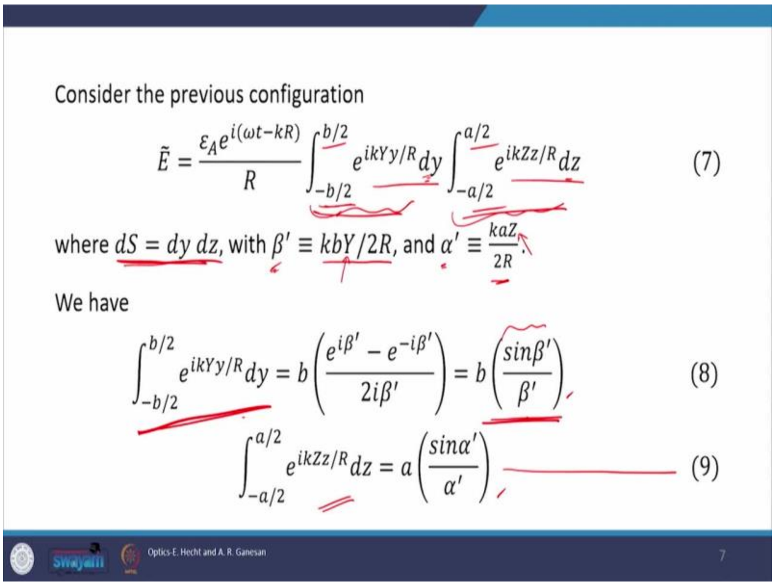

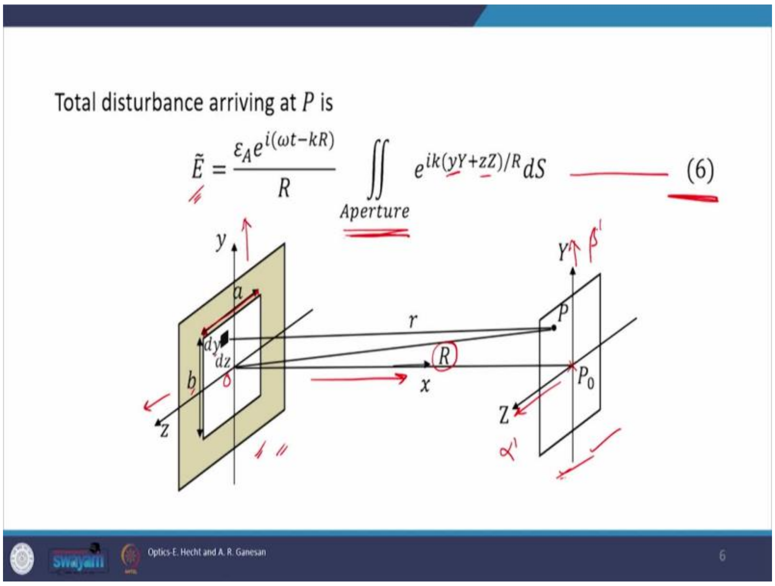

Since we are integrating over the aperture and aperture in our case is a rectangular aperture therefore, we can safely separate y and z dependent term. Aperture in the y direction is b length long. Therefore, the limit of integration will vary from - b/2 to b/ 2 and integration in z direction will vary from -a/2 to a/2. And now we can separately integrate these two integrals. And the dS is now replaced with dy into dz which is nothing but the area element, elemental area element.

We introduced two new parameters β' and α' here, β'= kb Y/2R while α'= KaZ/2R. Make it a point that β' is along Y direction in screen plane and α' in is in Z direction in a screen plane. What I mean to say is that α' is pointing here and $\beta'$is pointing here along Z, sorry α' is along Z, let me correct myself α' is here and β' is here β' is along y.

Now, we have two integrals to be solved, let us start with the first one okay and after a bit of mathematics you see that this integration is equal to b sin β'/β' again sinc function here. Similarly, the second integration can also be solved which is given by equation number 9. Now, let us substitute these two equations 8 and 9 back into equation number 7.

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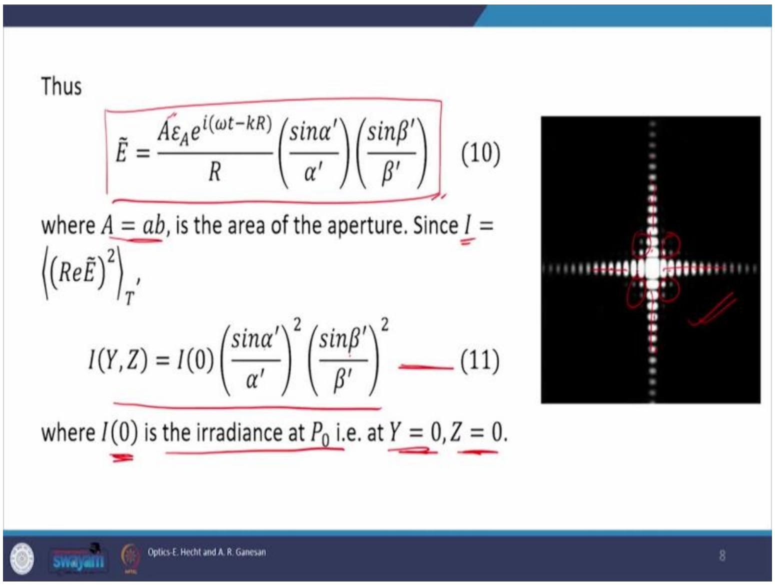

Then, the final field distribution at the point of observation P here is now given by equation number 10 you see here A which is nothing but ab which is area of the rectangular aperture. Now, once the field is known, we can easily calculate the irradiance and the irradiance is given by equation number 11.

Now, here you see that we have [(sin α'/α') (sin β'/β')]^2 and there is a constant which is I(0) which is irradiance at point P that is at the center of the screen whose coordinate is y is equal to 0 and z is equal to 0.

And this is how typically a diffraction pattern from aperture, particularly square aperture looks like. You see that they have bright and dark spot here and they decay down as you move up and down and left and right. Apart from this horizontal and vertical direction fringes, you also see a few fringes here too, okay, you see these fringes in the corner, see off axis fringes too.

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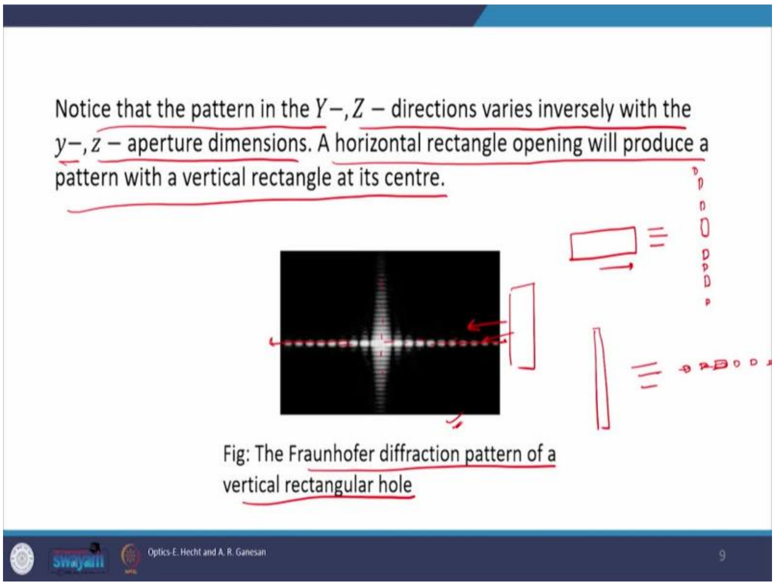

Now, notice that the pattern in Y and Z direction varies inversely with the y, z aperture dimensions, what I mean to say is that suppose this is your aperture which is elongated in one particular direction and then the corresponding pattern would be like this.

If the aperture is elongated in horizontal direction, you will get vertical aperture and similarly if the aperture is elongated in vertical direction, you will get horizontal pattern, this is how you get. A horizontal rectangle opening will produce a pattern with a vertical rectangle at its center and in this figure, this is what exactly you see, this is the Fraunhofer diffraction pattern of a vertical rectangular hole, rectangular hole is something like this and you see this horizontal pattern.

Although in vertical direction too, there or something, but it is extending this horizontal pattern or extending to the greater distances and they are more dominant. Now, what about maxima and minima?

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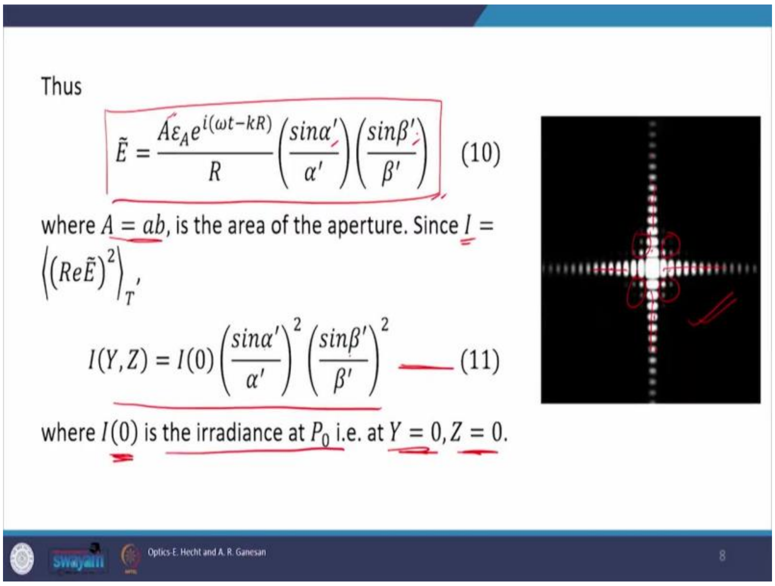

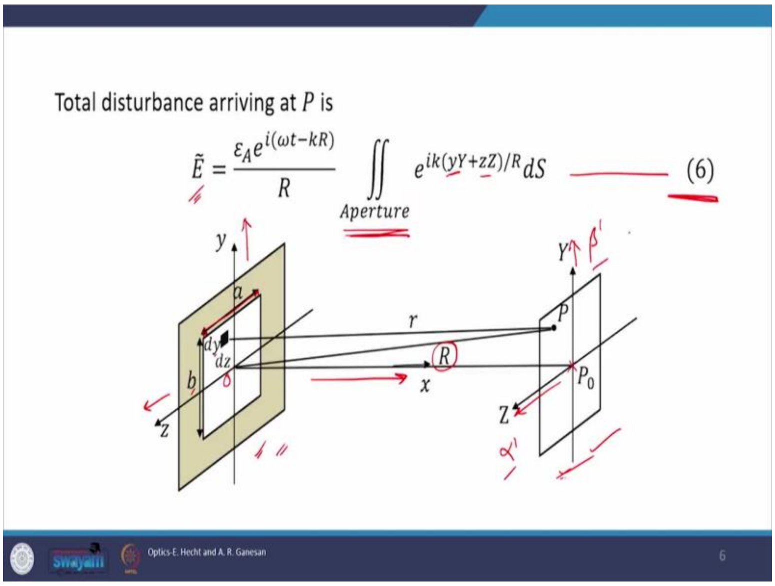

Now, you see that there are two parameters here in this expression of irradiance α' and β' which are associated with Z and Y axis. Now, it means this pattern appears along those axes.

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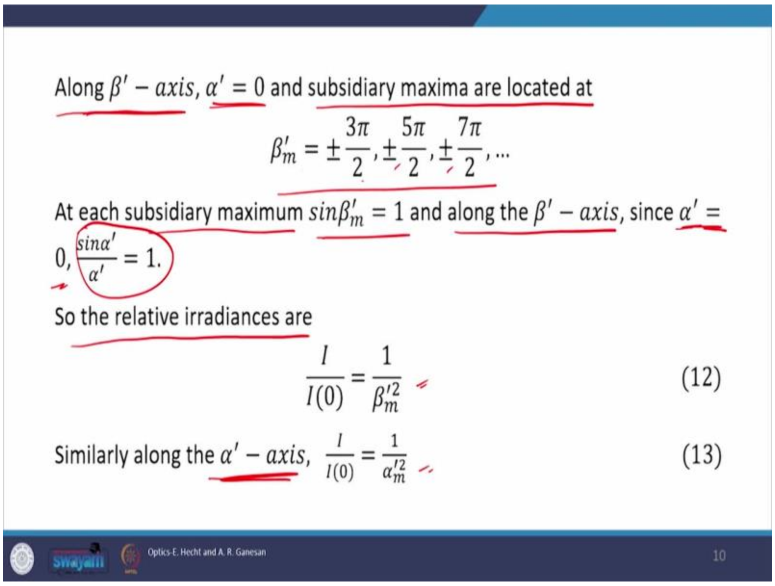

Of course, there are some patterns which are not on these axes too, but we can easily calculate them, how to calculate them? Let us first talk about along β' axis. Now, let us assume that β' and α' represents our new axis in screen plane. Now, along β' axis, α' would of course be 0. Therefore, the maximas would be located at β' values which are given by this here, $\beta_m' = \pm 3\pi/2$, then $\pm 5\pi 2$, then $\pm 7\pi/2$.

At each subsidiary maximum, the sin $\beta_m'=1$ okay and also along β' axis, since $\alpha'=0$ therefore, sin α'/α' would also be equal to 1 which is a sinc function. Since $\alpha'=0$ along β' axis therefore sinc function will assume value which is equal to 1.

Therefore, along β' axis, the relative irradiance would be given by this expression which is $1/\beta_m'^2$. Similarly, along α' axis the relative irradiance would be given by $1/\alpha_m'^2$.

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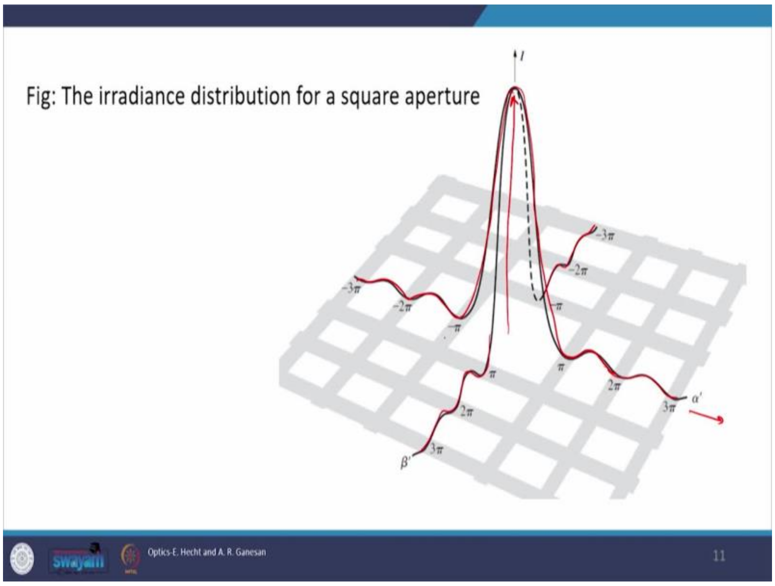

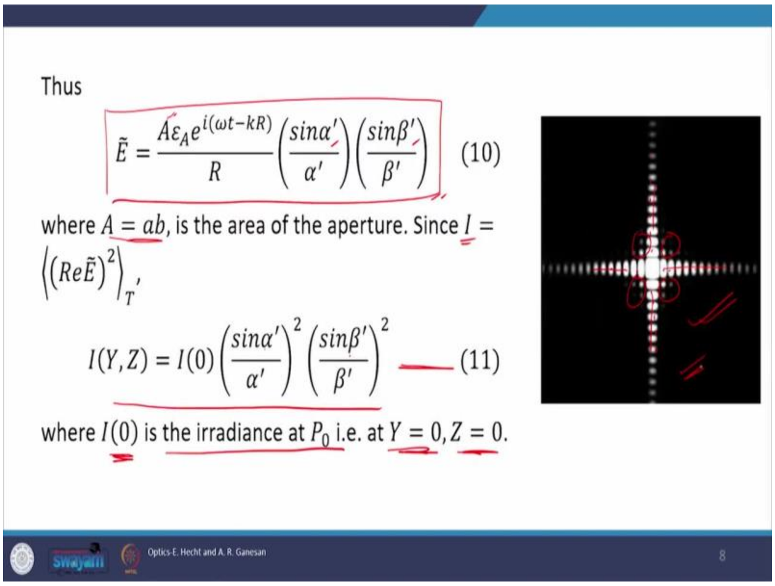

Now, if you plot this, let us assume that our aperture is square okay which is a special case of rectangular aperture then you will see very good intensity at the center of the screen and then there would be some pattern which would be nothing but sinc pattern, you see in α' direction, you get a sinc pattern which is like this.

Similarly, in the β' direction you will also get a pattern which is also a sinc pattern. Now, this pattern when you see in 2d you get this structure okay and this is how the diffraction pattern of a rectangular aperture look like. Now, with this I end my today's lecture, thank you for your patience.