Applied Optics
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Lecture 03
Ray Theory, Fermat's Principle

Hello everyone, welcome to this class of Applied Optics. Today we will learn more about geometrical optics. In previous class, we taught basics of geometrical optics, we learn about what is k? Which is your wave vector and then we learn about frequency, the definition of frequency, the definition of time period and then we learn about angular frequency of a wave, and then we saw how to represent a wave function in complex representation. Today, we will learn ray theory in this module one.

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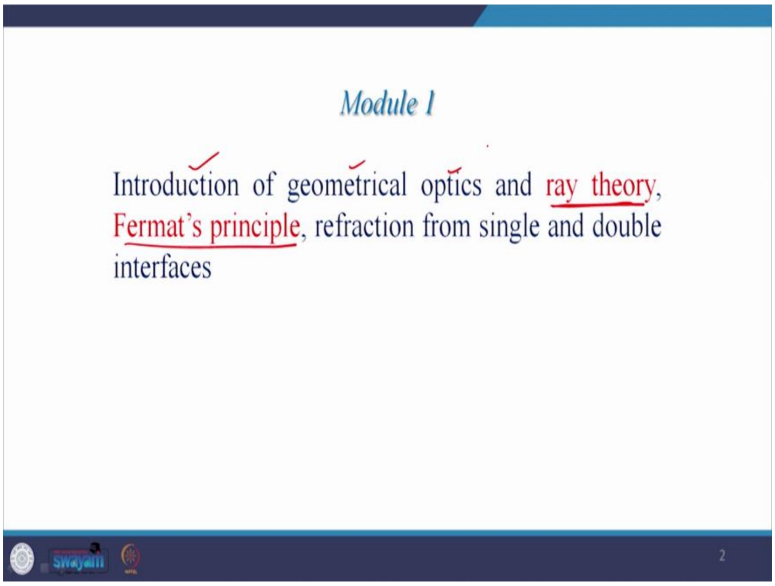

Today we will learn ray theory and Fermat principle, these parts are already covered introduction of geometrical optics we have done in the previous class and in module one. We will today learn about Ray Theory and Fermat's Principle.

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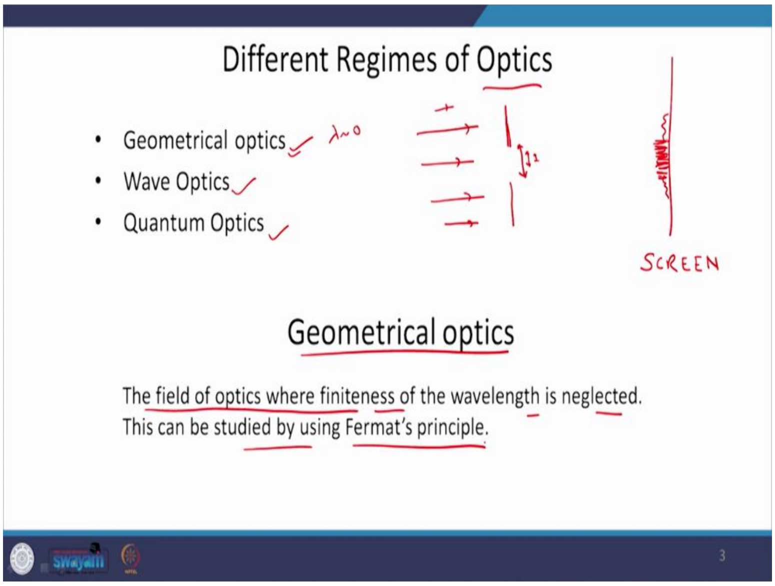

Now, as discussed before the different regimes of optics are geometrical optics, wave optics and quantum optics. In geometrical optics, we assume that wavelength, the extent of wavelength is almost zero, we neglect the finiteness of the wavelength. And this geometrical optics will talk about the part of the optics where diffraction is neglected. To understand what diffraction is, you can consider a sheet with an opening with an aperture and then you shine a parallel beam of light on this sheet. Then here on the screen, you will see that due to this opening you will see a bright spot in this region.

Now, if you start reducing the size of this opening, then what will happen is that this bright spot will extend in the shadow of reason here. The extension of this bright spot in the shadow region is, what we call diffraction and this phenomena is neglected in geometrical optics, we do not consider diffraction here and we will consider the light as a ray and wavelength is neglected, the finiteness of the wavelength is neglected.

And therefore, the geometrical optics is defined as the field of optics where finiteness of the wavelength is neglected and this regime of optics can be studied using Fermat's principle. In next slide, we will learn what a Fermat's principal and how does it governs the geometrical optics?

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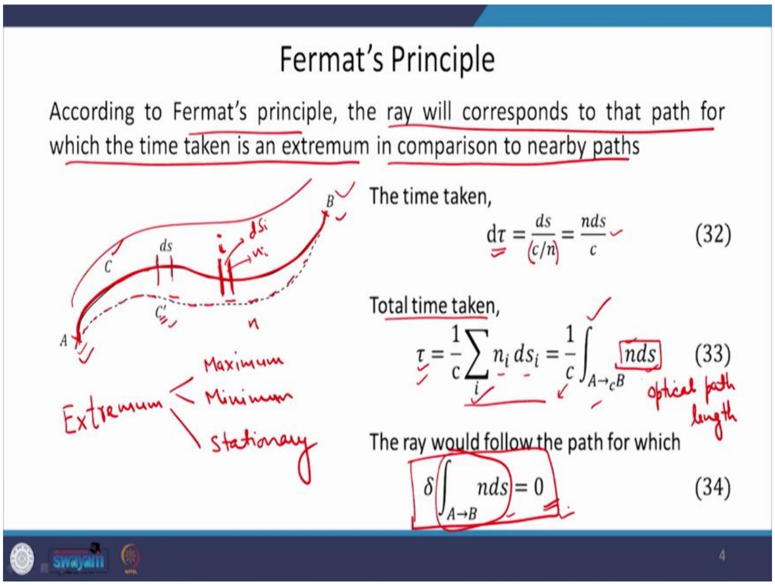

Now, according to this principle, the ray will correspond to that path for which the time taken is an extremum in comparison to nearby paths. What it says is that, suppose we have two points A and B given in a space, a ray which starts from point A may travel to point B following infinite many paths. Suppose, one of the path is given by C, suppose, this path which is given by a curve C represents one of these paths, another path is represented by this dashed line and which we named as curve C', C' represents the dashed path and continuous line is represented by curve C.

Now, let us assume that in this path AB on curve C there is an infinitesimally small part of the curve which we call as ds. Now let us calculate the time which a ray takes in traversing this infinitesimally small curve which is ds. Now, this time can be calculated once we know the speed and refractive index of the medium in which the ray is travelling.

Let us assume the refractive index of the medium is n then the speed of the ray in this medium would be c/n which is written here and therefore, if you want to calculate the time spent by the ray in travelling the length element ds then this can be given by this relation and which is ds/(c/n). where dτ represents the elemental time which the ray takes in covering distance ds. This relation can be written as $(nds)/c$. Where n is refractive index of the medium and c is the speed of the light in vacuum.

Now, once we know the time which a ray takes in traversing distance ds then we can also calculate the time the ray takes in traversing the whole curve path C and now, the whole time can be calculated by assuming several such small length element along curve C. Now, suppose there are very large number of these length elements and we can calculate the time for each of these length elements and then we can sum them up.

Now, once you sum them up, this will give you the total time the ray will take into traversing path C. This total time would be given by τ and which would be equal to $\frac{1}{c}\sum_i n_i ds_i$. Where ds_i is the path length element for strip i where i is any random strip and ds_i is the associated path length and n_i is the refractive index of that the medium within that strip.

I repeat, we choose a random strip or random part, random length element along the curve C. In this random length element, the length of this curve is ds_i, the refractive index of the medium within this length element is n_i and we call this element or we name this element as ith element.

Now to calculate the total time we will have to sum over all these i elements all the elements. Therefore, we are summing over $n_i ds_i$ and i runs over the number of elements, total number of elements. If the number of elements are very huge, then we can replace this summation with an integration, where these integrations runs from point A to point B along this curve C.

And this ds_i would then be replaced by ds and n_i would replaced by n and the summation is as I said is replaced by integration and this is how we can calculate the total time which a light ray takes in covering the whole distance from A to B along the curve C. Now, in this formula nds is the optical path length, ds is the distance, n is the refractive index. If you multiply n with ds it will give you optical path length.

Now, as I said, if a ray starts from point A and goes to point B it may take many paths and the fundamental question is which path the ray will prefer and as per the Fermat's principle the ray will prefer the path for which the time taken is extremum. What does extremum mean? Extremum means either maximum or minimum or stationary.

Now, we have this integration, c is a constant therefore, we are removing it from our further calculation we are just interested in optical path length and the question is which path the ray should follow. Fermat principle says we will follow a path for which the time is extremum.

Now here if this particular path is extremum, if the C is extremum and next to C path there is another path which is C' which is slightly varied from C and which is not extremum then any variation from path C will give you zero here, the variation path C should be zero as per the Fermat principle and if for a given path this variance is zero that particular path would be chosen by the ray and this is what Fermat said.

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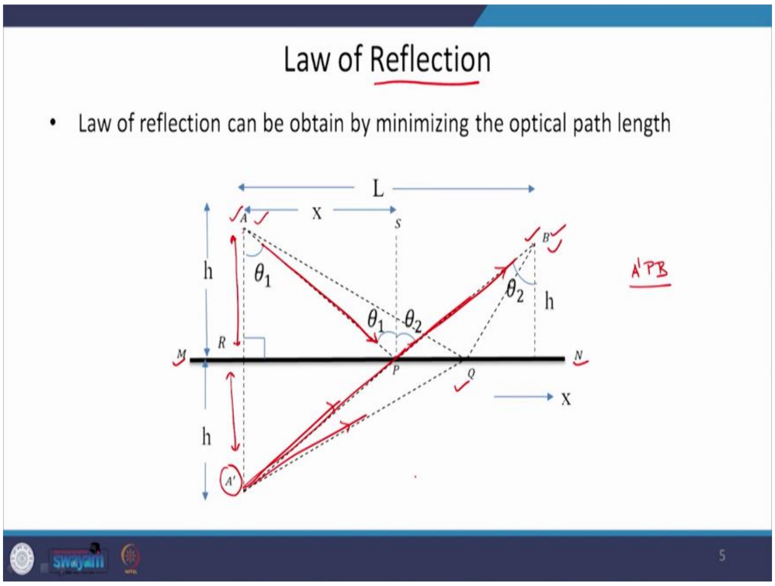

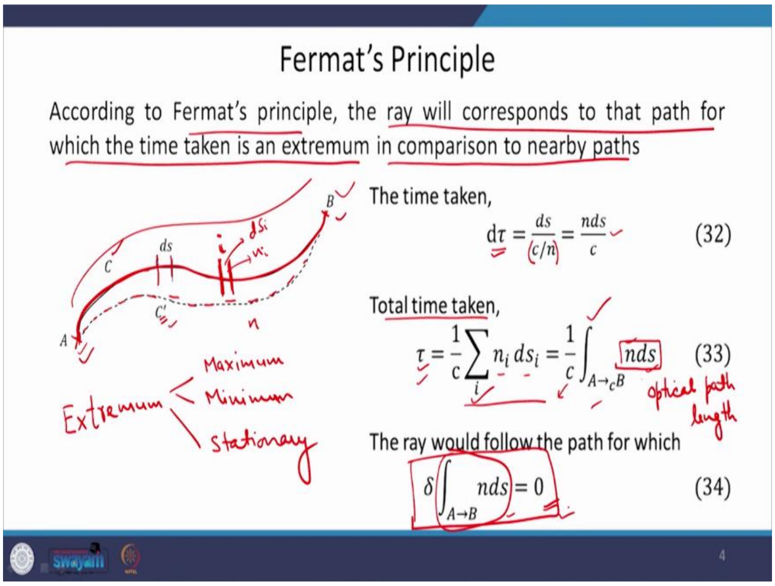

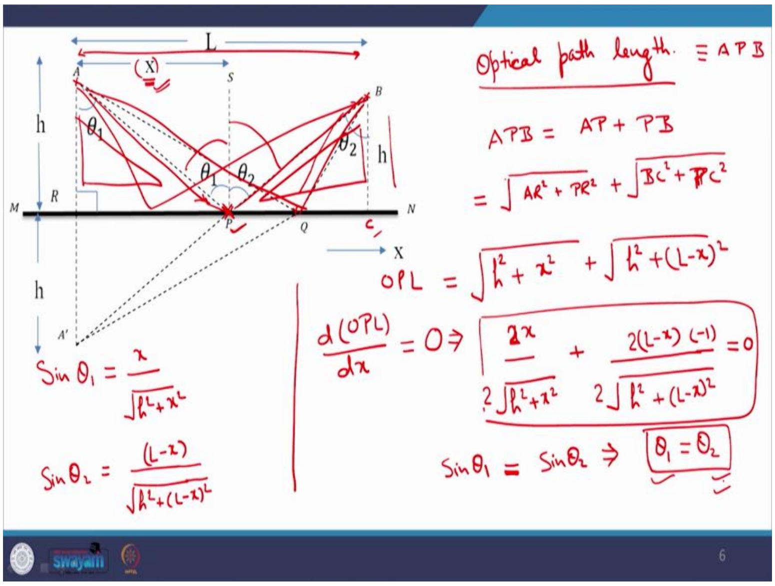

Now, as an application of this Fermat principal or we can say that we will understand the reflection and refraction under the light of this Fermat principle. To understand the phenomena of reflection, let us first consider a mirror which is designated by MN or line MN and this dark line is our mirror. A ray starts from point A and then it falls on the mirror and then after falling on this mirror, it goes to another point B. Point A is the start point and point B is finishing point.

Now, there are multiple path the ray may follow. The one path is APB, the other alternative path may be AQB. Again, the same question which path the ray should follow? Now, to understand this, we will again resolve to Fermat principle, but to really understand what exactly is happening here and how to decide whether a particular path is extremum or not.

Let us assume a point A' on the other side of line MN and this distance is equal to this upper distance. What I mean to say is that the distance AR is equal to the distance A'R. Now, if A' is mirror image of A then the rays which are starting from point A will also appear to start from point A' and these are the two possible rays which we discussed. These are the two possible paths which we discussed. And of course, the shortest path would be the path for which this line is a straight line and therefore, A' A, A' PB which is a straight line would be the shortest path which is an extremum.

And therefore, it follows this Fermat principle, it satisfies the Fermat principle and this path would be the extremum path and this is the path which the ray will follow. And another important point is that, if you want to prove this fact geometrically, then what you have to do is that calculate the optical path length. Now, how to calculate the optical path length in this case? The optical path length is the length of the path which the ray is following. Now, here in this figure the ray is starting from point A it is going to point B say and then it is reaching to the final point the destination point B.

For a while assume that the ray falls on the mirror at point B although there is an equal probability that it may fall on point Q, but for starting our calculation, we assume that the ray path is APB and we will check whether this ray path is extremum or not, the optical path length is here, the path length A, P and B. how to know whether APB is extremum or not?

Let us see, APB=AP+ PB, here what is AP? In this triangle, AP is hypotenuse and therefore, AP would be $\sqrt{AR^2 + PR^2}$. Similarly, PB in this triangle can be expressed as square root of, suppose, this is point C, let us see what is written in the previous slide, okay no name, suppose this point is C and it is equal to $BC^2 + PC^2$.

Now, what is $AR^2 + PR^2$, AR=h therefore, $AR^2 = h^2$ and PR is x therefore, $PR^2 = x^2$. Similarly, here BC is again equal to h and the point to be noted is that we have assumed that A and B are at the same height from the line MN. As is clear from the picture and PC is equal to L− x and what is L? L is the horizontal distance between the points A and B as shown in this picture here and this is our optical path length OPL.

Now, once optical path length is known Fermat said, it would be optimum or the optical path should be extremum, how to check whether it is extremum or not. let us differentiate it, we will take derivative with respect to the independent variable and what is the independent variable in our case. We can see here in this picture the point P is the point on the mirror MN. where the ray from A is falling and the ray may take several paths, it may fall at this point and then go to point B, it may fall at point P and then go to point B, it may fall on point Q and then go to point B.

Therefore, the variable quantity is x which is the horizontal distance from A to the point where the ray falls on this. The perpendicular to the points where the ray falls on the screen. PS is the perpendicular where the ray falls on mirror P and x is the horizontal distance between point A and S. It tells about the horizontal separation between point A and perpendicular to the point of incidence of ray on mirror MN, therefore, x is the variable here.

Therefore, to check whether this path is extremum or not, we will have to differentiate it with respect to x. We will differentiate this optical path length with respect to dx to check whether it is optimum or not and if it is optimum then this derivative must be equal to zero, let us do this and see what do we get.

Then from here we will get $1/(2\sqrt{h^2 + x^2})$ and in the numerator we will get 2x here and here we will get $\frac{2(L-x)(-1)}{2\sqrt{h^2+(L-x)^2}}$,this is what we get. This would be equal to 0. Now, if you see in the figure and check what is θ_1 then you can see that θ_1. Let us calculate sinθ_1, $Sin\theta_1 = \frac{x}{\sqrt{h^2+x^2}}$ and $Sin\theta_2 = \frac{(L-x)}{\sqrt{h^2+(L-x)^2}}$. This is how $Sin\theta_1$ and $Sin\theta_2$ are defined in the figure.

Now, we will see that, we will substitute $Sin\theta_1$ and $Sin\theta_2$ in this expression and from here what do we see is that this is equal to first term is equal to $Sin\theta_1$ and second term is $Sin\theta_2$

which gives $\theta_1 = \theta_{2,}$ which is exactly the law of reflection. In reflection, the angle of incidence which is θ_1 must be equal to the angle of reflection θ_2. The angle of incidence here in this figure this must be equal to the angle of reflection and this is what we got using Fermat principle.

It means, the ray path, if a ray starts from point A and lands up on certain point on mirror B and ultimately it gets collected at point B and then from this figure you can say or from this calculation you can say the point the ray will follow a path for which P is situated in such a way that angle of incidence is equal to angle of reflection. Now, once we studied reflection using Fermat principle.

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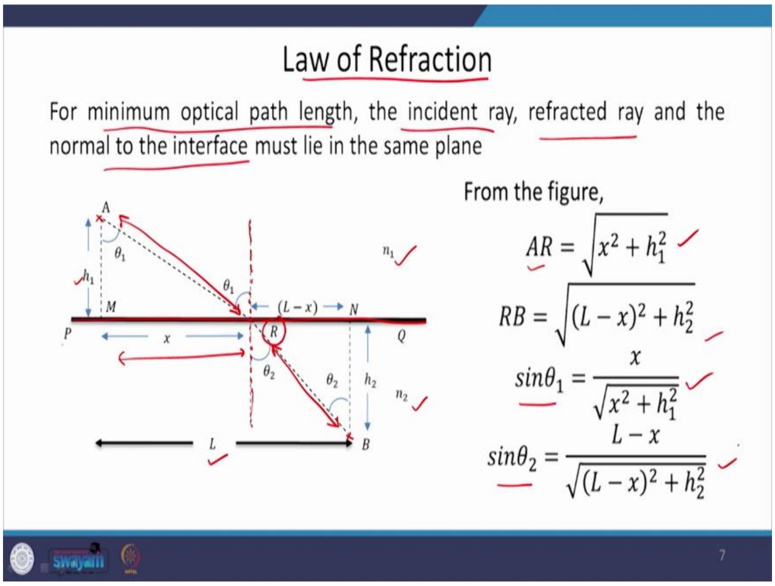

Now, we should study the second important law which is law of refraction. To study the law of refraction, as you know, we require two media. Now, we again draw a horizontal line which we name as PQ and this thick horizontal line PQ is an interface between the two media of refractive index n1 and n2 respectively, the medium of refractive index n1 lies above the interface PQ and the medium of refractive index n2 lies below the line PQ. Now the source or the ray start from point A, which is situated at a distance h_1 or at height h_1 from line PQ and then it passes through this interface PQ and then it is getting observed at point B, which is at that h_2 from line PQ.

Now, the ray which is starting from point A and ultimately reaching at point B. This may take infinite many paths, which path would be the extremum, which path the ray would follow. Now, for minimum optical path the incident ray, refracted ray and the normal to the interface this dash line that defines the normal to the interface, they all must lie in the same plane this is what is Snell's Law said. We will do some geometrical calculation as we did in the previous case and then we will see what Fermat principle gives us in this refraction case.

Now, here we assume that the ray which is starting from point A falls at point R on interface PQ and this point R is x distance away from line AM and the horizontal separation between point A and point B is L. Under this situation, let us calculate what AR is, AR is this distance.

Now, AR=$\sqrt{x^2+h_1^2}$. Similarly, we can calculate BR which is this distance, BR=$\sqrt{(L-x)^2+h_2^2}$. Once these two distances are known, we can calculate $Sin\theta_1$ and $Sin\theta_2$, where θ_1 and θ_2 are angle of incidence and angle of refraction respectively. Once θ_1 and θ_2

are known, then we can move forward for calculating the optical path length as we did in the last slide.

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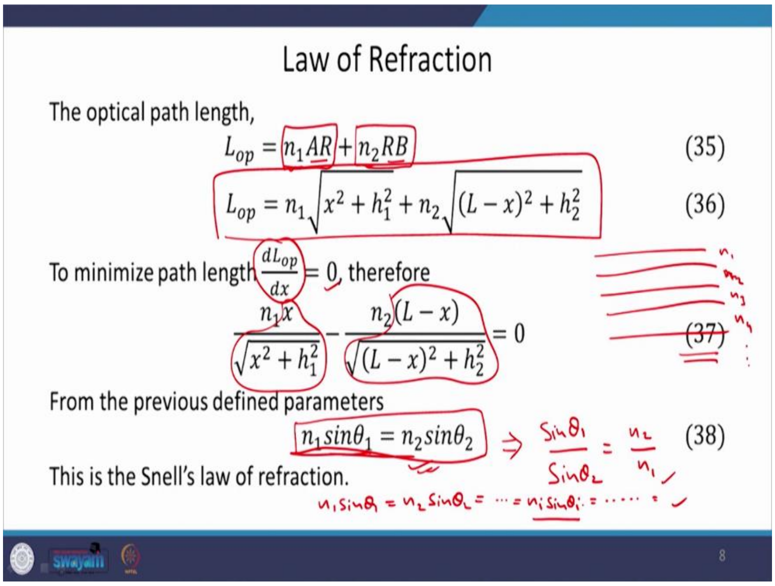

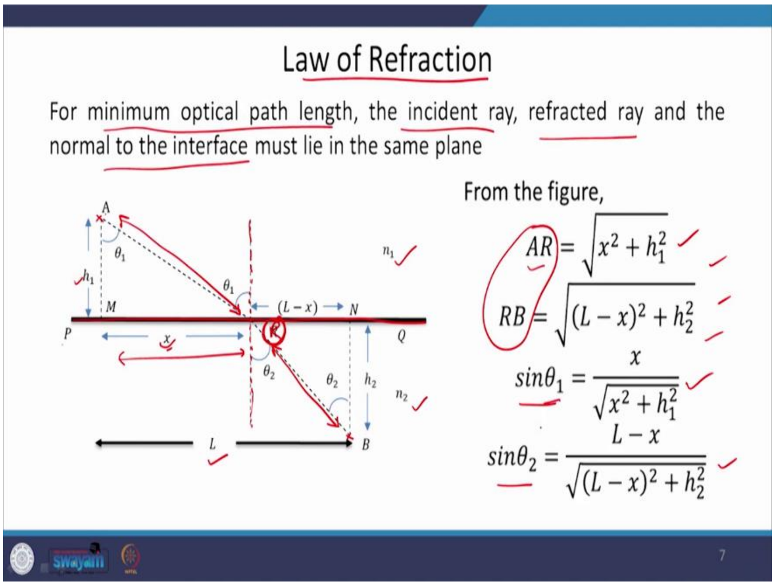

Now how to calculate the optical path length, the optical path length as visible in this figure would be AR+ RB, if you add them up, you will get optical path length. But, in this particular case, the upper medium and the lower medium are different and they have refractive indices which are different from that of AR and therefore, in addition to AR+RB we will have to multiply AR with n_1 and RB with n_2 which are their respective refractive indices and one is the refractive index in the upper medium and two is the refractive index in the lower medium.

Therefore, the effective optical path length in the upper medium would be $n_1 \times$ AR and the effective path length in the lower medium would be $n_2 \times$ RB and this is what exactly we did.

We calculated the optical path length in the upper medium and then optical path length in the lower medium and then sum them up to calculate the total optical path length. Once total optical path length is calculated, we will substitute for AR and RB from the previous slide here we have already calculated and this expression this equation number 36 will now give us the expression for optical path length.

Now, we will resolve to Fermat principle which says that if you take the time derivative or sorry if you take the derivative of the optical path length with respect to the variable which is x here, you can see in this figure x is the variable here because the point R may vary depending if you vary the path which the ray takes then the point R will vary and therefore, x is the variable and this is why we will differentiate the optical path length with respect to x and as per the Fermat principle this differentiation must be equal to 0 and once you do this, you will get equation number 37.

And we have already calculated the expression for $Sin\theta_1$ and $Sin\theta_2$ and if you substitute for this and this quantity, then you will get this expression and this is nothing but Snell's law. And when $n_1 Sin\theta_1 = n_2 Sin\theta_2$ so $Sin\theta_1/Sin\theta_2 = n_2/n_1$. It means that the ray will follow a path for which this relation holds and therefore, it is again established that Fermat principle is capable of deciding the path of the ray.

Now, suppose instead of having only two media we have a huge number of layered media. Suppose we have n_1, n_2, n_3, n_4, n_5. n_2 is here, n_3 is here, n_4 is here and so on. Then the Snell's law can be written as $n_1 Sin\theta_1 = n_2 Sin\theta_2 = n_i Sin\theta_i$. and so on and so forth. This is a generalized form. Therefore, we may say that $n_i Sin\theta_i$ is a constant. If you keep varying the refractive index in a layered manner in one direction only than n1, $n_i Sin\theta_i$ would be constant and what type of constant is that it is invariant, it is invariant, invariant of ray path.

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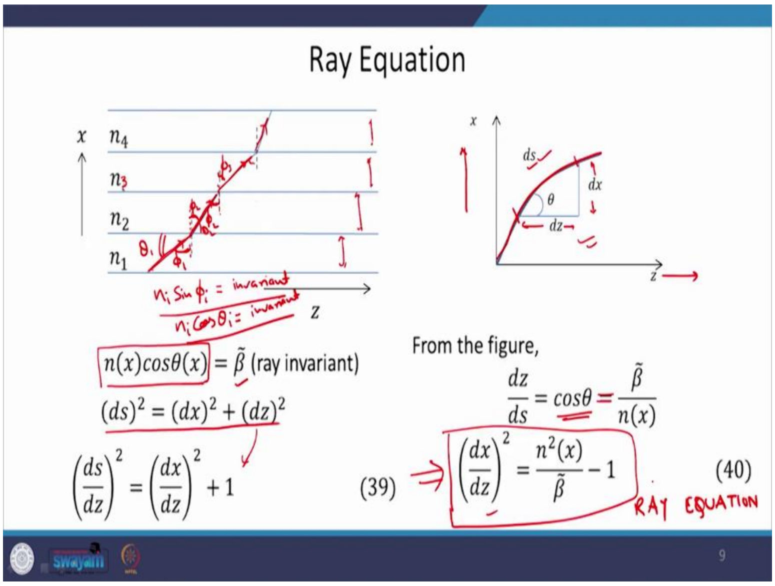

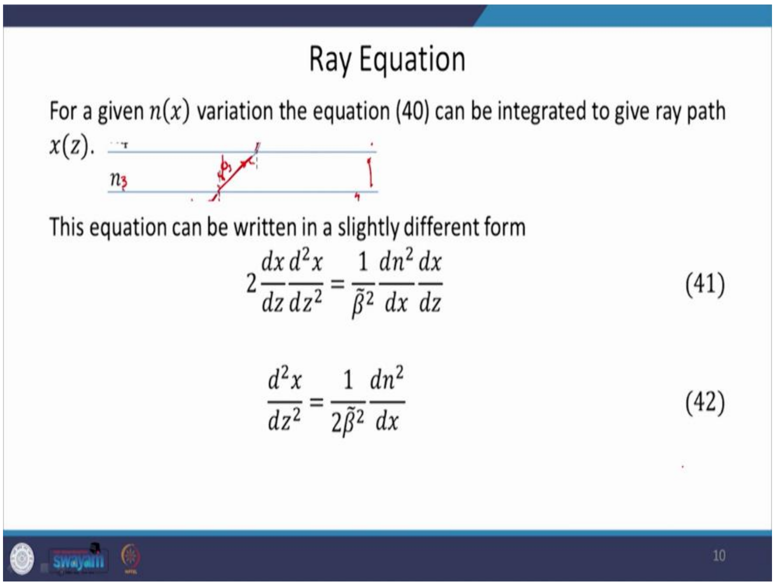

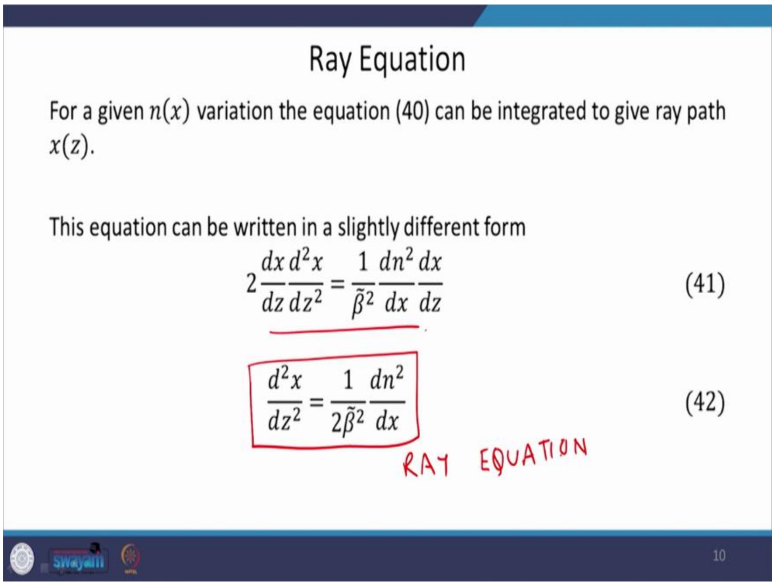

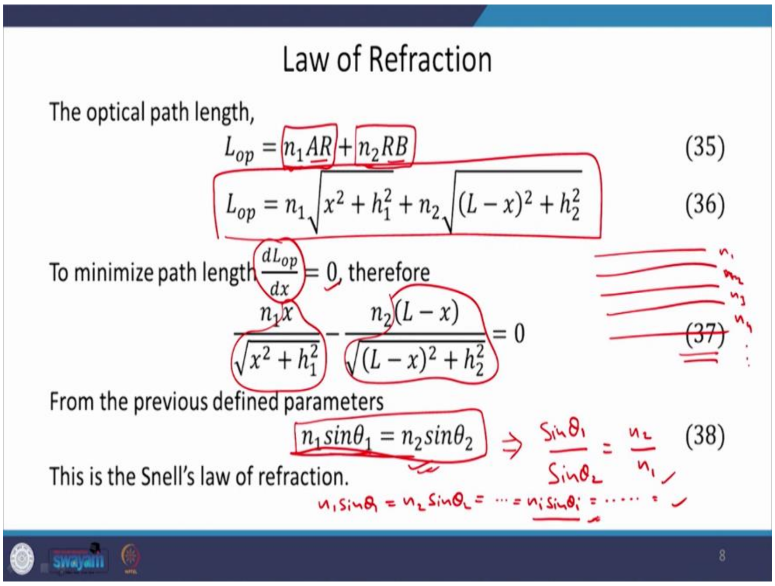

Now, once it is done we will now see this what is the application of this ray invariant quantity. Now, suppose as we discussed in the previous slide, we have layered media of different refractive index, the lower most medium has refractive index n_1 and then upper one has refractive index n_2, a bit upper one has refractive index n_3, n_4, n_5 and so on and refractive index is wearing only in vertical upward direction. Under this particular case, we can derive a ray equation which will tell the curve which our ray will follow, how to derive this?

Now, here as I said before n_1, n_2, n_3 and n_4 are the refractive indices of the front layer medium and this is the ray which will follow a particular path following your Snell's law or we can derive such a equation for the ray path if the thickness of these layer mediums are very small and if these thicknesses are very small then instead of having this discrete ray path we can rather draw a continuous curve which represents the ray path.

Now, suppose in vertical direction we have x axis and in the horizontal direction we have z axis then a small element of ray path may be represented by ds and it will be slanted at angle θ with respect to the horizontal direction and we may say that, that along horizontal direction the horizontal component would be dz and the vertical component would be dx of this length element ds. Now, we define now a quantity which is n(x)cos$\theta(x)$ and this quantity would be ray invariant as discussed before, how can this quantity be a ray invariant.

Now, if we launch a ray and then we draw a perpendicular at each interface then this angle would be the angle of incidence and this angle would be the angle of refraction in this medium the second one this angle would be the angle of incidence and this angle would be the angle of refraction. Now, suppose this is φ_1, this is φ_2 , this is again φ_2 and this is φ_3 and so on. Then

$n_i sin\varphi_i$ would be invariant. $n_i sin\varphi_i$ as discussed in this previous slide, it would be constant, it would not vary.

Now, if we write this invariant which we call the invariant in terms of cosine then it would be written in the following form $n_i cos\theta_i$. What is theta? theta is this angle, this is θ_1, this is θ_2 and so on, the other side of φ, $\varphi + \theta = 90$ degree.

Then this would be your invariant and if this refractive index variation is in only in one direction as I said before and the thickness of different layers are very small, then instead of writing this invariant quantity in discrete form, we can write it as a continuous variable therefore, we may write it as n function of x and cos θ where θ is a function of x where it is very slow there is a very slow variation in the refractive index along the x direction.

And then we define a quantity which is $\tilde{\beta}$ which says that this n(x)cos$\theta(x)$ is ray variant and this is invariant of a ray path. Once this is defined, then we can calculate the path length, the infinitely small path length along this ray path and this is using this diagram. We can say that $ds^2 = dx^2 + dz^2$. After a little bit of mathematics, we will get this relation and from the figure we can see that $cos\theta = dz/ds$. We know that $cos\theta$ is also related to $\tilde{\beta}$ (beta tilde) which is a ray invariant therefore $cos\theta$ may equivalently be retain as $\tilde{\beta}/n(x)$.

Now, using this equation 39 may be written as $\left(\frac{dx}{dz}\right)^2 = \frac{n(x)^2}{\tilde{\beta}} - 1$,, this is ray equation. Now, this ray equation which is represented by equation number 40 may equivalently be written in different form, how to write it in different form just differentiate equation 40 with respect to z once more, then we will get this relation and after a little bit of simplification, we will again get a modified form of this equation.

Equation number 40 and 42 they both are called ray equation and they trace the ray path. Now, we will take an example on this topic.

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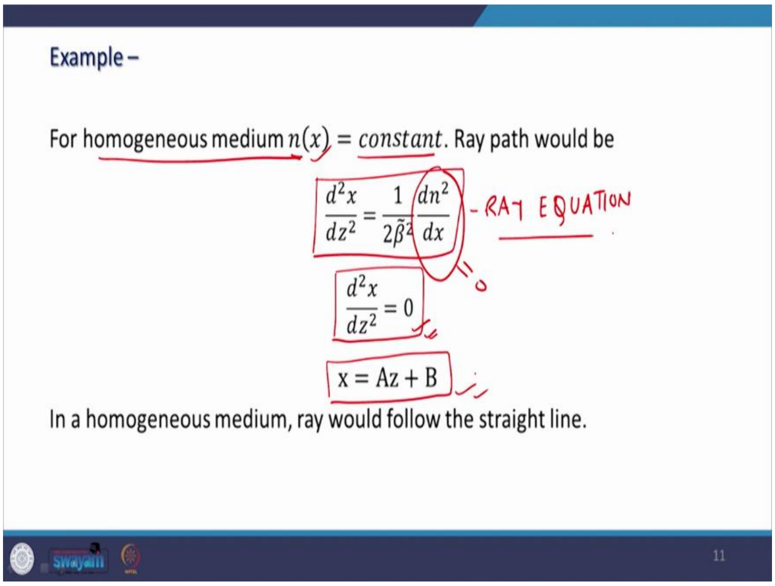

Now, suppose we have a homogeneous medium and for which refractive index which is a function of x would be constant because in homogeneous medium refractive index does not vary then how to decide the ray path? To decide the ray path, we will start with this ray equation. Now, since n is independent of x therefore, this term would be 0 and therefore, right hand side of the ray equation would be 0 this is what we are now left with after substituting n is equal to 0, we will get $\frac{d^2x}{dz^2} = 0$.

Now, if you solve this equation, how to solve this, integrate it twice then ultimately you will get a equation which is x= Az + B. where A and B are constants and this is an equation of a straight line which says that in homogeneous medium the ray would follow a straight line which is quite obvious what we observe in our daily life, light a torch, then the ray which is emanating from the torch will go in a straight line. It means that things are correct here. Now, this is all for today in my lecture here with ray equation. Thank you for your patience.