Applied Optics
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Module 3
Lecture 13
Superposition of Waves

Hello everyone, welcome to my class. Today, I will hold lecture number 13. Now you know that this is module number 3, and in this module, we have already covered concept of wavefront, and then Huygens's principle and its applications.

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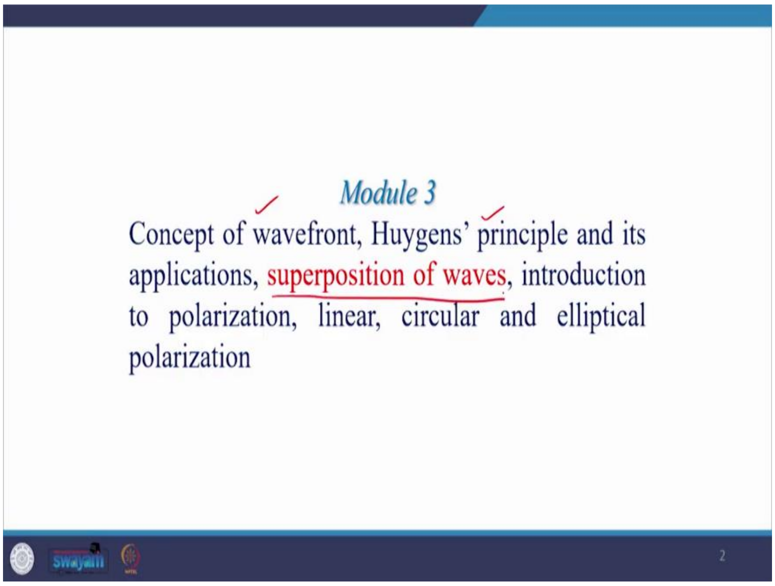

Today, we will talk about superposition of waves. Now, superposition means we are taking into account more than one waves, and what will happen if we have more than one waves, and they are overlapping in space and time.

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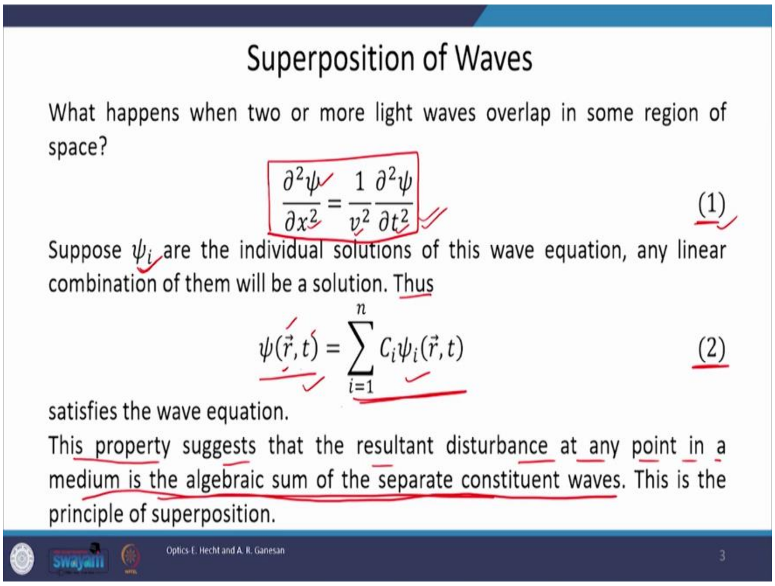

Now, we know that the dynamics of a wave is governed by wave equation, and this is the expression for a wave equation in 1D, and ψ is the wave function, x is the direction of propagation, t is the time, and v is the velocity of the wave in the medium. Now, suppose there are n number of waves, which are overlapping in time and space both. Now, say ψ_1 is the wave function for one of such waves, and this ψ_1 would be the solution of this wave equation 1. And, if there are n number of such solution, then their linear combination will also be a solution of this second order partial differential wave equation.

The properties of this wave equation is that, it is second order, it is partial, it is homogeneous equation. Now, the linear combination of all these solutions which are ψ is here represented by wave function ψ, which is function of space and time as you can see here. And this linear combination is represented by equation number 2.

Now, since ψ_i's are the solutions of wave equation 1. Therefore, the linear combination will also constitute a solution of this equation, wave equation 1. And therefore, equation number 2 that is ψ will also satisfy equation number 1. Now, this property suggests that the resultant disturbance at any point in a medium is algebraic sum of the separate constituent waves.

It means that, suppose we have two waves, which are propagating in the same medium, and they are overlapping in time and space. Therefore, the resultant wave will be algebraic sum of the two waves, and this is what being said here in these lines. And this is exactly the principle of superposition is. We have n number of waves, which are propagating in a particular medium, and if they are overlapping in space and time both, then the resultant wave will be algebraic sum of all these n number of waves. Now, we will exercise this super position principle, and see what would be the resultant? How would the resultant look like?

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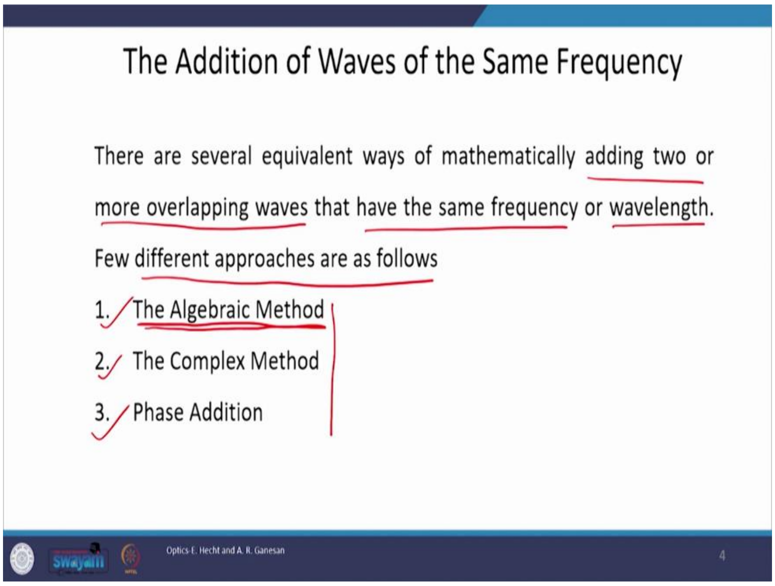

Now, to address this problem mathematically, several equivalent ways have been proposed to address this problem mathematically. Because this problem is nothing but addition of waves. Here, we are adding 2 or more overlapping waves, that is all what we required to do to come up with the resultant wave function. And what are the properties of this overlapping wave? They have same frequency, or the wavelength. They are propagating in some medium, and they are overlapping in time and space. Now, there are different approaches using which we can address this issue, address this problem.

The first one is algebraic method, where we represent wave, or wave function $\psi = Asin(\omega t - kz) = rcos(\omega t - kz)$. Second method is complex method, where we represent our wave function in complex form that is $\tilde{\psi}$, and then $\tilde{\psi} = \psi_0 e^{i(\omega t - kz)}$. This is the second formulation, or second representation of the wave.

And third is the phaser representation, in this phaser representation, or phase addition, we just represents the wave in a vector form and add them vectorially, and the resultant will give you the final addition. Now, out of these 3 methods, we will learn or we will talk about only algebraic methods, they are equivalent method therefore, we will pick one of the three.

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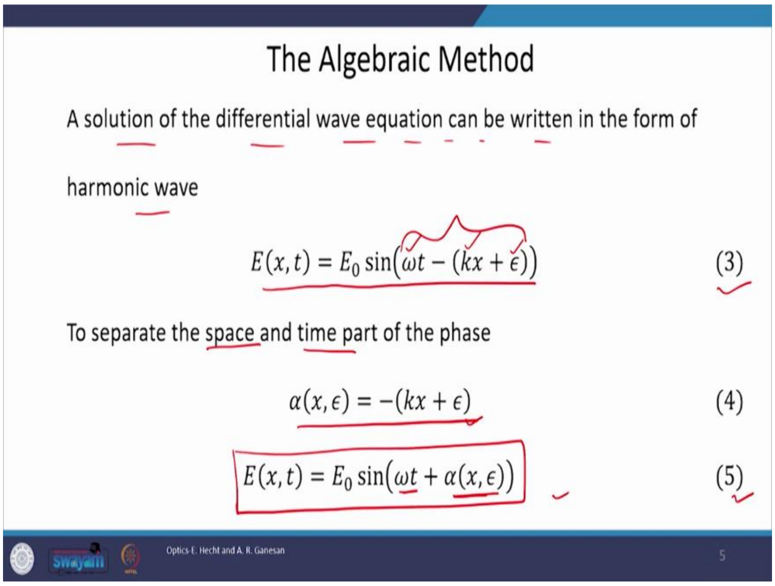

Now, in algebraic method, we will start with the solution of the wave equation, which is represented by equation number 1, which was there in our previous slide. Now, a solution of the differential wave equation can be written in the following form, and this form is given by equation number 3. Which is E, which is function of x and t is equal to E_0. E_0 is nothing but amplitude, and then $sin(\omega t - (kx + \epsilon))$, where k is our wave vector, we know ω is angular frequency, ϵ is initial phase, any initial phase. This is the general solution of the wave equation 1.

Now, you see that in the phase part, there are space dependent component, and time dependent component. Now, let us separate the space and time part in phase. The space part can be written as α, and α would depend upon x and ϵ. And therefore, the expression for $\alpha = -(kx + \epsilon)$. Now, if we do this, then the final solution, or final expression of the solution of the wave equation will look like this, and this is represented by equation number 5. α is the phase part, which depends upon the space and it also includes the initial phase. And ωt is the time dependent part.

This is one of the solution of the wave equation, and any harmonic wave can be represented like this by equation number 5. Now, once we know the expression of a harmonic wave, we will consider several such waves, and then let them overlap, or let them superimpose. And then we will calculate or we will try to find the resultant. Here, how to find the resultant? The superposition principle says we will have to add them up, and the addition will give the resultant expression.

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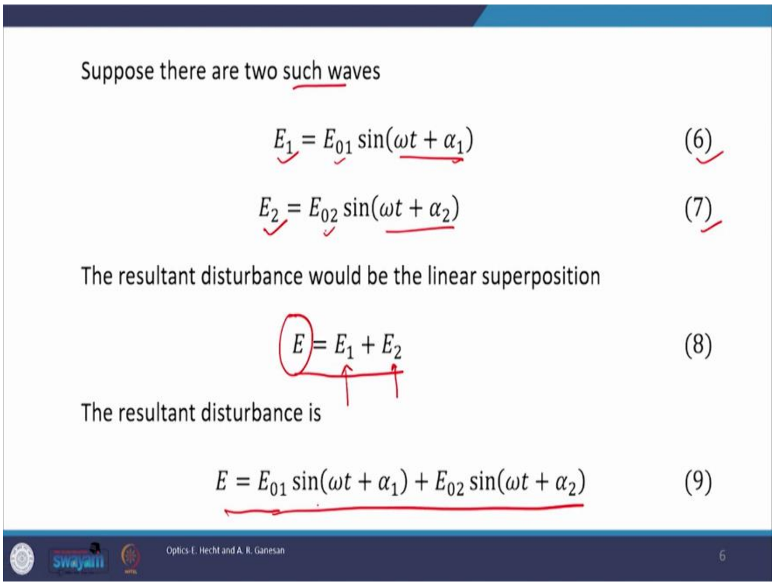

Now, suppose there are two such waves, first one is represented by E_1, and second one is represented by E_2. $E_1 = E_{01}sin(\omega t + \alpha_1)$, and $E_2 = E_{02}sin(\omega t + \alpha_2)$, where E_{01} and E_{02} are nothing but amplitudes of these waves, and the phase parts are given here.

What would be the resultant? The resultant would be the linear superposition, because the linear superposition of $\psi_i's$ gave us the resultant ψ (psi), Therefore, we just added them up, we added E_1 and E_2, which gave us the resultant disturbance here. Now, let us substitute the expression for E_1 and E_2 from equation number 6 and 7. And after the substitution we get expression number 9, this is the expression of the resultant disturbance.

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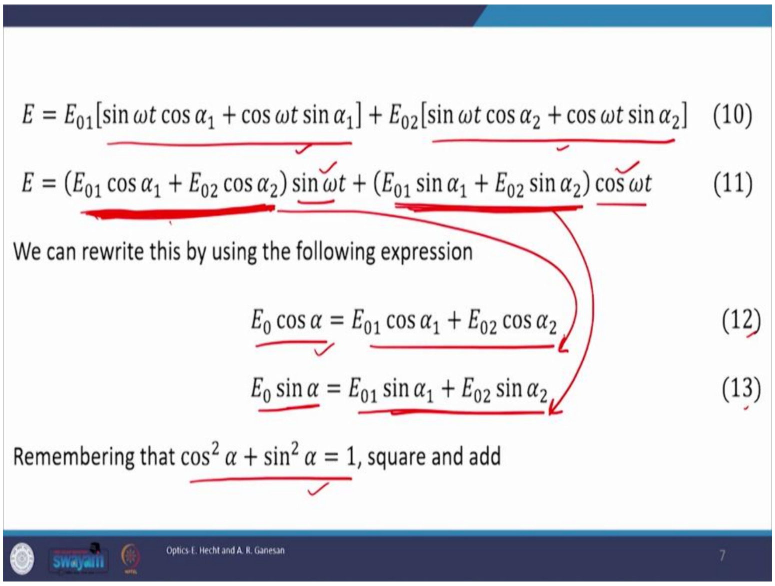

Now, let us expand the sin function and see what does it gives. Now, if you expand sin, then you will get these two expanded expression of sin by using formula sin (A+B), we will get these two expression for the sin function. Let us rearrange them, let us take $sin\omega t$ common and $cos\omega t$ common out of the given expression, then this is time independent component, and this is also time independent component, and time dependent components are $sin\omega t$ and $cos\omega t$. Now, since this term, the first term in the bracket on RHS of equation number 11 is independent of time, let us represent this time independent part by $E_0 cos\alpha$.

Similarly, the multiplication factor of $cos\omega t$ is replaced by $E_0 sin\alpha$. This is the term which is sitting with the second term in equation 11, and this is here. Here, we introduce $E_0 cos\alpha$, and $E_0 sin\alpha$. And replace these big brackets in equation number 11, and we know that $cos^2\alpha + sin^2\alpha = 1$. This holds, if you square equation number 12 and 13, and add them up, then you will see that this would satisfy this criteria.

From equation number 12, we can get the expression of $cos\alpha$, what we will have to do? We will have to just divide the left and right hand side of equation number 12 by E_0. And similarly, from equation number 13, we can get the expression for sinα, square them up, and then add them up.

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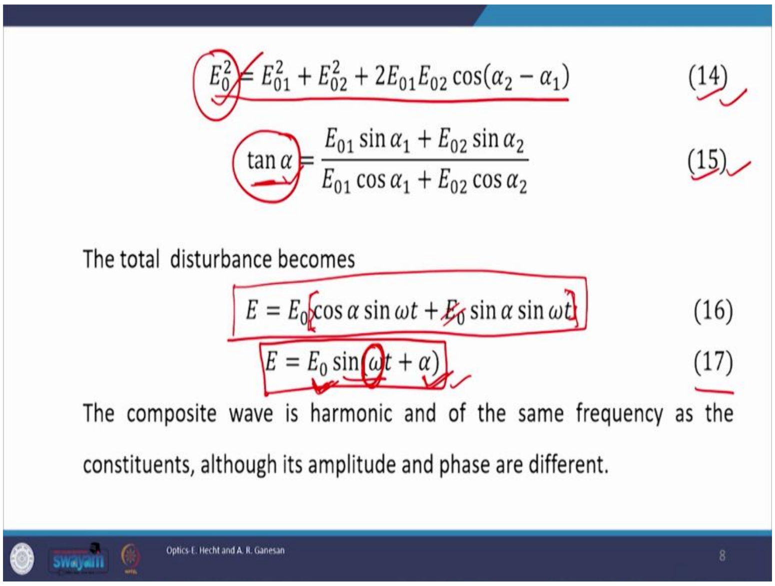

$$E = E_{01}[\sin\omega t\cos\alpha_1 + \cos\omega t\sin\alpha_1] + E_{02}[\sin\omega t\cos\alpha_2 + \cos\omega t\sin\alpha_2] \quad (10)$$

$$E = (E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2)\sin\omega t + (E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2)\cos\omega t \quad (11)$$

We can rewrite this by using the following expression

$$E_0\cos\alpha = E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2 \quad (12)$$

$$E_0\sin\alpha = E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2 \quad (13)$$

Remembering that $\cos^2\alpha + \sin^2\alpha = 1$, square and add

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After squaring and adding, we will get equation number 14. And if you divide these two equations, if you divide equation number 13 by equation number 12, then we will get the expression for tan α. Now, equation number 14 and 15, what do they give us? They give the expression of E_0, and the value of α. Which were the unknown in equation number 12 and 13, we replaced this term by $E_0 cos\alpha$, where both E_0 and α were unknown. Similarly, we replace the second term with $E_0 sin\alpha$, where again E_0 and α were unknown. By using equation 14 and 15 now, we can get the value of E_0 as well as the value of α, everything is known now.

Now, if we know this, let us substitute equation number 12 and 13, back into equation number 11. If we substitute them back into equation number 11. We will get this new expression for the total disturbance, and if you see then this expression if you take E_0 out of the bracket, and if you see this expression, then it is nothing but sin (A+B) formula. And therefore, the final expression reduces down to E is equal to $E_0 sin(\omega t + \alpha)$. Where α is given by equation number 14, E_0 is given by equation. Sorry, α is given by equation number 15, and E_0 is given by equation number 14. All the things are known in equation number 17. And this is the resultant disturbance in the medium.

If we let two waves of same frequency overlap in space and time, then the resultant disturbance will be given by equation number 17. What we see here is that, the resultant disturbance which is given by equation number 17 is again a harmonic wave. It is a sinusoidal wave, and it has the same frequency which is equal to the frequencies of the superimposing waves. And the amplitude is different from the earlier two superimposing waves.

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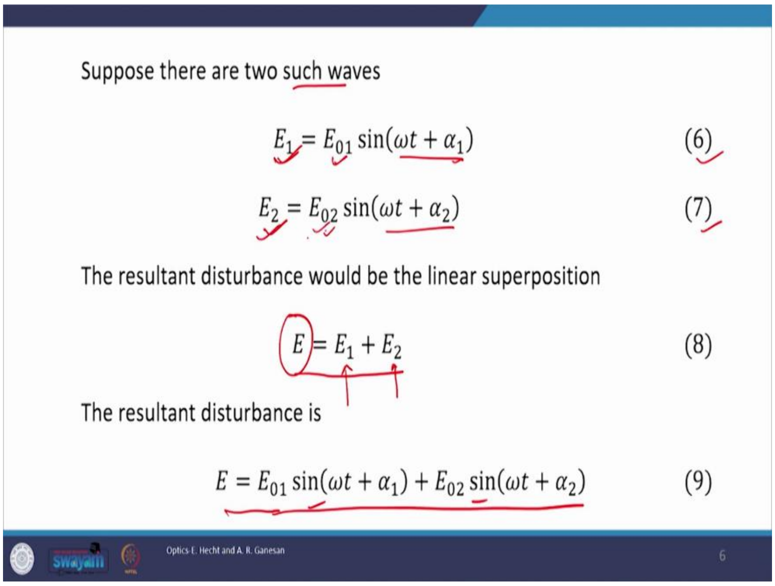

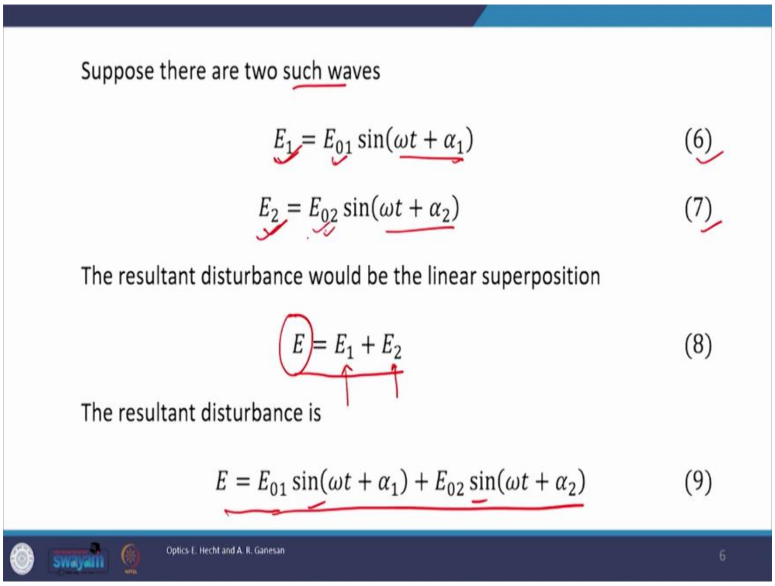

There were two waves, let us repeat. Initially there were two waves E_1 and E_2. They were having different amplitudes, but the frequencies were the same, and the resultant wave E has the same frequency, but a different amplitude. Here, this is what it is written here, the composite wave is the harmonic wave, and it has the same frequency as that of the constituents waves.

But its amplitude, and the phase are different, the alpha is different and E_0 is different. Here, amplitude and phase are different, while ω is the same here. Now, if you see equation number 14, then what are the informations which can be extracted out of equation number 14?

E_0 is the resultant amplitude, and the first term here is the square of amplitude of first wave, second term here is the square of amplitude of the second wave. But in addition to these two terms, we are having this multiplication of E_{01} and E_{02}, and this term is called interference term. This term gives rise to fluctuation in the resultant amplitude. Some sinusoidal variation will appear in the resultant amplitude, or intensity due to this term. And therefore, this term is called interference term. It means interference owes its origin in superposition principle.

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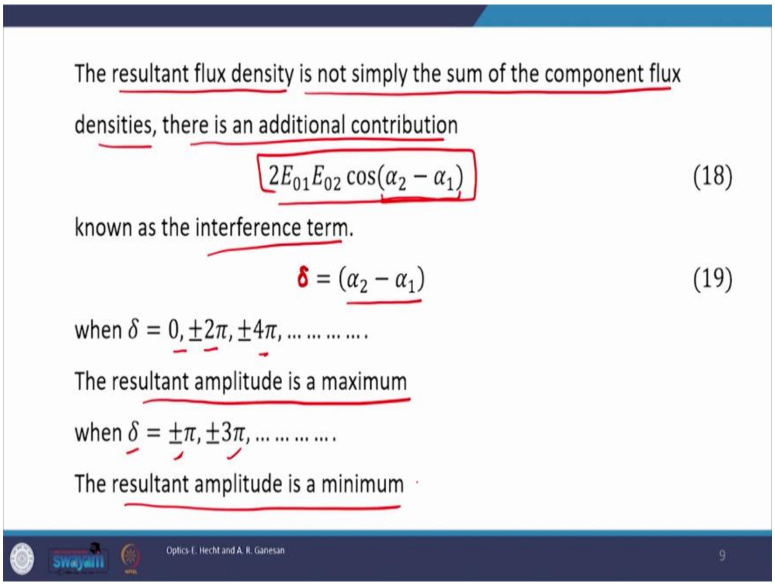

Now the resultant flux density is not simply the sum of component flux densities. Flux densities means amplitude square, which we saw in equation number 14. And there is this additional contribution which we known as interference term, this is our interference term. Now, let us talk about the phase part of this interference term, which is nothing but $\alpha_2 - \alpha_1$.

Let us represent this phase part by δ. This $\delta = \alpha_2 - \alpha_1$. Let us assume this δ (del) represents the phase of this interference term. Now, whenever $\delta = 0, \pm 2\pi, \pm 4\pi,$, the cosine function would be equal to 1, and this is the maximum value the interference term can achieve.

And due to this maximum value, the resultant amplitude will be maximum, and this we call a constructive interference. However, when δ is equal to odd multiple of π, that is if $\delta = \pm\pi, \pm 3\pi$, then the resultant amplitude will be minimum. Which correspond to destructive interference.

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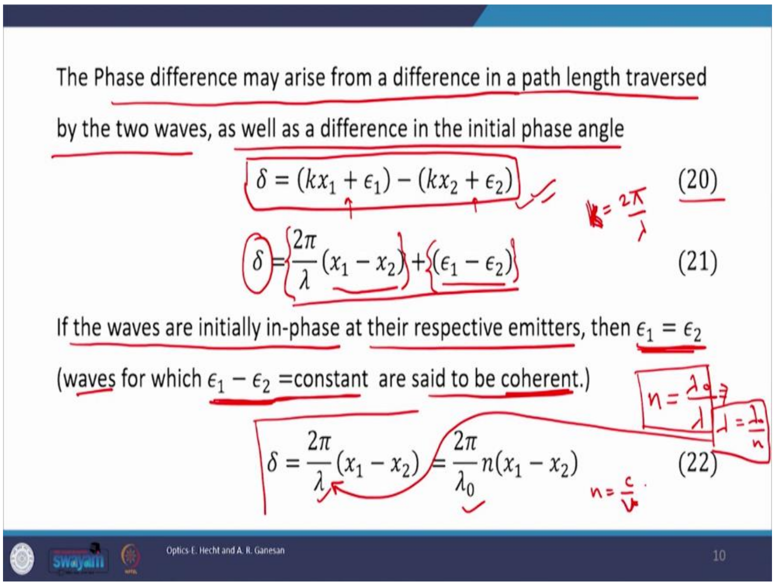

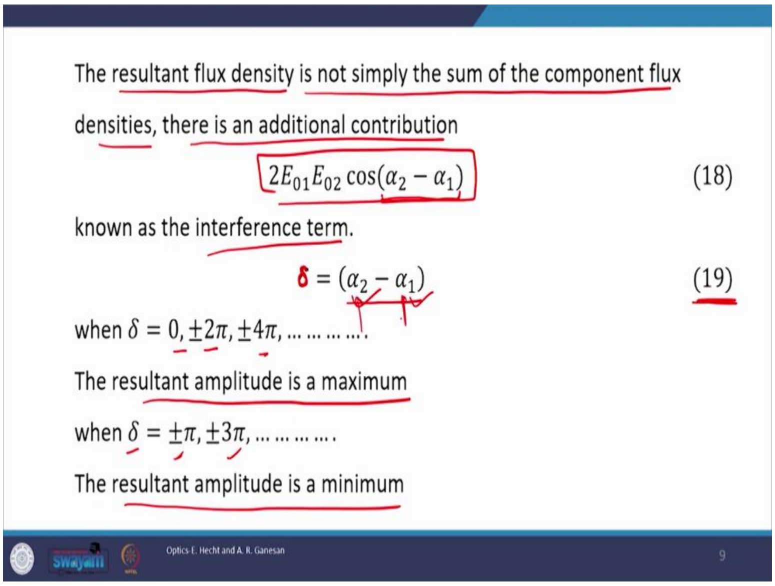

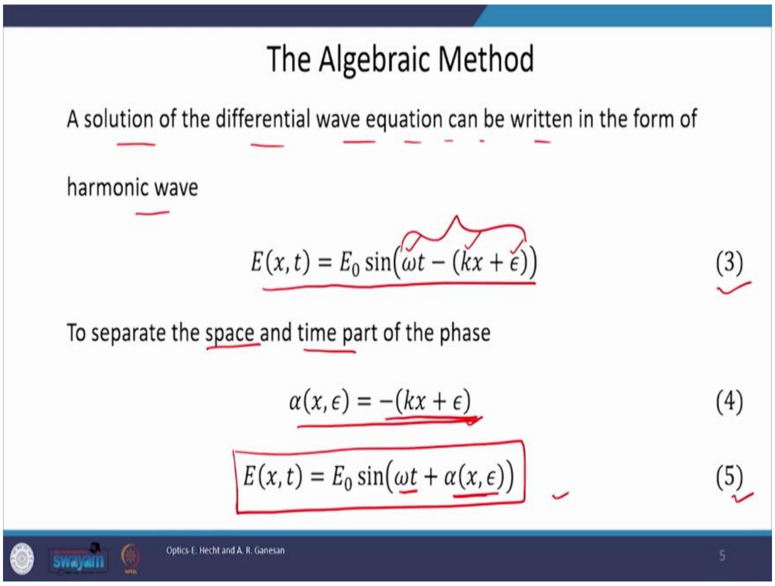

Now, the phase difference may arise from a difference in path length traversed by the two waves, as well as difference in the initial phase angle. Now, we are talking about this α, now we will substitute back the expressions for α_2 and α_1 in equation number 19. And then we will look into it, what are α_1 and α_2? Let us go back, , $\alpha = -(kx + \epsilon)$. Therefore, $\alpha_1 = -(kx_1 + \epsilon_1)$. Similarly, $\alpha_2 = -(kx_2 + \epsilon_2)$. Now, with this, let us go to again equation number 19, we will substitute this expression back in this equation number 19. After substitution, we will get this expression for the phase.

$\delta = \alpha_2 - \alpha_1$, which is phase difference. Phase is a relative quantity, it is always measured with respect to some reference value, whenever we say phase is equal to 0, it means the reference value is 0, and the phase has not evolved from there from that 0 value. Now, you see here after substitution for the α_1 and α_2, the δ, the phase difference between the 2 waves takes this form, which is given by expression number 20. Now, what is k? k is wave number and the expression for the wave number k=$2\pi/\lambda$. We know k=$2\pi/\lambda$, we will substitute for this expression of k in equation number 20. And this gives us equation number 21.

Now, in this phase component, in this phase difference in this expression of δ, the first term arises due to difference in x_1 and x_2, and second term arises due to difference in ϵ_1 and ϵ_2. What is $x_1 - x_2$? $x_1 - x_2$ is nothing but path length difference, x_1 is the path traversed by first wave, and x_2 is the path length traversed by the second wave, and $x_1 - x_2$ is path length difference, if you multiply this path length difference by wave number, which is $2\pi/\lambda$, this gives us phase difference. If you multiply path length difference by a wave number this path length difference then get converted into phase difference.

Therefore, this first term represents phase difference due to the path length difference. While the second term, which is $\epsilon_1 - \epsilon_2$, its owes its origin in initial phases of the two waves. ϵ_1 is initial phase of the first wave, ϵ_2 is the initial phase of the second wave, take the difference and this difference give you the difference in the initial phases.

And this is why if the waves are initially in the same phase at their respective emitters, then ϵ_1 would be equal to ϵ_2. In particular, waves for which this difference $\epsilon_1 - \epsilon_2$ is constant throughout, such a wave is said to be coherent, we will discuss more about this phenomena of coherence in later lectures.

But just to start with a wave in which if we have two waves, which have their own respective initial phases, if the initial phase difference is constant throughout the motion the waves are said to be coherent, waves are said to be coherent with each other. Now, for such a wave for which $\epsilon_1 = \epsilon_2$. The expression for δ, which is phase different modifies, and then it becomes $2\pi/\lambda(x_1 - x_2)$. λ is wavelength in the medium, and this λ embeds in self the information about the refractive index of the medium, and we know refractive index $n = \lambda_0/\lambda$. Therefore, we can safely write that $\lambda = \lambda_0/n$.

Now, we will pick this expression, as substitute it back here. And this gives $\delta = 2\pi/\lambda_0 n$ which is refractive index of the medium, and multiplied by $(x_1 - x_2)$, the path length difference. What is λ_0? λ_0 is the wavelength in vacuum, what is λ ? λ is wavelength in a medium of refractive index n. Here, and they are related by this relation, refractive index $n = \lambda_0/\lambda$. We can easily derive this relation from n is equal to c/v relation.

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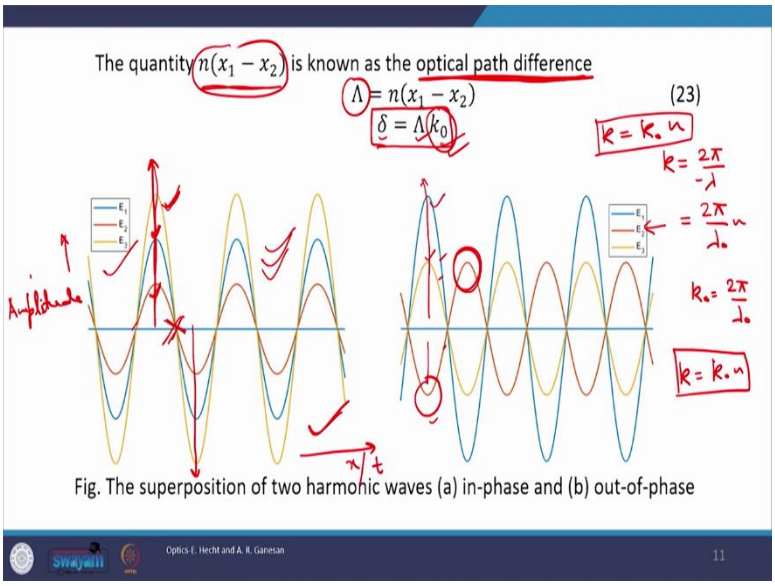

Now, this quantity n $(x_1 - x_2)$is path length difference in the medium, $(x_1 - x_2)$ is the path length difference in the vacuum, but if the waves are traveling in a medium. If superposition is happening within a medium, then the optical path length difference would be n$(x_1 - x_2)$, and this is why it is called optical path difference, and this optical path difference is expressed by Λ (capital lambda). Λ= n$(x_1 - x_2)$, and as I said if you multiply optical path length difference by wave number, then we will get phase difference. And this is what it is written here, the optical path difference is related with phase difference, and this relation is through this wave number k_0.

k_0 is wave number in vacuum, and k is wave number in media. k is related with k_0 through this relation, and we know that k=$2\pi/\lambda$, or k=$(2\pi/\lambda_0)n$, and $k_0 = 2\pi/\lambda_0$. Therefore, k= $k_0 n$, this is the relation which is also written here. Now, you see here in the figure on left, in this figure, there are 3 waves E_1, this is E_1, this is E_2, and this is E_3. They are of different magnitudes, there amplitudes are different, but they oscillate with the same frequency.

Now, if we assume that they are in same phase, if we assume that they are oscillating in same phase, then they will reach to their maxima simultaneously, and as well as they will reach to their minima simultaneously. Therefore crest and trough the boat will appear simultaneously for all 3 waves. And therefore, what you see is that the peaks of all 3 waves are coinciding here, the dips are also coinciding here, the waves are simultaneously crossing this 0 point. Here on the horizontal axis, we can consider either x or t, on the vertical axis amplitude is plotted.

Now, suppose one of the wave out of E_1, E_2, and E_3 is out of phase by 180 degree. Then what will happen? Suppose this red one in this right figure is out of phase, then you can see that when the green and yellow waves have their peak, the red one is having its dip, the peaks in first two waves coincide with the dip of the third one.

Therefore, if you sum these 3 waves, the resultant will be reduced as compared to that in this first case. Therefore, what we can say is that E_2 is which is represented by red colour, the wave E_2 is out of phase with wave E_1, as well as wave E_3. While, in figure 1, all 3 waves E_1, E_2, and E_3, they all are in phase, they are oscillating in phase. While in the right figure E_2 is oscillating out of phase, while E_1 and E_3 are in phase. This is all for today. Thank you for listening me.