Nuclear Astrophysics

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Module # 02

Lecture # 10

Reaction Rate

Dear students. Welcome to today's lecture, in which I am going to continue the topic I started in the previous lecture, which is the cross-section of a nuclear reaction. Let us have a quick recap on the previous lecture in which I have thrown some light on the sources of nuclear energy. I have tried to refresh your knowledge on the mathematical expressions of the Q value of a nuclear reaction.

I have written mathematical expressions for nuclear Q value, which considers masses of the nuclei, and atomic Q value, which considers masses of the atom. Because practically, it is not possible to measure the nuclear mass. So, we usually go for the atomic masses. I have suggested you download any app to get the basic properties of the nucleus.

For example, you can download International Atomic Energy Agency IAEA located in Vienna. Download that app by entering the atomic number and mass number or element symbol values. You can get almost all the information available till now. What are the energy levels of the nucleus, what are their properties, angular momentum, transition, and level schemes?

So, you can get almost all properties related to the nucleus from this app. It would do your job easy while doing the calculations-no need to go here and there. So, in the assignments, a few problems require an app. What is the difference between the atomic Q value and a nuclear Q value? How are they related?

Under what circumstances binding energy correction can be neglected and cannot be neglected. I have discussed by taking the help of positron decay as an example. Electron with negative charge ideally we should call negatron. Still, we usually use the word electron. The impression in our mind is that this is a particle with a negative charge. Still, ideally, we should use the word negatron.

But anyway, it is going on in our daily life one can use it without any issue. If a positron is involved in the decay, then the calculation of the Q value requires some special attention. I suggest you write down expressions for Q value considering beta minus decay, beta plus decay, and electron capture.

So, these three processes require a careful representation in terms of Q value. It makes the concept also easy for you. After that, I discussed the basic concept of a nuclear reaction cross-section. Like in quantum mechanics, the wave function plays an essential role in understanding the subject. Similarly, in this course, Nuclear Astrophysics, there were two objectives. First is the synthesis of new elements because the nuclear reaction is one of the goals of understanding nucleosynthesis. The second is the energy produced by stars. We can categorize these two as the primary objectives of this course because they help us understand the evolution of the universe and the synthesis of the elements in the universe.

The cross-section is the ratio of interactions per unit time, number of incident particles per unit area per unit time, and number of target nuclei within the beam. So, what does it mean when I say targeted beam? When nuclear reactions are happening in stars, there is no difference between target and projectile because both are moving in the stellar plasma.

So, one has to go to the centre of the mass system to compare the results obtained in the laboratory. Still, in the laboratory, we cannot have a nuclear reaction that is happening exactly the way reactions are happening within the stars. So, you have to consider a particle accelerator. You might have heard Van de Graaff generator, Pelletron, synchrotron, and cyclotron, which I will explain.

When a particle beam is generated from the accelerator it interacts with the target material. The interaction between the incident particles and the target nucleus is the nuclear reaction of our interest. How to get an idea of the probability of the occurrence of a nuclear reaction? To study any property of the nuclear reaction the fundamental parameter is cross-section whose units are centimetre square.

Depending on Earth's laboratory's target and projectile combination, you can use the barn, millibarn, micro barn, and nanobarn. So, how to mathematically express the quantity reaction cross-section that I have discussed in the previous lecture. In today's lecture, I am going to take the discussion forward that the Q value is energy produced in a nuclear reaction.

It could be negative or positive. If the Q value is negative, you have to supply the energy to the projectile to initiate the reaction. If it is positive, then energy is released in the form of some particle's kinetic energies of the reaction products. Still, here we are not interested in studying the nuclear reaction mechanism. The main interest is how this nuclear reaction helps us to understand the synthesis of elements in the universe and energy produced in stars.

So, we are shifting the focus from Q value to another quantity. What is that? How many reactions take place in a stellar plasma per unit time and per unit volume. The number of reactions taking place per unit time per unit volume is called the reaction rate. This is a fundamental quantity in Nuclear Astrophysics. The nuclear reaction rate is the number of reactions happening within a unit of time, and per unit, volume depends on two factors.

Number one cross-section of the nuclear reaction which means the probability of occurrence of the reaction. Number two is the velocity distribution of the particles seen in stellar plasma. In plasma, particles move in different directions. When there is a high temperature, it causes the particle's thermal motion, which gives the particles kinetic energy. When they collide, there is a probability for the nuclear reaction to occur. We want to understand the energy produced from the star within which all these reactions are taking place. So, it is quite natural to consider this reaction rate. The number of reactions per unit volume per unit time is one of the most important quantities to be measured. It depends on two quantities, cross-section of the nuclear reaction and the velocity distribution of the particles. At a fixed temperature the velocities of all particles are not the same. There is a distribution, and in today's lecture, I am going to give a few important mathematical expressions to calculate this velocity distribution and the reaction rate.

Let us see how to write down the expression for nuclear reaction rate and consider a reaction,

$$1 + 2 \rightarrow 2 + 3$$

This kind of reaction is happening within the stellar plasma. In Earth's laboratory, we take this 1 as a projectile and 2 as a target nucleus, and this 3 and 4 are the reaction products.

4 could be a proton, alpha, neutron or gamma. 3 could be some nucleus. If it is scattering, then 1 can be equal to 3, 2 can be equal to 4. Sigma is cross-section of the nuclear reaction. It depends on the energy of the particles. It can be written in terms of velocity. In the previous lecture, I have written expressions for the cross-section of a nuclear reaction.

I have written the number of interactions per unit time in the numerator. In the denominator, I have written the number of incident particles per unit time per unit area and number of target nuclei within the beam particle beam.

Rate of a nuclear reaction is the number of reactions per unit volume and per unit time.

$$\frac{N_R}{tV} = (\sigma N_t)\left[\frac{N_b}{VAt}\right] = \frac{\sigma N_t}{V}\frac{N_b}{At} = \frac{\sigma N_t}{V}\frac{N_b}{V}\vartheta$$

Number of reactions say N_R divided by per unit time per unit volume is the reaction rate. It is equal to cross section into number of target nuclei within the beam. The number of incident particles N_b per unit volume. The area of the beam is A. ϑ is velocity of the particles given by length over time.

The current density is the number of particles per unit area per unit time is given by

$$j_b = \frac{N_b}{tA}$$

That means number of particles within the beam over time into area.

So, this r_{12} can be written as number of reactions divided by time into volume.

$$r_{12} = \frac{N_R}{tV} = \frac{\sigma N_t}{V}\frac{N_b}{V}\vartheta = \sigma\vartheta\frac{N_t}{V}\frac{N_b}{V} = N_1 N_2 \vartheta\sigma$$

This is r_{12} and this can be written as N_1, N_2 velocity into sigma which is function of velocity ϑ where N1 is the target N_t divided by volume V and where N2 is number of projectiles N_b/V. They are the number densities of the interacting particles in units of particles per unit volume.

Now a thermodynamic equilibrium the velocity is not constant, but as I said earlier the distribution is there in the relative velocities described by a probability function P as a function of v and in this case P as a function of $\vartheta d\vartheta$ is the probability that the relative velocity of the interacting nuclei is in the range of ϑ and ϑ + d ϑ. So, how it can be defined starting from 0 to infinity P as a function of ϑ d ϑ = 1.

$$\int_0^\infty P(\vartheta)d\vartheta = 1$$

Now let us generalize the reaction rate for a distribution of relative velocities by writing

$$r_{12} = N_1 N_2 \int_0^\infty \vartheta P(\vartheta)\sigma(\vartheta)d\vartheta = N_1 N_2 \langle\sigma\vartheta\rangle_{12}$$

Just I am rewriting this equation velocity into probability for velocity distribution and sigma as a function of energy that is velocity. $\langle\sigma\vartheta\rangle_{12}$ is called as reaction rate per particle pair and this N1 and N2 is the total number density of pairs.

Remember they are non-identical. For identical you need to rewrite this expression. For identical case we can write like the total number of density of the pairs is given by

$$\frac{N_1(N_1 - 1)}{2} = \frac{{N_1}^2}{2}$$

As N_1 goes large we can write like N_1 square divided by 2.

And this leads to reaction rate for identical case as

$$r_{12} = \frac{N_1 N_2 \langle\sigma\vartheta\rangle_{12}}{1 + \delta_{12}}$$

δ_{12} is called as Kronecker symbol. The number of reactions per unit time per unit volume is important because this decides the abundance evolution of elements in the universe.

Please remember this sigma $\langle\sigma\vartheta\rangle_{12}$ is reaction rate per pair and this quantity contains the information related to nuclear physics. Normally in practice we multiply this with Avogadro's number and the units of this reaction rate is given as centimetre cube per mole per second.

Now you know that in the stellar plasma the kinetic energy of nuclei is available from thermal motion. Therefore, these reactions are called as thermonuclear reactions. When some initial stuff is available say starting with a hydrogen because of the gravitational attraction all the protons they come together. The gravitational attraction makes the condensation of this hydrogen matter and after certain stage this gravitational contraction leads to the production of thermal pressure. Thermal pressure leads to the occurrence of reactions between the nuclides and other particles such as protons. Now the energy liberated from these nuclear reactions will try to stabilize the star because the reactions are initiated by the thermal motion, we call these reactions as thermonuclear reactions.

In most cases the velocities of the nuclei can be described by Maxwell–Boltzmann distribution. The probability for the occurrence of a nuclear reaction depends on the sigma, relative velocity. The probability expression P of $\vartheta d\vartheta$ taking the case of Maxwell–Boltzmann distribution is

$$P(\vartheta)d\vartheta = 4\pi\vartheta^2\left(\frac{m_{12}}{2\pi kT}\right)^{3/2} e^{-m_{12}\vartheta^2/2kT}d\vartheta$$

This expression you know very well if you know the basics of statistical mechanics otherwise you can go through the details of Maxwell–Boltzmann distribution, Fermi–Dirac distribution and Bose-Einstein distribution.

Why Fermi–Dirac distribution is not taken here and how Maxwell–Boltzmann distribution is considered in the motion of the positive ions in the star. Maxwell–Boltzmann distribution gives the probability that the relative velocity has a value between ϑ and $\vartheta + d\vartheta$.

And the value of k in terms of Kelvin is

$$k = 8.617 \times 10^{-5} eV/K$$

Remember

$$m_{12} = \frac{m_1 m_2}{m_1 + m_2}, \qquad E = \frac{m_{12}\vartheta^2}{2}, \qquad \frac{dE}{d\vartheta} = \vartheta m_{12}$$

If you take the differentiation with respect to velocity you will get velocity into m_{12}.

Now let me rewrite the expression for probability P as a function of E.

$$P(\vartheta)d\vartheta = P(E)dE = \left(\frac{m_{12}}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{E}{kT}} 4\pi \frac{2E}{m_{12}}\frac{dE}{m_{12}}\sqrt{\frac{m_{12}}{2E}}$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{(kT)^{3/2}}\sqrt{E}e^{-E/kT}dE$$

Value of velocity at which the distribution has a maximum

$$\vartheta_T = \sqrt{\frac{2kT}{m_{12}}}$$

Corresponding to this the energy is k into T. So, the energy distribution has a maximum at

$$E_T = \frac{kT}{2}$$

Now the reaction rate per particle pair

$$\langle\sigma\vartheta\rangle_{12} = \sqrt{\frac{8}{\pi m_{12}}}\frac{1}{(kT)^{3/2}}\int_0^\infty E\sigma(E)e^{-E/kT}dE$$

Please remember the units are centimetre cube per mole per second.

So, please take care of the units and number and accordingly try to derive this relation from the previous expression. So, clearly the reaction rate depends on the cross section which is different for each nuclear reaction. So, in this lecture what I have tried to show you how to mathematically express the reaction rate occurring in the plasma. Though, earlier you are more confident of depending on the Q value to understand the energy released from the star.

But it is not like that because so many particles from different stages of the star they are undergoing reactions and the fundamental quantity of interest is number of reactions taking place per unit time per unit volume. So, that three-dimensional case one has to consider though Q value of the nuclear reaction helps us in designing some experiment. So, in today's lecture please try to derive the expression for reaction rate per particle pair after multiplying with the Avogadro's number.

This formula which involves the Maxwell–Boltzmann distribution, probability distribution, product of probability distribution and the sigma they are going to play a very important role to understand the physics aspect of the star’s evaluation. Thank you so much for your attention. In the next lecture, I will discuss more mathematical formulae related to the charged particle reactions and reactions induced by photons. Thank you so much.