Introduction to Atmospheric and Space Science
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Lecture - 54
Particle Motion in Homogeneous Electric and Magnetic Fields

Hello, dear students. So, in continuation with our earlier discussions so far we have consider plasma a collection of charged particles as a single particle.

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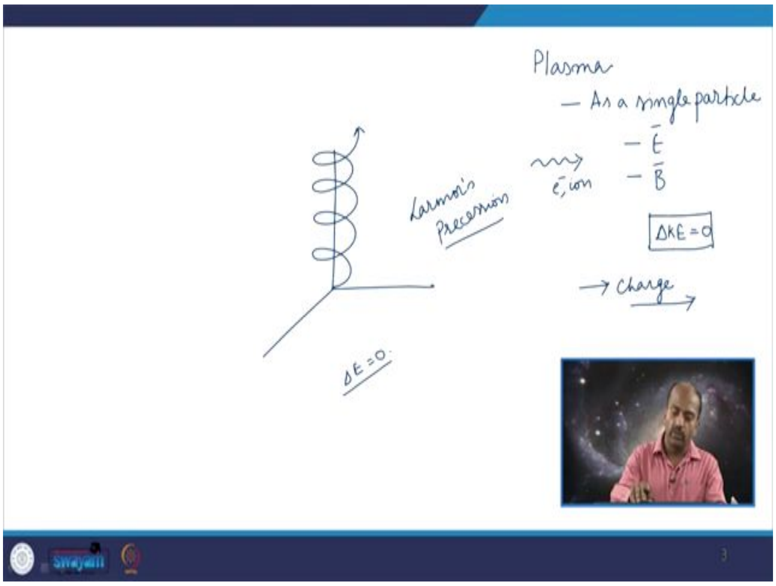

So, we were dealing with the particle description of plasma. So, like I said plasma can be dealt with in three different ways which are particle description of plasma, statistical description of plasma and. So, what differs in these descriptions is that, the way we treat plasma is basically different.

So, what we do is in today's class, we will continue our discussion of single particle moment or motion in various different types of fields. So, to begin with, we have considered plasma movement in an isolated homogeneous static electric field and then we have seen isolated homogeneous static magnetic field. we have considered the field to be is in a particular direction and we have considered a particle such as electron or ion to be experiencing these different types of fields.

What we realized is that, in both these pictures the net amount of work done that is change in the kinetic energy or change in the total energy was maintained to be 0. This is what we have noticed and in addition to this what we have seen is that if the particle is entering the an isolated electric field, we have realized that it will accelerate along the direction of electric field and this acceleration of this velocity linearly increasing with time we will lead towards charge separation of both the charges and which will conserve the total energy.

And when we considered the only the magnetic field, we have seen that the particle will gyrate along the direction of let us say, will gyrate like this. And this direction of this gyration was also called as Larmor's Precession. Then we have realized that a different types of particles, let us say ions and electrons will gyrate in opposite directions and we have also seen that the net amount of work done in this picture will be 0 that is change in the energy is 0. Now, so, this was one example that we have seen.

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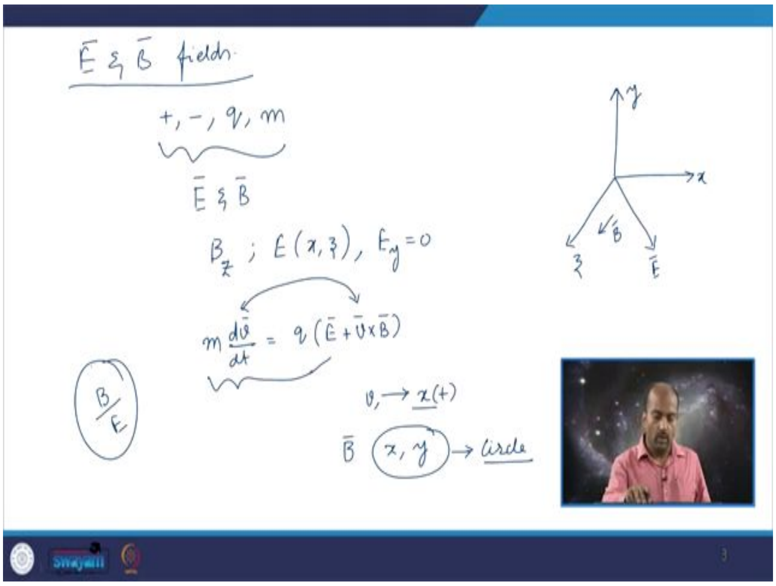

So, in today's class, in continuation to this discussion what we will do is, we will consider perpendicular electric and magnetic fields. So, we will consider an example in which there is an electric field and there is also a magnetic field such that they are perpendicular to each other and you allow a single particle to experience this type of field.

So, if we allow a non zero electric field to be present along with the magnetic field to our earlier description, the motion of the particle will be a combination of a motion of the particle in the presence of an electric field and motion of a particle in the presence of a magnetic field. Now, the particles let us say particles are of two types. Let us say depending on their charges it can be a positively charged particle or it can be a negatively charged particle and these two particles will have some amount of charge let us say, any particle will have some amount of charge and will have some amount of mass.

Now, this is subjected to a field arrangement in which E and B are perpendicular to each other let us consider the electric field to be present in the x and z plane. Let us consider this is the coordinate system that we consider; x, z and y. Now the magnetic field is present across along the z axis and the electric field is present along the x and z plane.

So, this is the so in the x z plane you consider, the electric field is present and the magnetic field is present along the z axis. So, the magnetic field is present along the z axis. Now, when the electric field is present only along the x z plane, we take B to be B z and E to be existing along x z plane that means, these components of electric field along these directions will be nonzero and the electric field along y axis y direction will be 0.

Now, as you see the z component of the velocity is unrelated to the transverse component and getting been treat separately. Then the equation of motion, how do you write the equation of motion? We will write simply the equation of motion as m d V by dt is q times E plus V cross B. What do we have here? So, what we have done so far is that we have dealt with this equation separately.

So, once when there was only electric field and once when there was only magnetic field. In both the types of solutions, we have realized that we can write the velocity or we can integrate this velocity to get the position as a function of time. So, in the second case for example, in the second case when there was only a magnetic field, we have realized the solution of x and y which will satisfy this differential equation resembled something which is circular motion in nature.

So, now what we are about to do is, we are about to combine these two things; that means, we are about to combine the description of a particle in an isolated electric field static electric field and the description of a particle in a magnetic field. Now, let us see. Now in order to do that we will need the solutions of both the pictures, let us say solution of equations of motion in E as well as B. In both the pictures we will need the solution. Then we will combine these two solutions in a suitable form so, that we can get the final picture.

Now, let us say; when the field is along only z axis. So, we have considered the field to begin with I am just recollecting the equations that we have derived in the earlier classes so that I can use them for combining them.

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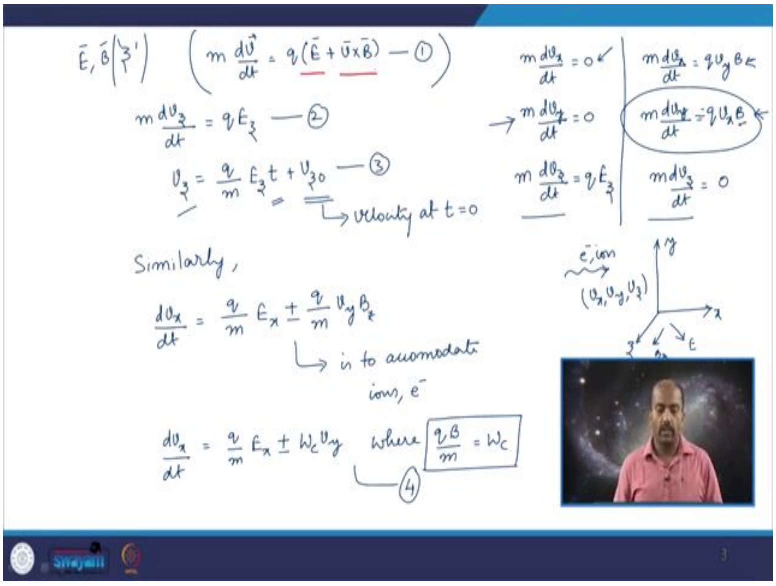

In the uniform electromagnetic field, particle's motion is represented by the following equations –

$$\frac{d^2 v_x}{dt^2} = -\omega_c^2 v_x$$

$$\frac{d^2 v_y}{dt^2} = -\omega_c^2 \left(\frac{E_x}{B} + v_y\right)$$

having solutions

$$v_x = v_\perp e^{i\omega_c t}$$

$$v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B_z}$$

where, $\omega_c = \frac{qB}{m}$, is the angular frequency.

When the electric field or magnetic field was only along the z axis, we have written the equations of motion as m d V x by dt is equal to 0, m d V y by dt is equals to 0, m d V z by dt is equals to q E z.

Then the second set of equations m d V x by dt is equals to q V y B, m d V y by dt is equals to q V x B with a minus and m d V z by dt is equals to 0. So, we will have to combine these two equations for a successful description of the picture. So, the z component of velocity will not be influenced by the magnetic field why is it? Let us redraw the figure for our reference. So, we have the coordinate system like this. So, we have z here x y z so, for our reference. So, the magnetic field is in this direction and the electric field is in this plane; so, B z. Now the velocity components are V x, V y and V z as long as the velocity component is parallel to the magnetic field it will not be influenced.

So, when you want to write an equation of motion for the z component of velocity so; obviously, z component of velocity has to be this plus this; that means, the z component of velocity is simultaneously influenced by the electric field and the magnetic field naturally. Every component of velocity is influenced by the electric field as well as magnetic field, but here what we have is the particle does have a three dimensional velocity let us say the particle. So, this can be an electron or ion.

So, the particle will have the three dimensional components of velocity, but when the magnetic field is parallel to a velocity component that particular part will be 0, so, here it is. Now, if I write the equation of motion. So, the acceleration is only due to the electric field. So, the z component of velocity will change with respect to time only due to the electric field, but not due to the magnetic field.

So, if it is the case I will write let us say m d V z by dt is equals to q E z or let us say you write for example, m d V by dt is equal to q times E plus V cross B. Let us call this equation as 1 and a component equation of this becomes 2, this is only the z component.

So, what does it suggest? It suggests that acceleration is linear with respect to time. So, we can simply integrate this equation to obtain velocity as a function of time V z is equals to q by m E z t plus V z 0 which is just a constant. So, here what we have got is the velocity as a function of time. So, what does it indicate it indicates that V z changes linearly with time and the initial velocity.

So, this is the initial velocity, the velocity at t is equals to 0,. Now similarly, we can write the other components of velocity. Similarly, what we can write is d V x by d t is equals to q by m E x plus minus q by m V y B or B z you can simply say B z.

Now, what have I done d V x by dt let us say d V x by dt is 0 here. In this it is 0, but if you have electric field component E x which is not 0 using this relation similarly, you can write m d V x by dt is equals to q E x which is what I have written. Then in addition to this then a effect of magnetic field also is to be taken into account.

So, d V x by dt becomes q by m V y V z. So, I have effectively combined, let us say this equation and this equation these two equations I have combined. Now, the reason that I have included plus minus is to accommodate ions with a plus and electrons with a minus. So, t this expression I can rewrite as q by m E x plus minus omega c V y.

So, q B by m where q B by m is omega c which is the frequency term which is associated with the Larmor's motion of the particle in the presence of a magnetic field. Let us say this is a very important equation let us say this equation is, this we are going to call as equation let us say equation 4.

So, if you do the similar thing across the other direction let us say E y.

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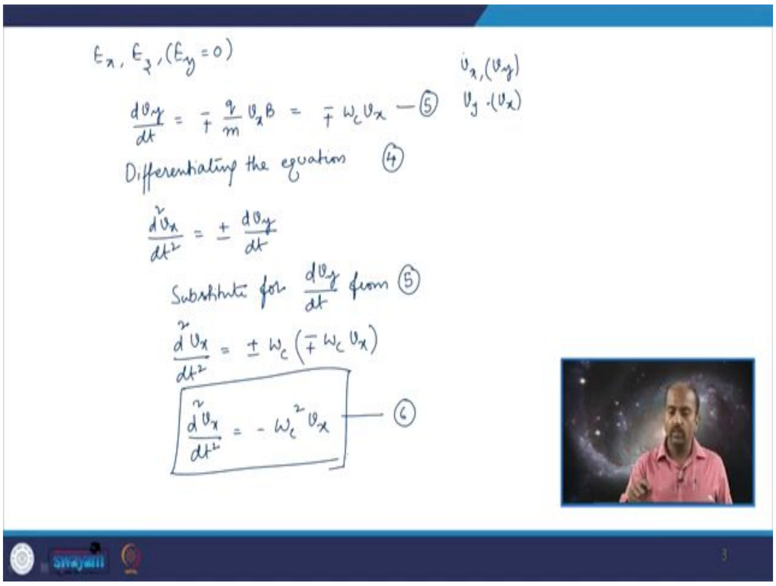

Now, we has we have already seen that E x is nonzero E z is nonzero and E y is 0. So, we need not, we do not bother about this term. So, this term is 0. So, the motion of the particle along the y axis is going to be only due to the magnetic field, but not due to the electric field.

So, in that case we can write d V y by dt is equals to minus plus because, we already have a minus there q by m V x B. See, here you already have a minus here in this equation. So, the motion of the particle or the rate of change of velocity of the particle along the y axis will just depend on the magnetic field, but electric field it will depend it can depend, but we have chosen a field such that the electric field along y axis is 0 we have made a choice like that.

So, in that case it is q V x B. So, this is the only term which will contribute for the net acceleration. So, this we will rewrite as minus plus omega c V x. So, very important thing that we should notice is that, in the acceleration equation of V x you have the V y term appearing and the acceleration V x dot has V y and V y dot has V x.

So, this kind of coupled equations that we have got. So, now let us say we call this equation as this is equation number 4 and this is equation number 5. Now, let us differentiate the above equation, differentiating the equation number 5. Let us say, how do we get? We will write d square V x by dt square is equals to plus minus d V y by dt.

Now, substitute for d V y by dt from equation 5. What we will get is d square V x by dt square is equals to plus minus omega c times minus plus omega c V x which will be d square V x by dt square is equals to minus omega c square V x. Let us call this equation as equation number 6.

So, we have got a second order differential equation in V x. So, we have removed the V y term, we have replaced V y in terms of V x again. Now similarly, we can do the same approach for equation number five as well.

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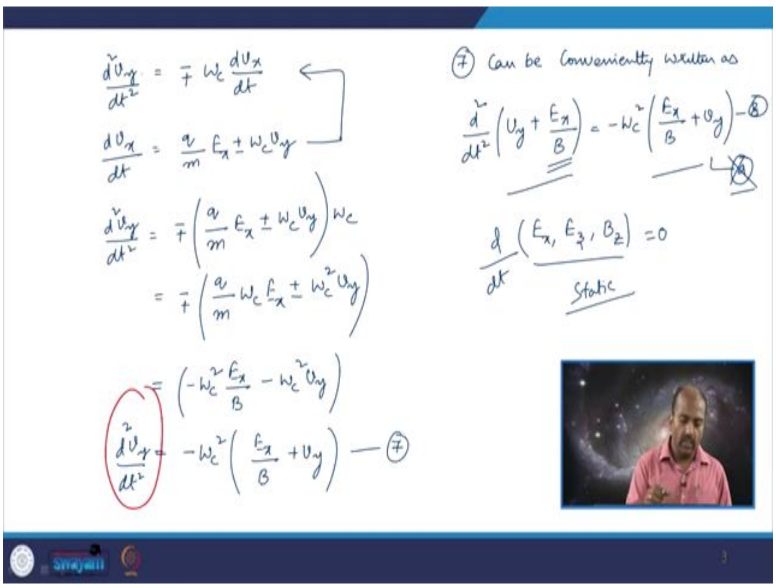

Which we can write as d square V y by dt square is equals to minus plus omega c d V x by dt. Now, here we have an additional term.

So; that means, that here the additional term is which comes only from d V x by dt. So, d V x by dt was written to be q by m E x plus minus omega c V y using this inside. Let us say substituting this into this equation, what we can write is d square V y by dt square is equals to minus plus q by m E x plus minus omega c V y omega c which will be equivalent to q by m omega c E x plus minus omega c square V y. I will write omega c square E x by B minus omega c square V y or simply which will be finally, written as d square V y by dt square is minus omega c square times E x by B plus V y. let us call this equation as equation number 7

The equation number 7 can be conveniently written as d square by dt square of V y plus E x by B is equals to minus omega c square times E x by B plus V y. So, now let us call this equation as a. Now, what have we done by for the sake of convenience what I have done is let us say?

So, we know that the electric field E x E z and the magnetic field B z they are perpendicular to each other of course, but what we have taken is that these are homogeneous in nature or this as static in nature; that means, they do not vary with respect to time. So, d by dt of these quantities will be 0. So, taking this advantage there is no harm in writing d by dt of E x by B. So, this term is eventually 0.

But to get the solution of this differential equation, it becomes easy to have the same multiplier on the left and right hand side. So, if it is the case. So, what have I got I have got one second order differential equation which we called as 6 and the second order differential equation that we have got is called as a let us say. So, let us for convenience or let us say for understanding, let us write these two equations together. So, that we can understand what do they mean as a whole.

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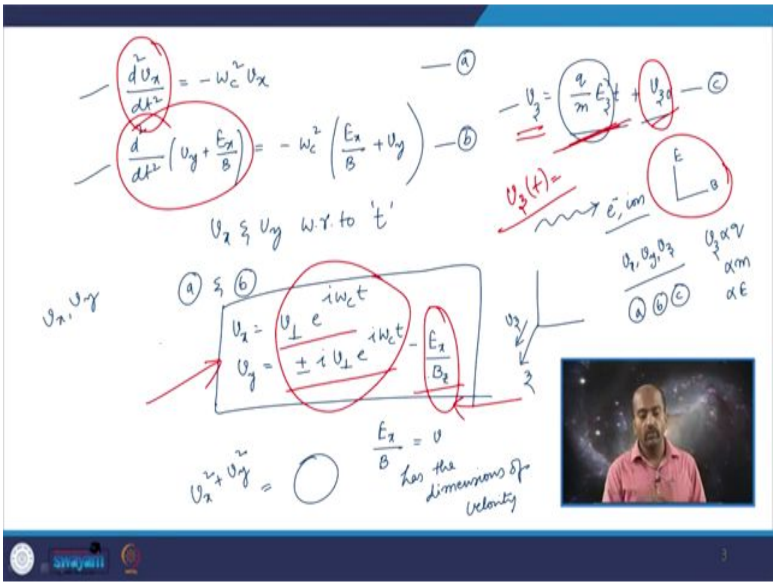

So, d square V x by dt square is equals to minus omega c square V x is the first equation that we have got and similarly the second differential equation is d square by dt square of V y plus E x by B is equals to minus omega c square into E x by B plus V y. Now, let us say this is equation a and this is equation b. So, this is not required.

So, this could I have been called by as equation number 8 sorry for that. Now, let us look at these two equations and try to understand what they are trying to convey. So what is the first thing that you notice about these two equations, the first thing that you notice about these two equations is that, they are second order differential equations and what will be the subject solution the solution will be V x and V y with respect to time.

So, what is particular about this? What is particular about this is that V x and V y will tell you how the velocity of this. So, what happened to V z is so, in combination with these two we have we also have V z , V z is omega c q by m E z times t plus V z naught. So, this is a second order differential equation, rather this is solution itself let us call this equation also for the reference c.

Now, what they represent is that V z is of course, changing linearly with respect to time and when a particle experiences when let us say an electron or ion which is the part of a plasma experiences E cross B field; that means, when electric and magnetic fields simultaneously exist let us say in such a picture if the electron or the ion comes across such a field. The velocity V x ,V y, V z will behave as shown in equation a, b and c.

So, they are not just velocity components which are simply changing linearly with respect to time, but they mean more than that. what I mean is that the velocity components will behave so, as to satisfy these three differential equations simultaneously. Now, this equations a and b can be solved to get V x as V perpendicular e to the power of i omega c t and V y as plus minus i term V perpendicular e to the power of i omega c t minus E x by B.

Now, if you see carefully this term E x by B has the dimensions of velocity we know it . So, if you solve these two equations, the solutions of these two equations will be like this V x and V y. V perpendicular is the perpendicular component of velocity , omega c is the Larmor's gyration frequency t is the time E x is electric field across x axis B is the field.

So, B could have been written by as B z. So, there is no harm in that. Now, what do these equations represent? So, generally if you want to find out the trajectory, you can simply square it and add it. So, it is to find let us say V x square plus V z square, then you will realize what will be the shape of this particular curve in addition to So, this should have been V y.

Now, the most important point what we have to understand here is that, V z see the particle let us say in this frame of reference, the particle has a linearly changing velocity component V z with respect to time along the z axis. Along the z axis the velocity is just changing linearly with respect to time you agree this is a linear. So, this is just y is equals to m x. So, this is the linear changing parameter.

Now, the particles velocity along V z depends. So, V z depends on pretty much on let us say depends on charge, depends on mass and depends on the electric field. Now most importantly, the velocity component along z axis changes linearly with respect to time you agree, but when you look at these two equations, let us call this equation, let us rewrite this equations for convenience it has been.

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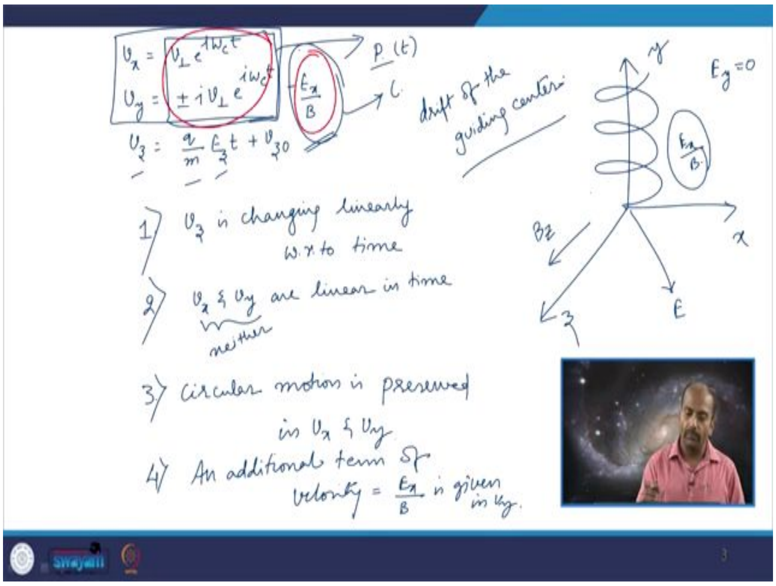

So, let us say V x is equals to V perpendicular e to the power of i omega c t V y is equals to plus minus i V perpendicular e to the power of i omega c t minus E x by B. And V z is equals to q by m E z times t plus V z 0. Now, what are we trying to achieve here, we are trying to understand how does the particle move when it has both the electric and magnetic fields.

So, this is the field alignment that we have taken. So, this is the x axis this is the y axis and this is the z axis E along y axis is 0 the magnetic field is along z axis the electric field is in the x z plane. Now, these three equations indicate. So, V z component depends only on E z. So, V z is changing linearly with respect to time. So, 1 inference 1 V z is changing linearly with respect to time.

So, what else you can say about V x and V y? So, V z is changing with respect to time of course, how about V x and V y. So, V x and V y neither of them are linear in time V x and V y neither of V x and V y are linear in time. So, they are not changing. So, if you see you have an exponential term appearing in the beginning.

Now,what you can realize is that in the first part let us say if you expand this and then take a square and add, what you can realize is that V x and V y are still the circular motion is preserved in V x and V y. So, in addition to the circular term, so the circular motion is appearing because of the first terms in V x and V y. In addition to the circular motion, you have an additional velocity component which is E x by B.

So, the inference 4 is that an additional term of velocity which is E x by B is given in V y. So, what you can say is that the V x component of velocity and V y component of velocity are of course, making the particle executes circular motion, but this circular motion is superimposed with another component of velocity which is E x by B.

So, this superimposed component of velocity is generally called as the drift of the guiding centres. So, this drift of guiding centre is due to E x and as well as B. Now, let us learn more about this drift velocity. Now, we can say that the Larmor's motion is preserved in a perpendicular E and B field in addition to this you have an additional component which you call as the drift of the guiding centres.

Now, let us say to get the more detailed analysis of this study.

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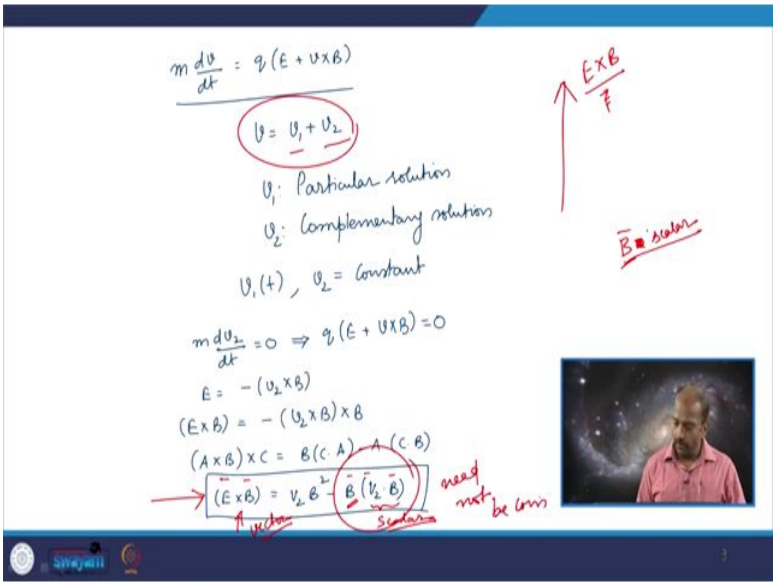

Let us say generally, when you have a differential equation as this m dv by dt is equals to q times E plus V cross B. If you have a differential equation like this you generally write the solution as V is equals to V 1 plus V 2. Here V 1 is called as the particular solution and V 2 is the complementary solution or homogeneous solution things like that.

Now, what you will realize is that when you solve it, V 1 will be a function of time and at the same time, V 2 will be a constant V 2 will just be a constant. How do you get V 2? Let us say here also you can see what I am trying to say is that see, this is the solution of this particular differential equation the Lorenz equation, where we have the charge and mass appearing on the right hand side.

So, you see these two parts I have already underlined them. So, you have to understand. So, this part is the particular solution and this part is the complementary solution. As you see the particular solution is very much dependent on the time and the constant or the complementary solution is time independent. Now generally, if you have a differential equation as this what is the standard method for you to obtain the complementary solution.

The complementary solution is simply obtained by equating it to 0; that means, by equating the rhs to 0 that is it. So, you can write. So, now, since we are dealing with only with V V 2 m d V 2 by dt is equals to 0 right that implies q times E plus V cross B is equal to 0. So, I can write E is equals to minus V 2 cross B.

Let us take the curl of this as this minus V 2 cross B cross B. Now, so, we have a vector triple product which is like A cross B cross C which is to be written as B times C dot A minus A times C dot B. So, using this I will write E cross B as equal to minus E cross B is equals to V 2 B square minus B times V 2 dot B.

Now, this is a very peculiar part. So, let us say before I analyze this I will tell you what I am trying to do again. What I am trying to do is, I have these three differential equations let us say these three second order differential equations d square V x d square V y and we already have a solution V z. The solutions of d square V x and d square V y are going to be V x and V y

What is the origin of this second order differential equation? The origin of this second order differential equation is simply this equation number 1 right. So, origin of the second order differential equation is simply the force expressed in terms of electric and the magnetic components. Now starting from here, I have been able to obtain 2 second order differential equations d square V y by dt square and d square V x by dt square.

So, the solution of these two differential equations has to tell me, how the particles velocity will change with respect to time when you have electric and magnetic fields. Now, I have been able to solve this equations to get the solutions as this V x is equals to this and V y is equals to this plus this. So, what I realized is that V V z of course, the z component of velocity is changing linearly with respect to time plus a constant value; that means, this is the initial value with which the particle has started or when the particle starts to experience the magnetic field and electric field this is what the velocity to begin with.

So, in addition to this, the velocity at any instant of time. So, V z at any instant of time is this plus a linear component which is changing with respect to time. So, V z is clearly defined there is no ambiguity in V z, but what is V x and V y doing. So, V x and V y when I get the detailed solution, you can try yourself by substituting this taking the second order derivatives and substituting this into these two equations you can see for yourself whether these equations satisfy or not in any case.

What these equations represent is that V x and V y have terms which are indicating circular motion; that means, the circular Larmor's precession of the particle is preserved. So, in addition to this, you have another term which is also velocity by the way. So, B is E by c. So, that is still the velocity. So, now, in order to ex obtain the explicit nature of this second term we go back to the original equation we started from. The original equation the solution of the original equation is generally in two terms which are V 1 and V 2. V 1 is the term which indicates time dependence and V 2 is the term which is constant in time.

\So, we have the same thing, we have time dependent terms and we have a constant term right. So, just to get that what I have done is that I have taken a cross product of this and now by resolving it what we have seen is that.

Now, if you look at this. So, E is in one direction the electric field is in one direction the magnetic field is in another direction let us say. Now, here E cross B is a vector. Now, what you see is that, the second term on the right hand side V 2 dot B is a scalar and this is a vector. Now, what I understand is that. So, this term scalar multiplied with a vector will still be a vector in the same direction.

So, the scalar multiplication is not going to change the direction of the vector. Now, E cross B in is in a particular direction and this B multiplied. So, this is not the cross product that I am trying to right here let us say B multiplied. So, the scalar, its magnitude will change, but its direction will not change. So, this will be in a direction which is not in the direction of E cross B.

So, this vector let us call this vector as F this vector F will be in such a direction which is perpendicular to E as well as B; that means, that it will not be in the direction of B certainly not in the direction of B. So, when you are talking about the direction of E cross B and the vectors in the direction of E cross B this term can be neglected need not be considered.

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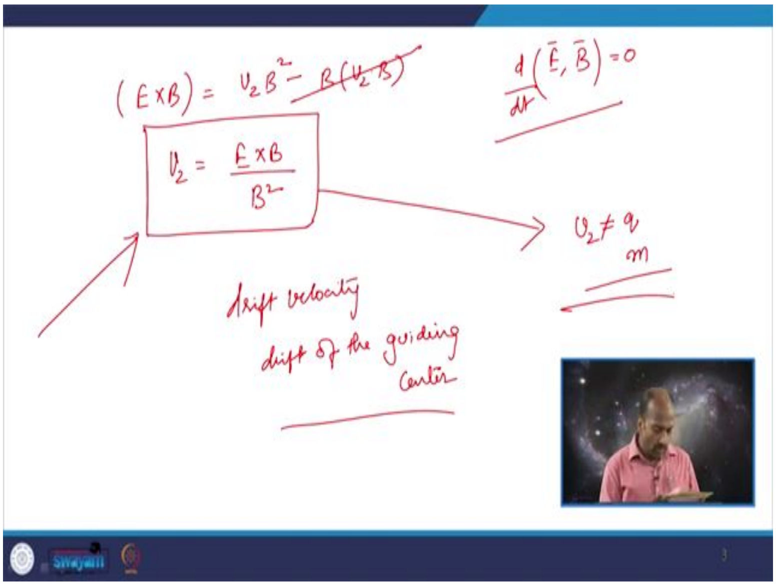

Now, what does it mean it means that so, if you simply write. So, E cross B is equals to V 2 B square minus B into V 2 dot B. So, this term gets neglected and we will write V 2 is equals to E cross B by B square. So, this is called as the drift velocity or drift of the guiding centres.

Now, whenever there is a charged particle movement in the presence of an electric field, we call the velocity that is attributed to the particle as the drift velocity. So, this is the drift velocity that is attributed to the guiding centre. Now one very particular thing about this is that. So, there is no dependence of time in this. So, we have not taken we have taken electric and magnetic field to be static in nature.

So; that means, that there is no time dependence in this in this solution and V 2 I mean the most important thing is V 2 does not depend on q and does not depend on the mass. So, the velocity the drift velocity that is attributed to the particle the guiding centre of this motion it neither depends on the charge nor does it depend on the mass.

But unlike in the electric field or in the magnetic field in the electric field let us say, the electric field is if it is along the z axis we have seen that most of the cases when it allows charged particles to be experienced. Then it there will be a polarity and depending on the polarity charge separation will take place and the second electric field will be set up. And in the magnetic field the effect the polarity was in terms of the charge again. The positive charge particles were gyrating to the right and the negatively charged particles were gyrating to the left.

So, the field was specific to the polarity of the charge. It is not that all the particles were experiencing the same effects no, which the field was specific or biased to the polarity of the charge. Now in this case it is not like that in this case it is the velocity that is the drift velocity is independent of the charge and mass of the particle.

So, this was something about the charged particle movement in the presence of electric and magnetic fields. So, in the discussions I had, what we will see we will try to draw trajectories for the particles movement in these different types of fields and we will try to explore more on these topics.

Thank you.