Introduction to Atmospheric and Space Sciences

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Lecture - 45
Chapman's Theory of Layer Production – Continued

Hello, dear students. So, we will continue our discussion on Chapman Theory of Layer Production. So, in the last class we have understood the basic assumptions behind the Chapman's theory. We have also seen how the production rate is a combination of the decreasing number density from the bottom and decreasing intensity of solar radiation from the top. So, how we can combine these two.

So, in the last class we have stopped at dx is equal to dh secant chi. Let us call this as equation 1. So, when the solar radiation passes through the atmosphere a fraction of intensity is absorbed and scattered by the atmosphere.

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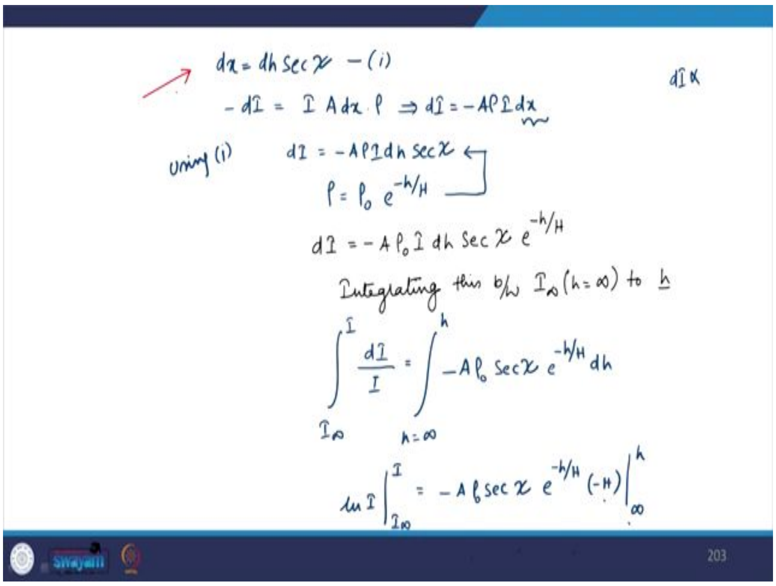

So, in continuation what we can see is that so, the decrease in intensity of radiation is directly proportional. I mean, the decrease we are talking about the decrease in intensity is directly proportional to let us say dI or let us say since there is a decrease, is directly proportional to the intensity of radiation. Let us say I, volume of the atmospheric layer we write it as A times dx a is the area and the density of the atmosphere, or we can simply write dI is equals to minus A rho I dx.

So from the earlier equation, we can replace dh in terms of dx as; so, dI is equals to minus A rho I dh secant chi. We are using. So, we know that the density of the atmospheric species varies exponentially as a function of height and the scale height. So, we can substitute this into this equation and then we can write dI is equals to minus A rho naught I dh secant chi exponential minus h by H.

We have just substituted the functional dependence of rho into the dI. What is dI? dI is the rate at which the intensity decreases from the top. How does it depend? It depends on the area or the volume that it encounters and the density which is a representative of the number of atoms which are present in that volume, and the incident intensity itself.

So, this is just the Beer-Lambert's law,. So now, we have got the rate at which the intensity decreases; in terms of angle as well,. Now let us substitute, let us integrate this equation. Integrating this between I infinity that is, at h is equals to infinity to a particular height h.

Let us do the integration. So, integral I infinity to I dI over I is equals to integral h is equals to infinity; h is equals to infinity means at the top of the atmosphere when the radiation is just entering into the atmosphere; where the intensity is maximum which is being referred to as I infinity minus a rho naught.

Now, rho naught is the density at the bottom that is h is equal to 0 or the surface of the earth. Secant chi, chi is the solar zenith angle e to the power of minus h by capital H dh. So, getting this we will get ln of I between the limits I infinity to I is equal to minus a rho naught secant chi e to the power of minus h by capital H into minus H from infinity to the height h.

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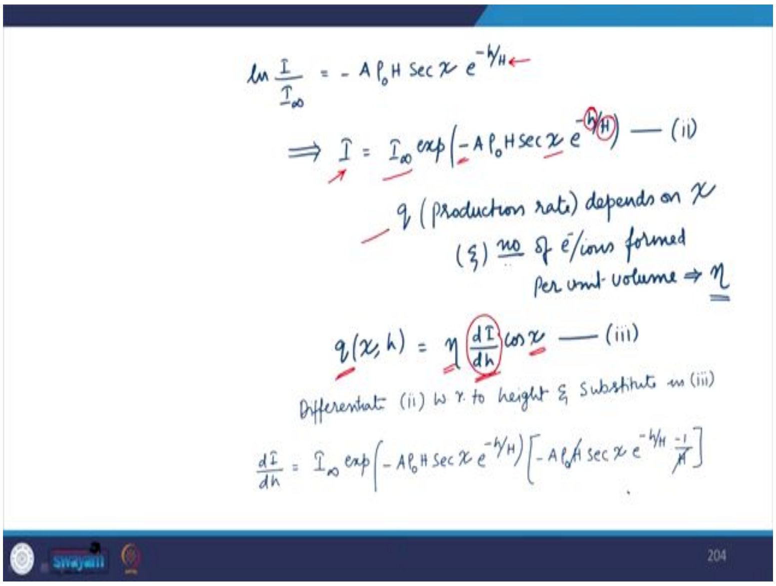

Intensity of radiation at height h in the atmosphere is given by

$$I = I_{\infty} \exp\left(-A\rho_0 H sec\chi e^{\frac{-h}{H}}\right)$$

Where, I_{∞} is incident intensity and A is the volume of the atmosphere.

So taking it further, we can write that ln of I by I infinity is equals to minus A rho naught H secant chi e to the power of minus h by H. Or we can write I is equals to I infinity times exponential minus A rho naught capital H secant chi into exponential minus h by H.

What do you see? You see an exponential outside this bracket and you also see an exponential inside the bracket. Let us call this equation as 2. What was the equation number 1? Equation number 1 was between the oblique height to the vertical height and the angle in between .

Production rate of ions let us or electrons depends on the solar zenith angle. So, we are introducing now q which is called as the production rate, depends on; what does it depend on? It depends on chi or number of electrons slash ions formed, per unit volume energy absorbed by the atmosphere.

And this is called as let us say number of electrons per unit volume we are going to call it as eta. So the production rate, the rate at which electrons or ions are released or produced depends on the angle at which the radiation is incident and also the number of electrons and ions formed per unit volume. So, this number is referred to as eta.

So we can write q, which is a function of chi the solar zenith angle and the height; that means, we are trying to calculate the production rate at a particular height. Which is equals to eta times dI by dh cos chi. Let us call this as equation 3. Now let us differentiate equation 2; now equation 2 which is we will differentiate equation 2 with respect to height and substitute in equation 3.

What will we get ? dI over dh the rate at which the intensity decreases with respect to height, I, we have already is I infinity exponential minus A, I am just differentiating this A naught A rho naught H secant chi e to the power of minus h by H exponential x is equal to exponential x times whatever that multiplies x. So, we can get rid of this straight forward.

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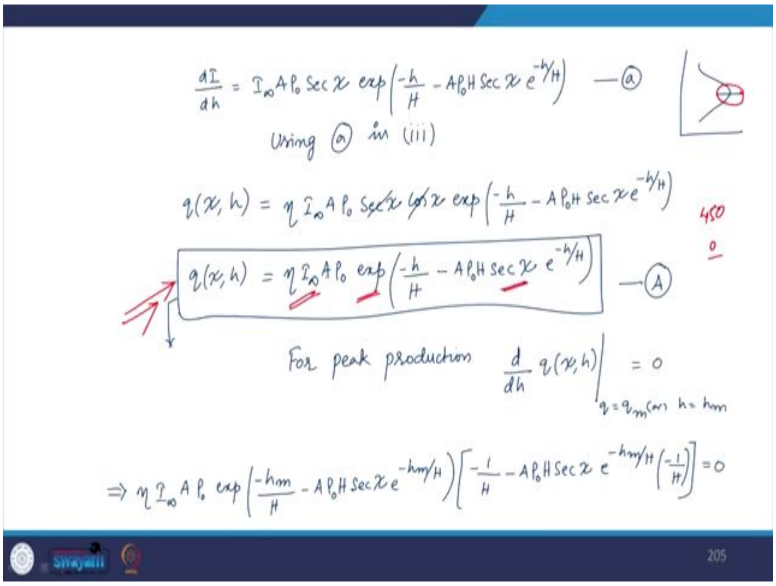

Production rate

$$q = \eta I_\infty A\rho_0 \exp\left(-\frac{h}{H} - A\rho_0 H sec\chi e^{\frac{-h}{H}}\right)$$

where, H is scale height and η is the no. of electrons/ ions formed per unit volume.

Then we will write dI over dh is equals to I infinity A rho naught secant chi upon substitution simplification, it is h by H minus A rho naught H secant chi e to the power of minus h by H. What is this? This is not an equation let us say we call this as a.

Now, using a an intermediate equation, in 3; what is 3? 3 is here. We will get q the rate of production as a function of the angle of incidence or angle of oblique incidence and the height. So, using this expression we should be able to evaluate the production rate at any height or at every height.

Which is eta is the number that is produced A rho naught secant chi cos(chi) exponential minus h by H minus A rho naught h secant chi exponential minus h by H. Which upon simplification, this will get cancelled. Eta I infinity a rho naught exponential minus h by capital H minus a rho naught H secant chi e to the power of minus h by capital H. What is this? This is the rate of production as a function of height and the solar zenith angle.

now let us talk about. So, this is just an expression which is which is going to be very much relevant ahead. Let us call this expression as A, capital A. Now this is the rate of production for let us say for maximum or for peak production. What is peak production? This is the peak production. Let us say if you see this is the peak production at which the rate of the number of electrons are maximum. The number of electrons that are released maximum. So, at peak production we can say that this is for mathematically you can see that its the maxima?

So, d by derivative with respect to height of q should be 0 where when q is equals to Q m should be 0. Or h is equal to H m, H m is the peak production height Q m is the peak production rate. So, we take a derivative of this with respect to height and then we can equate it to be equal to 0. Which implies that eta I infinity A rho naught exponential minus H m. So, I have substituted h for H m minus A rho naught H secant chi e to the power of minus H m by capital H minus 1 by capital H minus A rho naught H secant chi e to the power of minus H m by H into minus 1 by H is equal to 0.

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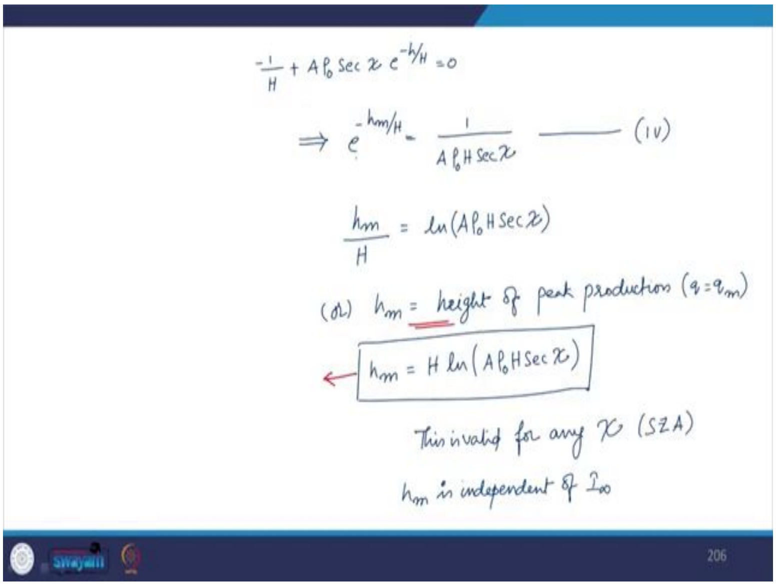

Height of peak production

$$h_m = \text{H} \ln (\text{A}\, \rho_0\, H\, sec\, \chi\,)$$

Where, χ is solar zenith angle.

Or we can simplify it, or minus 1 by capital H plus A rho naught secant chi e to the power of minus h by capital H is equals to 0 or e to the power of minus H m by capital H is one over a rho naught h secant chi. Let us call this equation as 4. So, we can write H m by capital H is equal to ln A rho naught H secant chi, or H m; what is H m? H m is the height of peak production or where q is equals to Q m that is it.

So, H m is H times ln of A rho naught H secant chi. So, this is valid for any solar zenith and any chi and we can see clearly that the peak production height H m is independent of I infinity. The magnitude of I infinity. So, if you look back what we have done so far is we have taken in this expression for, I have taken this expression which is, just the relation between angle and the length coordinate in oblique as well as in a vertical direction then we have taken how the intensity drops as it travels.

So, its it drops because we have a minus; I mean we have a minus because the intensity decreases with respect to height as you are traveling from the top to the bottom. So, we thought that it will depend on the available density, the volume and the intensity to begin with. So, there we got the dI, then we integrated this expression for dI and we got, intensity at any particular point as a function of the incident intensity, after having traveled a distance h in an atmosphere which has a scale height capital H.

This is the most important, I mean if you want to understand really. So, the this expression tells you how the intensity what is the intensity when it has been I infinity after it has traveled the distance of small h in an atmosphere with a scale height capital H at an angle chi. That is it,. So, you will realize that it is exponentially decreasing as it travels with because, you have a minus you will realize that it will exponentially decrease as a function of height from the top,. This was a just about the intensity.

Now introduce the production. So, intensity is decreasing, what is happening due to which intensity is decreasing? What is happening? The production is happening. So, how does the production depends on intensity? Production depends on the rate at which intensity is decreasing. So, more is the rate at which intensity is decreasing more is the production. So, production is directly proportional to the rate at which intensity is decreasing. So, that is why you have q which is the production rate and you have a direct proportionality with the rate at which the intensity is decreasing with respect to height.

And what else the production will depend on? Production will also depend on how many number of atoms are produced per unit loss of intensity with respect to height. The unit loss of intensity with respect to height is given by dI by dh, the number of electrons or ions produced per unit volume is the efficiency let us say, is eta and chi is; obviously, the angle. So, the dependence of chi is maintained to be to just justify the fact that intensity or the production will not be the same if the intensity is traveling and vertically or in oblique direction. So, it will be different.

So my objective is; so, we have already have the intensity as a function of height. So, we take a derivative of this and we substitute it into q. We got the q expression which is which is this. What is this? This expression is the production rate expression. Now we wanted, see this production rate is of course, a function of height. So, you substitute various different heights.

Let us say we start from 450 kilometers and you go up to 0 kilometers. So, there are some there are some exponential terms which have guaranteed the decrease in trend with respect to the height, but we also have some terms with the secant let us say.

So, the due to these two terms I mean you want to get a linear way. But so, at some point. So, this if you plot this take random variables and take height you can take the capital H to be let us say 10 kilometers, if you plot this you will realize that its not a straight line it will be a curve something looks something like this. So, we want to see what happens at this height which is the peak production height.

So, d we take a derivative because it is mathematically maxima, the maxima is the derivative of the function will become 0 at that particular point, at that turning point. So, we take a derivative d by dh of q which is equals to 0. We take the derivative which appears to be lengthy then upon simplification, we will realize making the derivative to be 0 and substituting the height small h to be H m which is the height of the peak production.

So, H m is equal to h times ln A rho naught H secant chi. So, this is the height of peak production. So, H m is , let us write here, we already written here the H m is the height of the peak production. Now, so the height of peak production does not depend on the incident intensity. Incident intensity was there when we were discussing this, but at some point the height of peak production now is independent of the peak production height.

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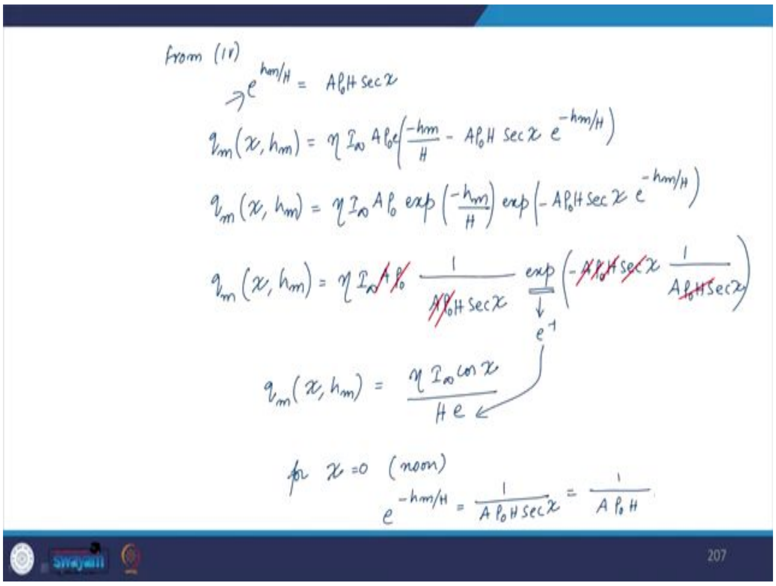

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Peak production rate

$$q_m(\chi, h_m) = \frac{\eta I_\infty \cos\chi}{He}$$

Now, let us say from equation 4, we can write that e to the power of h m by capital H is a rho naught h secant chi. The peak production q m as a function of the solar zenith angle and the peak production height is eta I infinity A rho naught minus h m by capital H minus A rho naught H secant chi e to the power of minus h m by capital H. We will rearrange this equation for simplification q m peak production as a function of chi and h m is equals to eta I infinity A rho naught exponential minus h m by capital H times exponential.

So, here there is an exponential. So, this is what comes here exponential minus a rho naught H just a rearrangement, there is nothing that we should worry at this point h m by capital H. Now, Q m using h m, but that we have derived already as a function of chi and h m is equals to eta I infinity A rho naught 1 by A rho naught H secant chi into exponential minus A rho naught H secant chi into 1 by a rho naught H secant chi.

So, we have used this that is it,. So, things will get cancelled, this will get cancelled A rho naught and A rho naught H secant chi. So, as a result what you get is q m chi h m is equal to eta I infinity cos(chi) divided by H e. So, this exponential this will e to the power of minus 1 which comes here. Now, for let us say for chi is equal to 0; that means, at noon. So, e to the power of minus h m by H is equal to one by A rho naught H secant chi which is 1 by A rho naught H.

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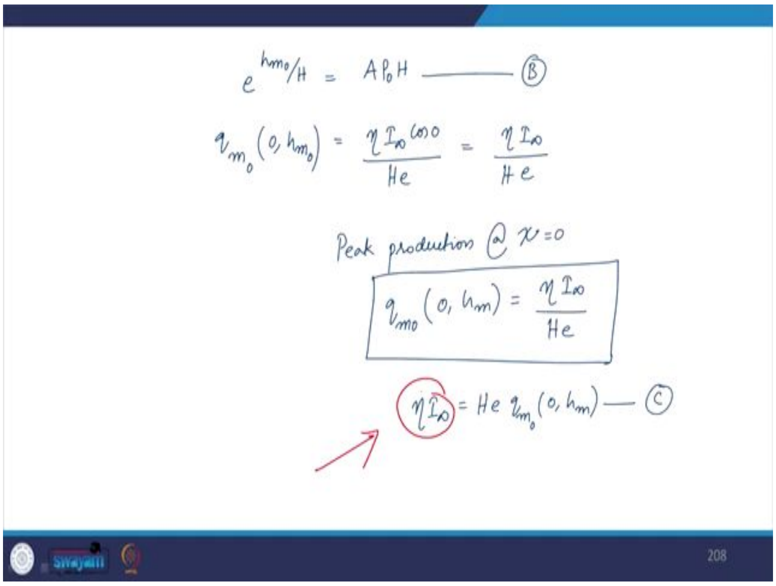

Peak production rate at solar zenith angle $\chi = 0$

$$q_{m_0}\,(0, \mathrm{h_m}) = \frac{\eta I_\infty}{H\,e}$$

Or e to the power of h m 0; h m 0 is at chi is equals to 0 otherwise it was h m. A rho naught H, let us call this equation as capital B,. Now which is q m at noon at chi is equals to 0. So, chi is; obviously 0 here and at what height? At h m 0 is eta I infinity cos(0) divided by H e. So, which is eta I infinity divided by H e. So, what have we got? We have got the peak production at chi is equal to 0 is q m 0 at 0 h m is eta I infinity divided by H e.

So, let us say is a very important equation the peak production at the noon time depends on the I infinity, the scale height let us say this is just a number and eta. So, you can rewrite this equation eta I infinity is H times exponential into q m 0 which is a function of 0 and h m. Let us call this equation as C.

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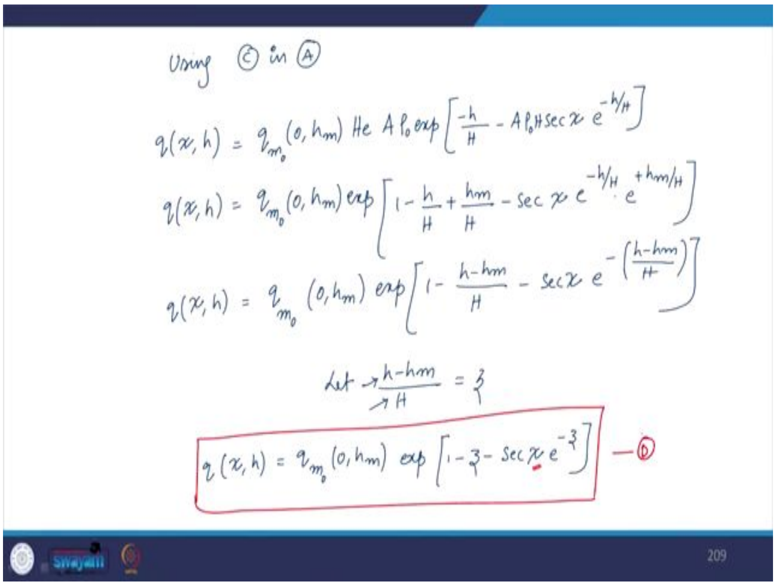

Production rate in terms of Peak production rate

$$q(\chi, h) = q_{m_0}(0, h_m)\exp(1 - z - \sec\chi\, e^{-z})$$

where $z = \frac{h - h_m}{H}$

Using C in; where do we have A? C is this, this is C where do we have A? equation capital A. that is the equation A. So, using this where what are we trying to use? We are trying to we are trying to substitute for Eta I infinity in this expression; what will we get? We get q just a production as a function of height is equals to q m 0 h m H e that we got from the equation C.

Then the equation A itself rho naught exponential minus h by H minus A rho naught H secant chi e to the power of minus h by H. So, we will write q as a function of these two variables taking this inside taking this H e inside exponential 1 minus h by capital H plus by capital H minus secant chi e to the power of minus h by H into e to the power of minus by H or q of chi comma h, h equals to Q m 0 0 into exponential.

Rearranging these terms 1 minus h minus by H minus secant chi e to the power of minus h . Let us call h minus h m by h. So, the normalized height above the peak production height, the height above the peak production height, normalized with respect to the scale height. Let us call this as z.

So, then we can write q the rate of production as a function of the zenith angle and the height is q m 0, what is q m 0, q m 0 is the peak production which happens at the noon when chi is 0. 0 and h m times exponential 1 minus z minus secant chi e to the power of minus z . So, this is the expression that we want to derive; what is this expression? this expression let us see.

This expression tells you, what is the peak production? What is the production at any given height as a function of production that could happen during the known or when chi is equals to 0? What does it depend on? It depends on the solar zenith angle, depends on the height and depends on the scale height.

So, this is what let us call this equation as let us say D. So, we have been able to derive using the Chapman's theory, we have been able to derive how the production changes with respect to height. So, there are 4 or 5 major inferences that one can draw based in the atmosphere based on this equation. We will look at that in the next session. So far we have derived a generalized expression for the production rate as a function of height and as a function of the maximum production.

So, we will stop here, we will continue the discussion which is going to be the inferences on this expression.