Introduction to Atmospheric and Space Sciences

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Lecture - 39

Radial Growth of Droplets by Diffusion

Hello, dear students. So, in today’s class we will continue our discussions on the formation of cloud droplet or on the formation of rain droplet from the cloud droplet right. So, far in our discussions of atmospheric thermodynamics or in cloud physics, we have seen what is the process of water vapor condensing to form tiny droplets of water, a collection of which we call as clouds right.

So, we have seen that what are the two effects which play very important role in the formation of clouds or the droplets; the two effects were called as the solute effect and the curvature effects. These effects are important in the sense that they will try to inter compete in the formation of the droplets right.

So, we have seen that the one will try to increase the saturation vapor pressure and another will try to decrease the saturation vapor pressure. We also seen how these two effects can be combined and which term is important and under which circumstances is this term important we have seen all this right. So, in today’s class we will try to derive a mathematical expression for the Radial Growth of a Droplet.

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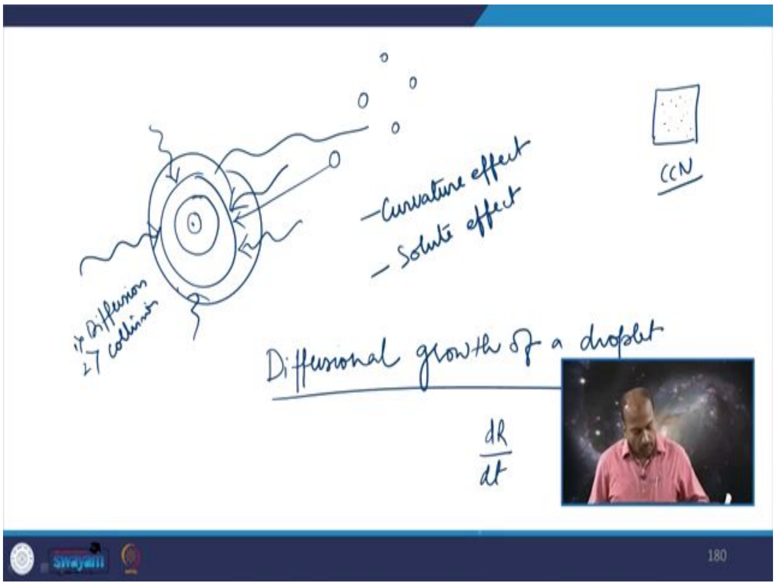

So, now we see that the once the water vapour let us say if you have water vapour, if you have small tiny particles which you call as the cloud condensation nuclei which are available. So, these cloud condensation nuclei will not allow the mixing ratio or the amount of moisture to increase beyond the saturation.

So, the moment when the saturation is reached or when it is equal to 99 percent at least they will immediately behave as the surfaces over which condensation can happen and as a result if the condensation happens you will get the droplets right.

So, the ideal situation where you will require several hundreds of relative humidity is never actually needed right. So, having said that so we have seen what are the curvature in the solute effects these are very important in understanding the cloud formation. So, curvature effect and the solute effect right we have also obtained an expression for that right.

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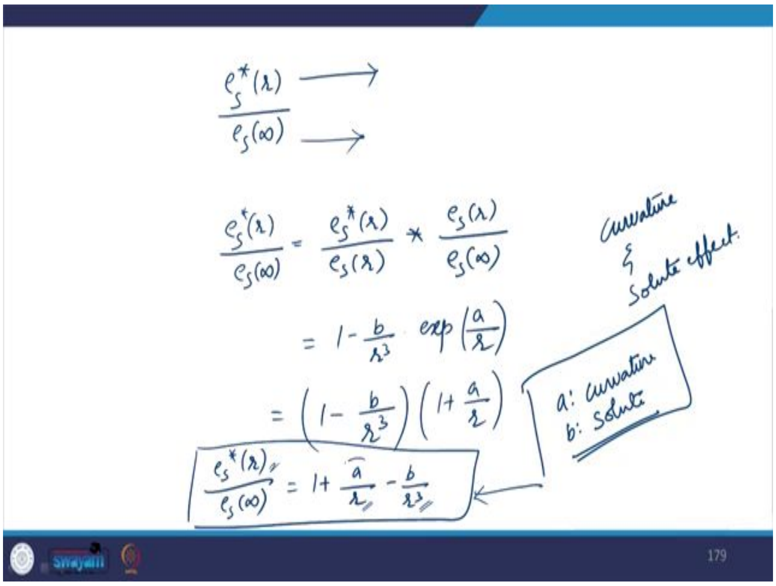

$$\frac{e_s^*(r)}{e_s(\infty)} \begin{matrix} \longrightarrow \\ \longrightarrow \end{matrix}$$

$$\frac{e_s^*(r)}{e_s(\infty)} = \frac{e_s^*(r)}{e_s(r)} * \frac{e_s(r)}{e_s(\infty)}$$

$$= 1 - \frac{b}{r^3} \exp\left(\frac{a}{r}\right)$$

$$= \left(1 - \frac{b}{r^3}\right)\left(1 + \frac{a}{r}\right)$$

$$\boxed{\frac{e_s^*(r)}{e_s(\infty)} = 1 + \frac{a}{r} - \frac{b}{r^3}}$$

Curvature & Solute effect.

a: curvature
b: solute

So, this is the combined expression, what will be the saturation vapor pressure with respect to a curved surface in the presence of a solute right. So, we know what are these, a and b terms right.

So, in today's class what we will see is, we will see the diffusional growth of a droplet. So, what are we trying to do here? So, we have seen that once the tiny droplet forms in the sizes of typical micrometers. So, what this droplet will do is, this droplet has to grow to larger sizes.

How does it grow? It grows by two different mechanisms. Let us say, one is by diffusion where water molecules from the surroundings will diffuse into this and they will contribute in the increase of the size of this droplet right.

And another mechanism is let us say is by collision and coalescence. What it does is by collision let us say. What it does is, if the droplet has acquired some size or if the droplet is moving it will come in contact or it will collide with several other molecules or tiny droplets. They will fix themselves they will stick to the droplet and they will contribute in the growth of this particular droplet right.

So, today what we will do is we will try to see how the radius of a droplet will change with respect to time. So, we are going to obtain an expression for the rate of change of the radius of a droplet with respect to time. Eventually, what we are trying to look at is how the droplet will grow in size with respect to time, how it will change right.

So, let us say for this discussion what we will do? We will start from the standard Clausius Clapeyron equation from which we will try to understand the radial growth of a droplets.

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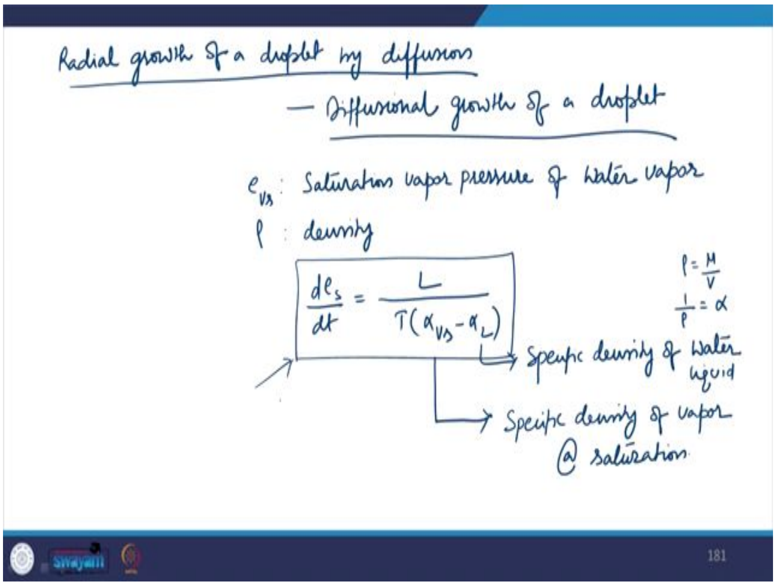

Radial growth of a droplet by diffusion

— Diffusional growth of a droplet

e_{vs} : Saturation vapor pressure of water vapor

ρ : density

$$\boxed{\frac{de_s}{dt} = \frac{L}{T(\alpha_{vs} - \alpha_L)}}$$

$\rho = \frac{M}{V}$

$\frac{1}{\rho} = \alpha$

Specific density of water liquid

Specific density of vapor @ saturation

So, the topic is the radial growth of a droplet by diffusion. You may also see it as diffusional growth of a droplet right, this is the topic right.

Now, let us assume that e vs is the saturation vapor pressure of water vapour; e vs is the saturation vapor pressure of water vapor and rho is the density right. So, we can write that de s by dt according to the Clausius Clapeyron equation is latent heat divided by temperature into alpha vs minus alpha L right so, where alpha L is the specific density; specific density of bulk water or liquid. So, alpha is the specific density we know that the density is mass per unit volume and 1 by rho is alpha right.

So, alpha L is the specific density of liquid and alpha v s is similarly the specific density of vapor at saturation, this is more important right.

Now, let us say we take this equation this is going to be the starting point we take this equation we will try to rearrange this and we will try to incorporate many other terms which are relevant for the droplet, for the vapor or for the radial growth how did the mass changes.

So, you have to understand that if the droplet is growing in size it’s mass is also increasing. So, this radial growth at some point also involve the mass right.

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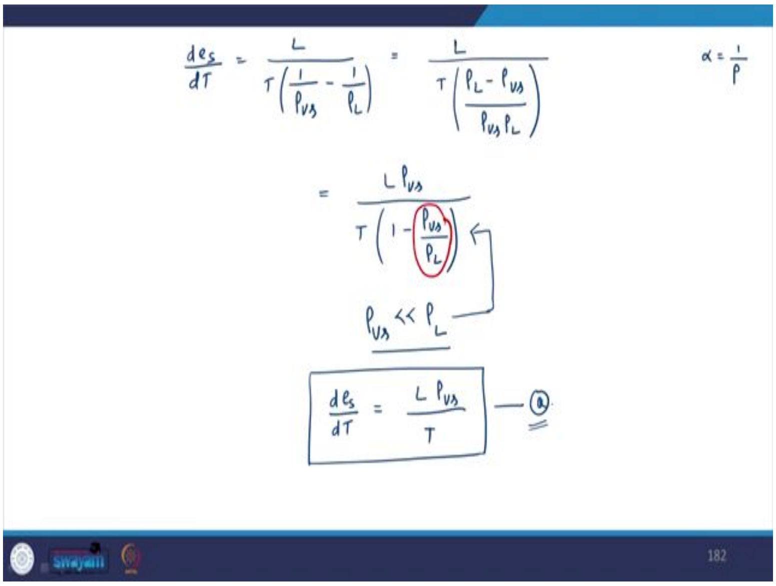

$$\frac{de_s}{dT} = \frac{L}{T\left(\frac{1}{\rho_{vs}} - \frac{1}{\rho_L}\right)} = \frac{L}{T\left(\frac{\rho_L - \rho_{vs}}{\rho_{vs}\rho_L}\right)}$$

$\alpha = \frac{1}{\rho}$

$$= \frac{L\rho_{vs}}{T\left(1 - \frac{\rho_{vs}}{\rho_L}\right)}$$

$\rho_{vs} \ll \rho_L$

$$\boxed{\frac{de_s}{dT} = \frac{L\rho_{vs}}{T}} \quad — ⓐ$$

So, we can rewrite the expression for our convenience. So, which is d e s by d T is L by T into 1 by rho vs the density of water vapor at saturation minus. So, what I have done is I have used alpha the specific density as 1 by rho.

So, I am going to just use the simple rearrangement of this equation. So, which is basically algebraic rho L minus rho vs divided by rho vs times rho L or L rho vs divided by T into 1 minus rho vs by rho L.

So, what is rho vs rho vs is the density of saturated vapor. So, and rho L is the density of the liquid. So, for obvious reasons we can say that the density of water vapor at saturation will be very small in comparison to the density of the bulk liquid right.

So, we can use this into this equation right and we will say that so d e s by d T is simply equal to rho L times rho vs by T ok. Let us call this equation, what have we done so far, we have to just use the Clausius Clapeyron equation between the density of water and the density of vapor at saturation. So, we have gotten rid of the density of liquid because the term is appearing in the denominator which is very large in comparison to the term that appears in the numerator.

So, we can neglect these entire rho vs entire rho vs by rho L and as a result we can write the change of the saturation vapor pressure with respect to the temperature is equal to the latent heat times the density of water vapor at saturation divided by the temperature right.

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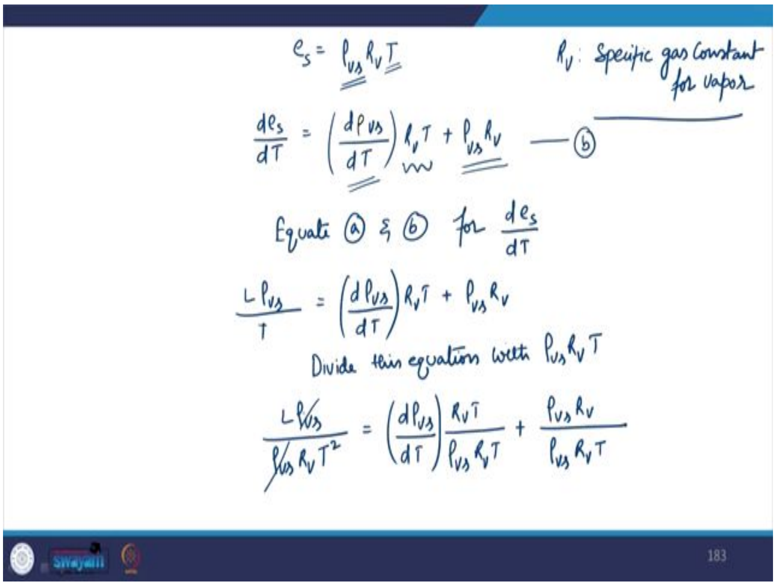

$$e_s = \rho_{vs} R_v T$$

R_v: Specific gas constant for vapor

$$\frac{de_s}{dT} = \left(\frac{d\rho_{vs}}{dT}\right) R_v T + \rho_{vs} R_v \quad \text{— ⓑ}$$

Equate ⓐ & ⓑ for $\frac{de_s}{dT}$

$$\frac{L\rho_{vs}}{T} = \left(\frac{d\rho_{vs}}{dT}\right) R_v T + \rho_{vs} R_v$$

Divide this equation with $\rho_{vs} R_v T$

$$\frac{L\cancel{\rho_{vs}}}{\cancel{\rho_{vs}} R_v T^2} = \left(\frac{d\rho_{vs}}{dT}\right) \frac{R_v T}{\rho_{vs} R_v T} + \frac{\rho_{vs} R_v}{\rho_{vs} R_v T}$$

Now, we have to understand that the water vapor satisfies the ideal gas equation.

So, how does it look like, e s, e s is the pressure because it is a partial pressure you call it as the vapor pressure e s is the pressure that is exerted by the vapor on the surface on the bulk water, is rho vs at saturation density of vapor at saturation times R v, you use the gas constant for this specific vapor not the universal gas constant.

So, this is the specific gas constant which is for the water vapor. So, R v is the specific gas constant for vapor right.

So, we can write that we take a derivative of this equation with respect to the temperature. So, this is e s is always the vapor pressure to indicate that we use e instead of p. So, this is equal to d e vs by d T times R v T plus rho vs into R v.

So, this is just a derivative the derivative of the vapor pressure saturation vapor pressure with respect to temperature is so this is now a function of temperature. So, this term appears here R v T appears as it is. Then the rate of change of temperature with respect to the temperature is 1, so the remaining part comes as the second term in the equation.

So, let us call this equation so we have this equation as let us say we have this equation as a. Now, let us call this equation as b. So, this is b.

So, now let us equate; let us equate equation a and b equate a and b. So, why are we equating? So for the rate of change of vapor pressure with respect to the temperature. What we will get is. So, L times rho vs by T is equals to d rho vs by d T times R v T plus rho vs times R v.

Now, let us divide this equation with rho vs R v T, which we will get as L rho vs divided by rho vs R v T square is equals to d rho vs by d T into R v T divided by rho vs in to R v T plus rho vs into R v divided by rho vs R v T so we can cancel this.

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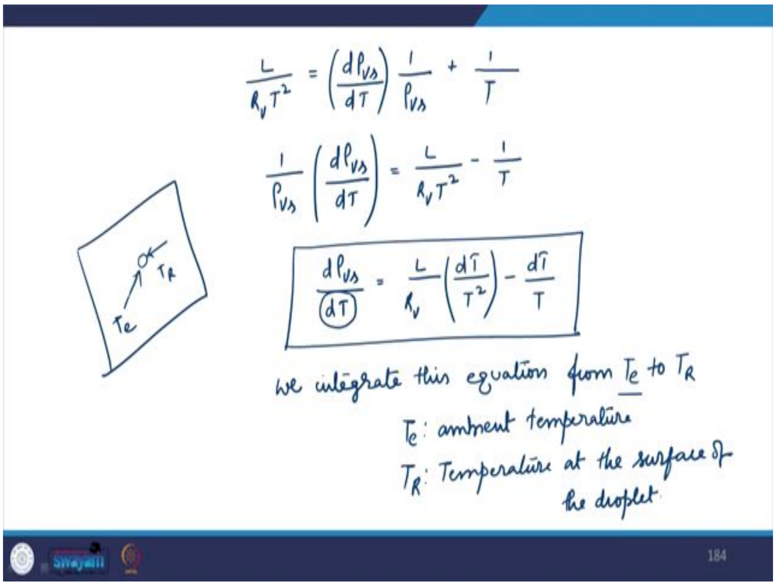

$$\frac{L}{R_v T^2} = \left(\frac{d\rho_{vs}}{dT}\right)\frac{1}{\rho_{vs}} + \frac{1}{T}$$

$$\frac{1}{\rho_{vs}}\left(\frac{d\rho_{vs}}{dT}\right) = \frac{L}{R_v T^2} - \frac{1}{T}$$

$$\boxed{\frac{d\rho_{vs}}{dT} = \frac{L}{R_v}\left(\frac{dT}{T^2}\right) - \frac{dT}{T}}$$

we integrate this equation from $\underline{T_e}$ to T_R

T_e: ambient temperature

T_R: Temperature at the surface of the droplet

So, you can rearrange this expression for simplicity as L by R v T square is equals to d rho v s by d T into 1 by rho vs plus 1 by T or 1 by rho vs d rho vs by d T is equals to L by R v Tsquare minus 1 by T right or d rho v s by d T is L by R v L by R v times d T over T square minus d T by T ok.

Let us say, we keep this equation we are going to use . So, let us see what we have got, we have got the rate of change of vapor density at saturation with respect to the temperature the change of vapor density at saturation with respect to the temperature is equal to L is the latent heat that is released when condensation happens, R v is the specific gas constant d T is the change of temperature, T is the temperature itself minus d T by T.

So, where have we started we have started at this point. So, we have started with this basic equation which gives you, how the saturation vapor pressure will change as a function of change in the pressure or change in the density with this right. So, we have used this equation and we have rearranged this equation in just so that we have vapor pressure on the left hand side and we have the density on the right hand side right.

So, in addition to this equation a, we said that the vapor the water vapor obeys the ideal gas law. So, we taken the ideal gas law in terms of the specific gas constant we have we have taken a derivative we have equated these two terms and we equated these two terms to obtain the rate of change of saturation vapor density with respect to the temperature right.

Now, let us say we integrate this equation now we integrate this equation from a temperature. So, this is with respect to temperature right. So, this is the denominator or the variable with respect to which the change is recorded is the temperature. So obviously, the integration has to be carried over temperature from let us say we call a temperature T e to T R

So, what is T e where T e is the ambient temperature, T R is the temperature at the surface of the droplet. So, we have a picture in which you have an ambience let us consider is a single dropper for the case of simplicity. So, the temperature of the ambience is T e the temperature at the surface of the droplet is T R. So, we are integrating from this to that right.

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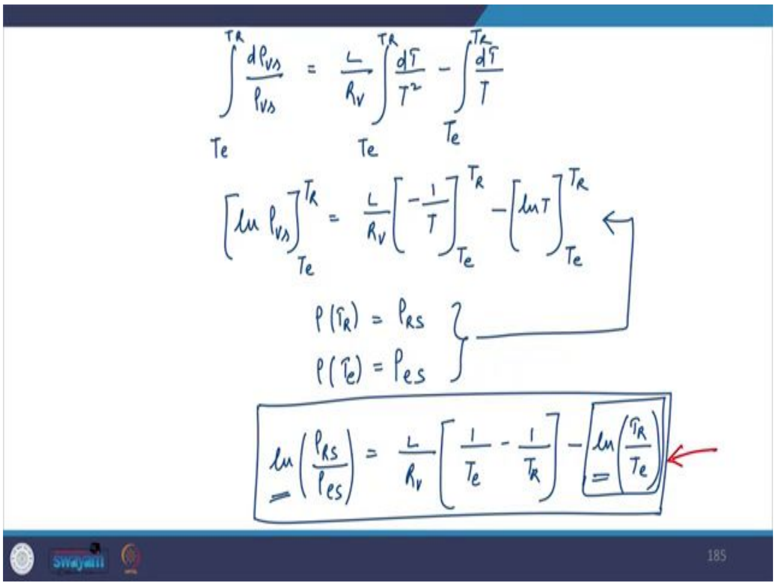

$$\int_{T_e}^{T_R} \frac{d\rho_{vs}}{\rho_{vs}} = \frac{L}{R_v}\int_{T_e}^{T_R} \frac{dT}{T^2} - \int_{T_e}^{T_R} \frac{dT}{T}$$

$$\Big[\ln \rho_{vs}\Big]_{T_e}^{T_R} = \frac{L}{R_v}\Big[-\frac{1}{T}\Big]_{T_e}^{T_R} - \Big[\ln T\Big]_{T_e}^{T_R}$$

$$\left.\begin{aligned}\rho(T_R) &= \rho_{RS}\\ \rho(T_e) &= \rho_{es}\end{aligned}\right\}$$

$$\boxed{\ln\left(\frac{\rho_{RS}}{\rho_{es}}\right) = \frac{L}{R_v}\left[\frac{1}{T_e} - \frac{1}{T_R}\right] - \boxed{\ln\left(\frac{T_R}{T_e}\right)}}$$

Now, let us say if you do that. So, that will be integral d rho vs by rho vs, this is equals to L by R v integral d T over T square minus integral d T over T. So, the integration limit is so it has to be converted from temperature to density T e to T R or let us say ln rho vs from T e to T R is equal to L by R v minus 1 by T between T e to T R minus ln T from T e to T R right.

Now, let us call the density at the temperature or the droplet as rho R s saturation at the radius and density at T e as rho es we are going to use this into this equation then we will rewrite it as ln of rho R s now it is obviously at saturation but where is it, the density is now being calculated at the radius of the droplet where the temperature is identified to be T R that is I am conveniently calling as rho R s by rho es. The density of the ambience at saturation right is equals to L by R v times 1 by T e minus 1 by T R minus ln T R by T e ok.

Let us say, we consider this as a intermediate result which we have to use right. So, what have we done now we have got an expression for the rate of change of density of vapor at saturation in terms of changes of temperature and the temperature and the latent heat and the specific gas constant. We are in a position that we have integrated that, to obtain parameters with respect to two points let us say one in the ambience and one on the droplet itself.

Now, when will diffusion happen diffusion will happen when there is a density gradient of course. And when the water molecules which are present in the vapor phase stick to the droplet and the droplet will grow right. So, let us say we will take this equation forward right. Now we have two long terms appearing here right.

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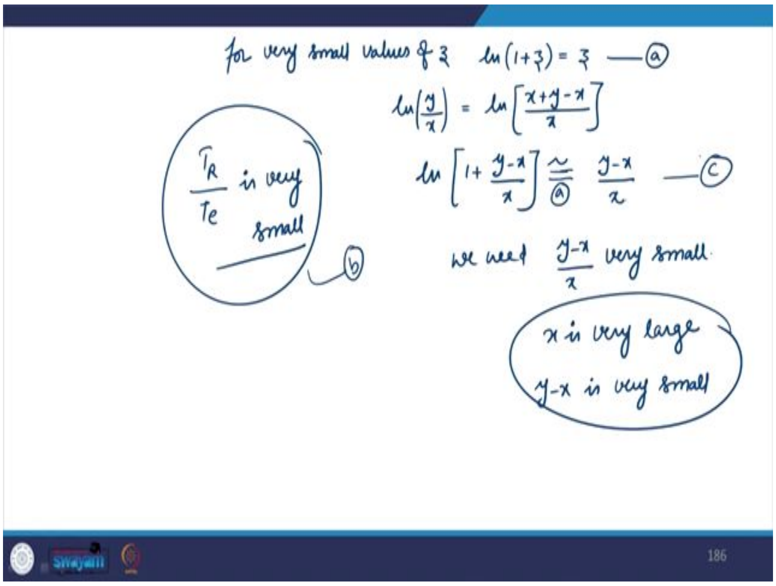

for very small values of z $\ln(1+z) = z$ — ⓐ

$$\ln\left(\frac{y}{x}\right) = \ln\left[\frac{x+y-x}{x}\right]$$

$$\ln\left[1+\frac{y-x}{x}\right] \underset{ⓐ}{\simeq} \frac{y-x}{x} \quad — ⓒ$$

we need $\frac{y-x}{x}$ very small.

x is very large

$y-x$ is very small

$\frac{T_R}{T_e}$ is very small — ⓑ

So, we will; we will use an approximation let say for what is the nature of approximation for very small values of z, z is just a variable ok. We will write ln of 1 plus z as equivalent to z right.

Now in this case, so let us say we have ln of y by x. So, we can write it as ln of x plus y minus x divided by x or ln of 1 plus y minus x by x according to this according to this let us say let us call it as a again is due to a is equivalent to y minus x by x.

Now, for this to happen we have to we need for this approximation to work we need y minus x by x very small right, y minus x divided by x becomes very small when x is very large and or y minus x is itself is very small these two are the cases right.

Now,. So, why are we discussing this in the first place we are discussing this because we want to address how we can simplify this term ln T e by T R. So, for this approximation to work out we will simply have to demand that T R by T e is very small.

So, let us apply this apply this and apply this let us say we call this as b and let us say we call this as c.

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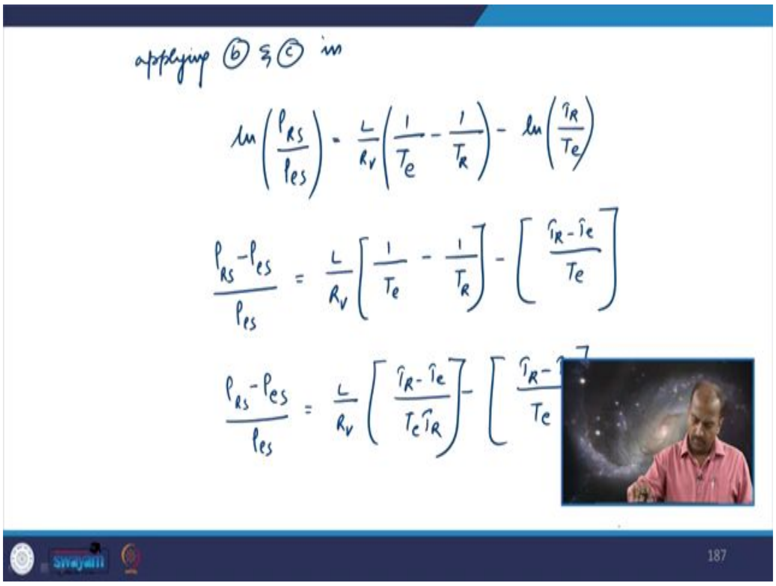

applying ⓑ & ⓒ in

$$\ln\left(\frac{\rho_{RS}}{\rho_{es}}\right) = \frac{L}{R_v}\left(\frac{1}{T_e} - \frac{1}{T_R}\right) - \ln\left(\frac{T_R}{T_e}\right)$$

$$\frac{\rho_{RS} - \rho_{es}}{\rho_{es}} = \frac{L}{R_v}\left[\frac{1}{T_e} - \frac{1}{T_R}\right] - \left[\frac{T_R - T_e}{T_e}\right]$$

$$\frac{\rho_{RS} - \rho_{es}}{\rho_{es}} = \frac{L}{R_v}\left(\frac{T_R - T_e}{T_e T_R}\right) - \left[\frac{T_R -}{T_e}\right.$$

Now, applying b and c in ln of rho R s by rho es it is equals to L by R v into 1 by T e minus 1 by T R minus ln T R by T ok. When I apply this we will get rho R s minus rho es divided by rho es is equals to L by R v into 1 by T e minus 1 by T R minus T R minus T e divided by T e.

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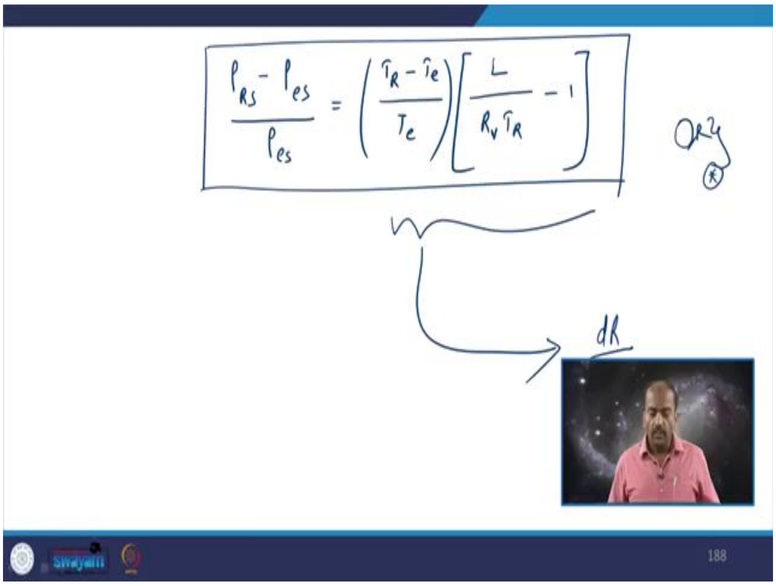

$$\boxed{\frac{\rho_{Rs}-\rho_{es}}{\rho_{es}} = \left(\frac{T_R - T_e}{T_e}\right)\left[\frac{L}{R_v T_R} - 1\right]}$$

Simplifying this algebraically, rho R s minus rho es divided by rho es is equals to L by R v into T R minus T e divided by T e T R minus T R minus T e divided by T e or rho R s minus rho es divided by rho e s is equals to T R minus T e divided by T e pulling this out I will write L by R v T R minus 1.

So, this is how this vapor pressure at saturation changes. So, what this expression tells you is how the vapor pressure at the radius or the difference of vapor pressure between the droplet surface and the ambience let us say this is the ambience and this is the droplet surface how did the difference of vapor pressure is with respect to the ambience how large is it or how small is it how it varies is given by the variation of temperature at the radius and the ambience multiplied by a term which is the latent heat divided by the specific gas constant of water vapor T R minus 1 right.

So, we will stop here. So, we will continue this and we will try to see how we can get from here do an expression for the rate of change of droplet radius with respect to the temperature right. So, we will stop here. So, we will continue this discussion this mathematical discussion in the next class. So, our objective is still to find an expression for the radial growth of a droplet of a cloud droplet.