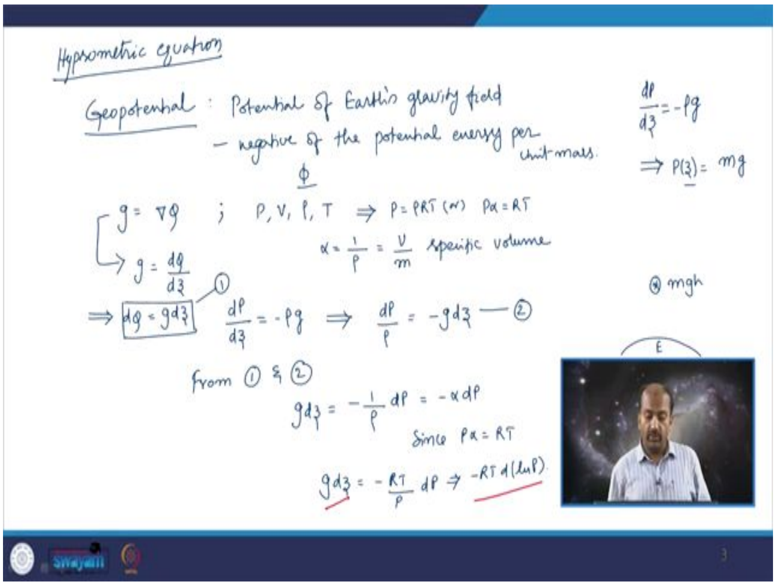
Introduction to Atmosphere and Space Science
Prof. M. V. Sunil Krishna
Department of Physics
Indian Institute of Technology, Roorkee

Lecture – 16
Hypsometric Equation

Hello dear students. So, in today's class in continuation with the last lecture, we have derived the Hydrostatic Equation and we have also understood the hydrostatic equilibrium, which is simply dP over dz is minus rho g, right.

(Refer Slide Time: 00:42)



So, this is the equilibrium; that means, the force the pressure gradient force is balanced by the earth's gravitational pull. A consequence of this equation is that, the pressure at any height z, is simply the mass of atmosphere multiplied by the gravity; that means, the weight of the atmosphere that is existing above this particular height is the pressure, we know this very well, right.

Now, in today's class we will try to use this hydrostatic equation and derive what is called as the hypsometric equation. Now here we define in this class, we will define what is called as the geo potential geo. What is geo potential? Geo potential is the potential of the earth's gravity field; geo potential is defined as the potential of earth's gravity field. Also, it is negative of the potential energy per unit mass; that means, it is negative of the potential energy per unit mass per unit mass very well.

So, if you know the geo potential, if you indicate geo potential with phi; you can write the gravity as g is equals to del phi. So, so as to write g without a negative, you define the geo potential as the negative of the potential energy per unit mass. So, it can also be say that said that, if you want to rise some object; if this is the earth surface and if you want to rise an object to this particular height, the amount of work done, that is, to rise this object against the gravity is generally referred to as the geo potential. So, geo potentially is the negative of the potential energy per unit mass. So, if you rise an object at this to this height, the potential energy at this height is m g h, right.

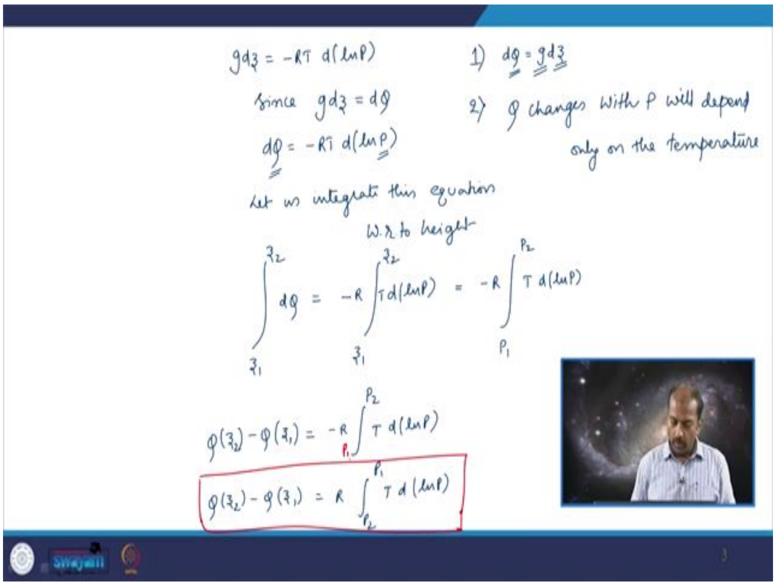
So, the potential energy per unit mass is negative of this, is generally called as the geo potential. So, you can write g is equals to del phi. So, any given thermo dynamical system can always be represented in terms of its pressure, volume, density, and temperature. So, you can combine these four variables and write that, P is equals to rho R T or simply P alpha is equals to R T. We already defined alpha is 1 by rho, that is volume per unit mass, which is also called as the specific volume; volume per unit mass is called as the specific volume, right.

Now, let us bring the hydrostatic equation which is d P over d z is minus rho g, ok. Now we will try to rearrange these equations, so that we can try to get the height or the geo potential of at a particular height, ok. Let see how we get it; I mean what we will start rearranging this one by one. So, as a consequence of this hydrostatic equation we can write that, d P over rho is equals to minus g d z.

So, just rearranging this, let us call this equation as let us say as call this equation as 2; g is equal to minus del g is equal to del phi or g is equals to or g equals to let say d phi over d z just across one dimension, the vertical dimension, that means, d phi is written as g d z, no. Before we get confused what I have done is; let us call this equation as 1, this equation as 2. So, we have introduced geo potential as the negative of the potential energy per unit mass, right. So, if phi is defined in such a way, the gravitational pull, the acceleration due to gravity g is the del phi, the gradient of the this potential is just the gravity. So, g in one dimension across the height is d phi by dz; that means, d phi is written as g dz.

So, this is one relation that we will use and the second relation is straightforward from the hydrostatic equation. The hydrostatic equation; I have got the density in the denominator, and on the right hand side you have dg dz, right. So, we now we can equate 1 and 2; from 1 and 2 we can simply write, g dz is equals to minus 1 by rho d P which is equals to minus alpha d P, right. Since P alpha is equals to R T, we can write g d z is equals to minus R T by P into d P which is equals to minus R T d of lon P.

(Refer Slide Time: 06:40)



We can take this relation to the next slide, that is to say g d z is equals to minus R T d of lon P, right.

Since g dz is simply d phi, we can write that d phi is minus R T times d lon P. So, what we can say is that, see here one very important relation is d phi is g dz; that means, the rate at which the potential changes with respect to the height is the gravity field, right. Then the second important consequence is that, the rate at which phi changes with P, right. So, this is what you see; the potential, the geo potential; rate at which the geo potential changes at any given surface or at any given pressure will depend only on the temperature, it is straight forward.

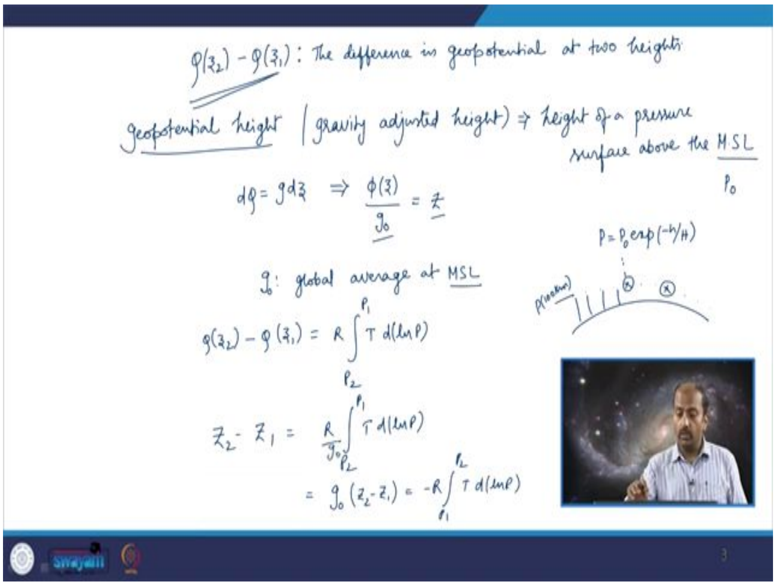
I mean capital R is a constant. So, the rate at which the geo potential changes with respect to the pressure will depend on the temperature. And the rate at which the potential changes with respect to the height is equal to the gravity, right. So, these two are very important conclusions, let say let us call this equation. Now; that means, let us now integrate this equation from with respect to height. Let us integrate this equation with respect to height. What do we get? Let us say the integration limits are z 1 to z 2 d phi is equals to minus R T integral again z 1 to z 2 d of lan P.

Now, temperature is not a constant; that means, as the height changes, temperature should not be treated as a constant. So, temperature should actually be in inside the integral, right. So, R times T d lon P, or we can also write this as this equation in terms in the limits of pressure; let us say, if you take the pressure at a height z 1 to be equal to P 1 and pressure at height z 2 equal to p 2, so it should be T times d of lan P, right. Or we can write phi of z 2 minus phi of z 1 is equals to minus R times integral P 1 to P 2 T times d lon P. If you get rid of the minus sign, phi of z 2 minus phi of z 1 is equals to R times integral P 2 to P 1 T d of lon P, right.

Now, this equation is called as the hypsometric equation; that means, so what does this equation tell you? Where did we start and what does this equation tell you about the height and the operation, right? So, we started by defining the geo potential which is the negative of the potential energy per unit mass. And if you define geo potential you bring in the dependence of phi over z which is equal to the g, right. Then you combine this equation with the state equation and you define g dz in terms of, in the units of temperature and pressure.

So, g dz is being equal to d phi; you can simply say that, the rate at which the potential changes with respect to the pressure will depend only on the temperature. As a consequence, if you integrate this between the suitable limits of pressure; what you will realize is that, the geo potential at a particular height the is equal to; geo potential at a particular height let us say the difference between the geo potential at two different heights is equal to the integral of lon of the pressure between P 2 and P 1.

(Refer Slide Time: 11:30)



So, what we can say is that, phi of z 2 minus z 1. So, here the left-hand side is phi z 2 minus phi z 1. So, what does this tell you? This tells you the thickness of the geo potential; I mean, if phi of z 2 is a particular height and phi of z 1 is a particular geo potential, this tells you the difference in the potential, the difference in the difference in geo potential at two heights.

So, here the idea is not about the pressure rather this is still in the units of geo potential, right; but at two different heights; so, the rate at which the geo potential changes is the gravity, right. So, we should always remember that. Now if it is the case; then this equation is called as the geo potential. Now at this point we can introduce what is called as; we have seen what is geo potential, we will now introduce what is called as the geo potential height. So, what is geo potential height?

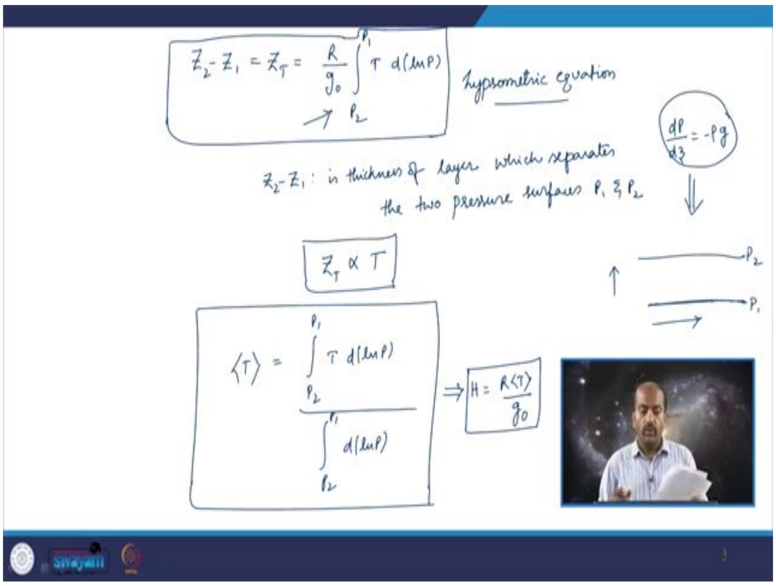
Geo potential height is the height of a particular value of potential, or the height which is normal the height of a particular pressure surface which is normalized with respect to the gravity at the sea surface, ok. Or this geo potential height is called is also called as a gravity adjusted height, gravity adjusted height. Or we can also say that, the geo potential height is the height of a pressure surface above the mean sea level. Now, the mean sea level is assumed to be at a pressure P naught.

Now and the most important thing is the, so here if you take the height of a pressure surface. So, now, you are assuming that the pressure is changing only vertically upwards; yes yours now you are saying that, the pressure is changing with respect to height only vertically upwards at so, at any given height. So, at any given height across this across the atmosphere, at any given height the pressure will always be the same any point. And if you go vertically upwards so only then the pressure will decrease, right. That means, if you normalize this, the height of this particular pressure surface; let us say if it is a P at 100 kilometers, if the pressure at 100 kilometers, if we normalize it with respect to the mean sea level, then you call that particular height as a geo potential height,.

So, now what is geo? So, d phi is defined as the as g dz. So, we can integrate this and we can write phi at a height z by g naught is equals to z. So, the height is normalized now with respect to g naught; g naught is the gravity, gravitational pull at the surface, ok. Or we can say that g naught is the global average at the mean sea level. Now using this in the hypsometric equation; so, what we have is, phi at z 2 minus phi at z 1 is equals to R times integral P 2 to P 1 times T of d lon P.

Now, we can write these equations and say that, the geo potential height z 2 minus geo potential height z 1 is equals to R by g naught times integral P 2 to P 1 T times d lon P or more simply is equals to g naught times z 2 minus z 1 is equals to minus R times P 1 to P 2 T times d lon P.

(Refer Slide Time: 16:08)



We will take it to the next slide; or we will say that z 2 minus z 1 is equals to z T is equals to R by g naught integral P 2 to P 1 T d lon P. So, this equation is called as the hypsometric equation. So, this equation is called as the, right very well.

Now, what is z 2 minus z 1? Z 2 minus Z 1 is thickness of layer of atmosphere which separates the two pressure surfaces at P 1 and P 2 that is it. So, now, what you can see is that, the thickness of this particular layer of atmosphere seems to be proportional to temperature, right. So, what we can also define a; So, now, you have. So, essentially the point that you should understand is; if you have two pressure surfaces.

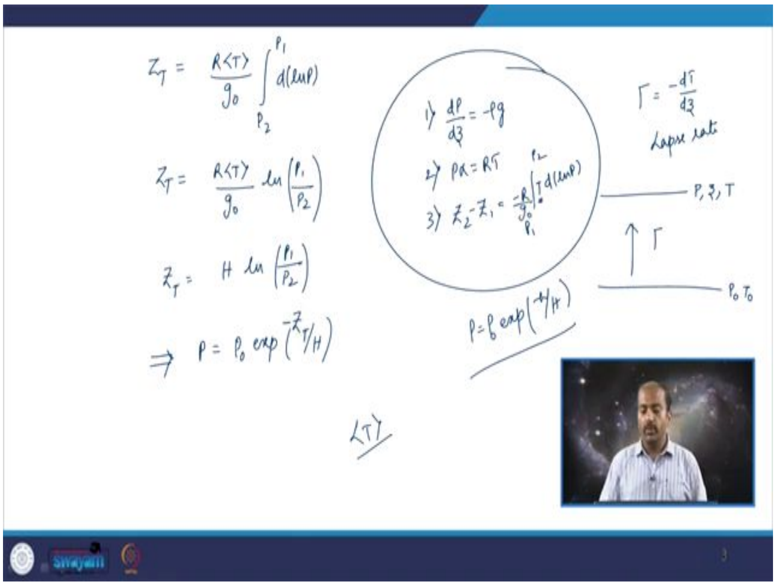
So, this surface I mean this if you define this to be a surface, any point on this surface is isobaric in this, this is the isobaric surface; that means, any point on this line is at the same pressure. So, there are no variations in the pressure as long as you are moving horizontally; but there is a variation in pressure when you move vertically, right. So, this pressure surface is defined at a pressure P 1 and P 2.

Now, if you have an atmosphere which is hydrostatic in nature; that means, if this height, if the pressure is varying as per this balance. If you have such an atmosphere, the hydrostatic balance imposes a condition on the pressure saying that; if you have such an atmosphere, the thickness of the atmosphere between the pressures P 1 and P 2 will be given by this relation, right. So, and the thickness seems to be dependent on the temperature. So, naturally if you have more temperature the, it will lead to the expansion of fair and as a result the thickness of this layer will increase, as it is evident from the equation itself, right.

Now, with this equation we can now define the average temperature or the mean temperature of a layer of atmosphere that is confined between two pressure surfaces P 1 and P 2, as the average temperature is integral P 2 to P 1 d lon P divided by integral P 2 to P 1 d lon p right. So, this is the consequence of they having this hypsometric equation; that means, if you have this equation, given any two pressure surfaces, you can always calculate, how spatially they are separate, how with how much or what is the special separation between these two pressure surfaces, right.

So, with this mean temperature; now you can also define. So, using this mean temperature now, within this column of atmosphere; you can define the scale height, capital H is equals to R T by g naught, this is R T by g naught, ok. So, the thickness of a layer of atmosphere separated by two pressure surfaces is given by like hypsometric equation, right.

(Refer Slide Time: 20:03)



Now, if you substitute this scale height into the Z T equation, Z T is equals T now. Now you since you have defined the average temperature, the mean temperature; the mean temperature of the layer is not going to change with respect to height. You have already taken average; I mean the temperature between these two pressure surfaces varies with respect to height.

So, this variation is brought in by taking an average. So, it comes now out of the integral and you have P 2 to P 1 d of lon P. So, now, Z T is equals to R times mean temperature by g naught times lon of P 1 by P 2 or Z T is simply the scale height times lon of P 1 over P 2. So, this means I mean we are. So, as a result you can write, you can summarize this as P is equals to P naught into exponential to minus Z T by H, right.

So, it naturally follows I mean this the, so the approximation of the hydrostatic equilibrium along with the geo potential and the. So, we have combined these three equations. So, d P over d z is equals to minus rho g; the second equation that we have been able to combine is P alpha is equal to R T; the third equation is Z 2 minus Z 1 is equals to minus R times by g naught temperature of into lon P 1 to P 2. So, combining these three equations, we have been able to realize that the pressure will indeed still valid to have this exponential behavior, right; so as long as the temperature is not changing with respect to height.

So, with this temperature coming out of this integral only allows you to have this relation, right. So, if t changes with respect to height, then it becomes complicated. So, now, that is the reason that you define an average temperature, right. Now using this hypsometric equation, we can get a very important result, very important derivation which you can. So, the idea is let us say; if you take the surface at the mean sea level or the surface at zero kilometers height; the pressure; if it is P naught and if the temperature is T naught. What we can do is?

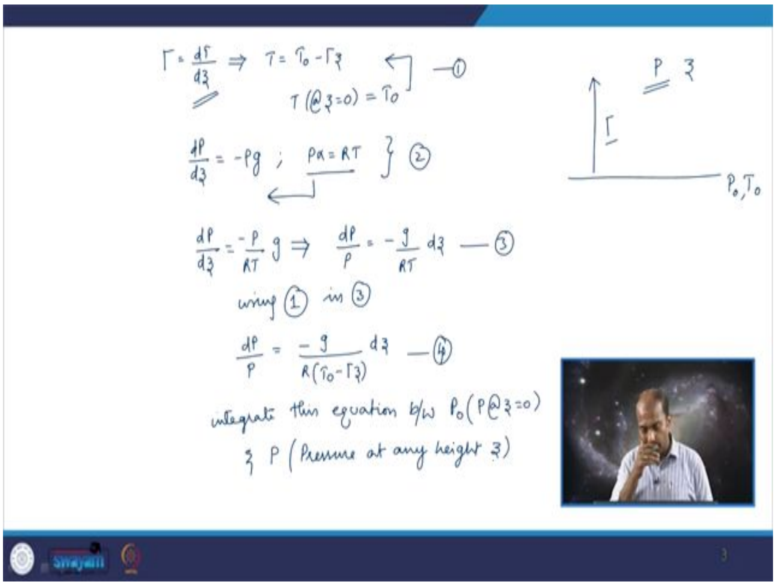
We can calculate, the pressure at any height let us say z and the temperature of that particular height T; if the temperature of the atmosphere is varying with a uniform lapse rate of gamma.

So, what is gamma? Gamma is minus d T over d z. What is this called as? This is called as the lapse rate of atmosphere; that means, it gives you the rate at which temperature decreases with height, let say we will do that. So, just to take quick look back, we started with the understanding of the basic idea of geo potential, how do you define geo potential. And using geo potential, we define g to be equal to the gradient of potential, we reduce it to one dimension; then we combined this with gas equation, then we got g d z is equals to minus R T d lon P.

Then we integrated this into in due to the suitable limits of height, we go to a relation in terms of the difference of geo potentials at two different heights. We converted these geo potentials into geo potential heights; thereby we realize that the geo potential heights or the thickness of an atmospheric layer in terms of its geo potential height boundaries was dependent on the temperature of this particular layer.

Then we defined what is called as the average temperature of this layer and this allowed us to define scale height in terms of average temperature times the gas constant divided by the gravitational pull at the surface or the normalized gravitational pull at the mean sea level. Then we realize that this could this equation could very well lead us to the very famous variation of pressure with respect to height.

(Refer Slide Time: 24:51)



Now, what we are going to do now is, we will derive a simple relation for the height of a pressure surface. I mean if you know the pressure to be some milli bar ok, with 300 milli bar, 200 milli bar with the hydrostatic balance in place; that means, there are no variations in pressure horizontally, where the pressure decreases exponentially only with respect to height.

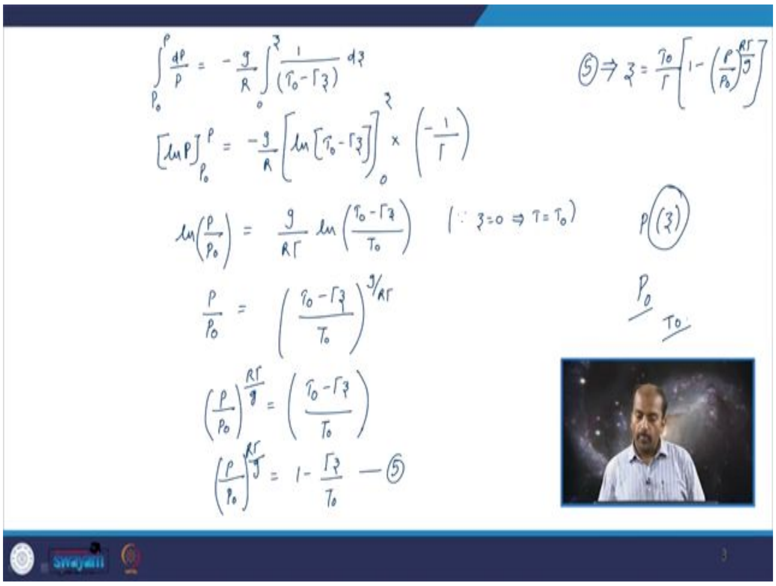
So, we will define what will be the height of a given pressure P; if you know the pressure at the surface to be P naught and the temperature at the surface to be T naught and this atmosphere the atmospheric temperature with respect to height decreases uniformly with the lapse rate gamma ok, with a uniform lapse rate or the constant lapse rate.

So, we can define gamma is equals to d T over d z; that means, temperature at any given height is simply T naught minus gamma z, right. So, what is what it means is that; if the temperature t at z is equals to 0 is defined to be T naught, then temperature at any given height is simply this; T naught minus gamma z. And what is gamma? Gamma is d T over d z and we have assumed that this rate is uniform; I mean it is not there are no sudden changes in the atmospheric temperature, ok. Now the other equation that we will use is d P over d z is equals to minus rho g and P alpha is equals to R T, right.

Now, we will just rearrange this equation and try to use this equation into this equation, right. And so, d P over d z is equals to; in the place of rho, we can write P by R T minus g; which can be written as, d P over P is equals to minus g by R T d z. That means, now if you substitute let say let us call this equation 1 using equation 1 and let us call this set of equations as 2 and this equation as 3. Using 1 in 3, we can say that; d P over P is equals to minus g times R times T naught minus gamma z d z.

Now, let us integrate this equation, integrate let say this equation is 4. Integrate this equation between P naught that is P at z is equal to 0 and P. What is P? Pressure at any given height, at any height; what is it? It is z, right.

(Refer Slide Time: 28:05)

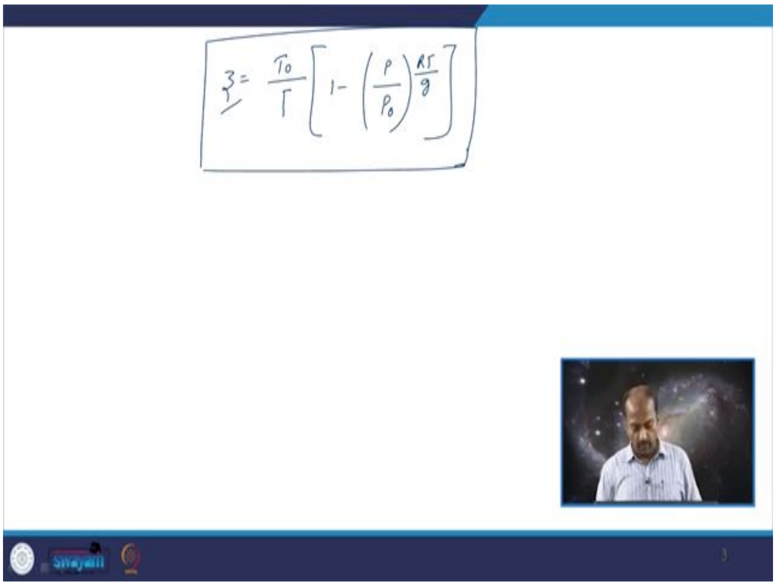


Let us integrate this equation which will look like integral P naught to P; integral limits in terms of pressure is equals to minus g by R, the constants come out and 0 to z to 1 by T naught minus gamma z and it is d z, right.

So; that means, lon of P between P naught to P is equals to minus g by R into lon of T naught minus gamma z within 0 to z into minus 1 by gamma, integrating this 1 by T naught minus gamma z. So; that means, lon of P by P naught is equals to g by R gamma times lon T naught minus gamma z divided by T naught; because since at z is equals to 0 temperature is simply equals to T naught, right.

Now, if you simply rearrange this equation, we can write P taking the logarithm to the right-hand side T naught minus gamma z divided by T naught times g by R gamma. Or P by P naught R gamma by g is equals to T naught minus gamma z divided by T naught, right. P over P naught; I am just rearranging the equations, so that I will simply get height at the end, right. So, let us call this equation as 5; and equation number 5 simply leads to writing height z as T naught by gamma into 1 minus P by P naught into R gamma by g.

(Refer Slide Time: 31:00)



Let us simply write it for the sake of discussion; so, T naught by gamma to 1 minus P over P naught R gamma by g, right.

Now, so what have we derived? I mean we have derived an expression for the height, where did we start. We started with the hypsometric equation. So, hypsometric equation gives you the thickness of atmosphere that is separated between two pressure surfaces, right. Now if I know the pressure at the surface to be P naught and the temperature at the surface to be T naught; and pressure of course, decreases exponentially with respect to height, temperature can decrease linearly. If I take the linear variation or uniform variation of temperature with respect to height with a constant lapse rate gamma; then combining the hydrostatic equation and the gas equation, I got this; that the pressure changes with respect to the height, ok.

I got this in terms of this equation then I integrated this equation between the suitable limits. So, my objective is to find; if I know the pressure at a particular height as P with respect to the pressure at the surface as P naught, then I want to find out this particular height in terms of P naught and T naught, ok. So, at the end what I have derived is the height of a pressure surface z is T naught by gamma. What is gamma? Gamma is the lapse rate; T naught is the temperature at the surface P by P naught into R gamma by g, right. So, this is a very important derivation which we can use in combination with the hypsometric equation, right.

So, this is where we stop. So, in the next class we will try to continue the discussion by using the hypsometric equation and hydrostatic equilibrium to understand the thermo dynamical aspects of the atmosphere, ok.