Introduction to Atmosphere and Space Science
Prof. M. V. Sunil Krishna
Department of Physics
Indian Institute of Technology, Roorkee

Lecture – 15
Hydrostatic Equation

Hello dear students. So, in our discussions on Fundamentals of Atmospheric Physics, we have seen how the pressure variation occurs with a with respect to height and how density varies with respect to height? right. So, in today's class we will try to we will try to use some aspects of pressure and density variations and we will also try to use some aspects of forces, surface forces and body forces that we have learnt to derive the very useful or very fundamental equation which is the hydrostatic equation.

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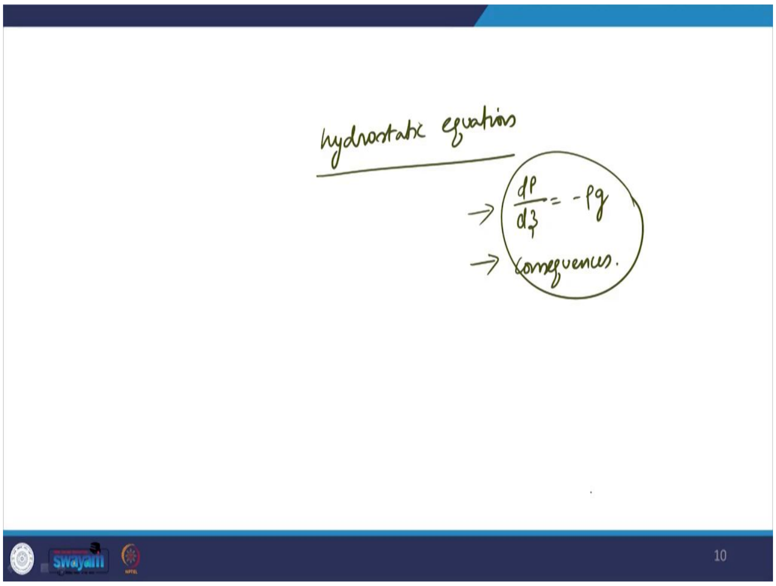

So, we will try to derive the mathematical derivation of the hydrostatic equation which looks something like this, dP over dz is minus rho g, ok. We will try to derive this and then we will try to see what are the consequences of this equation. In subsequent classes, we will also try to see what is the hypsometric equation and what does it convey about and at the about the atmosphere. So, we will have to we will try to use this many number of times in our discussions.

So this is the most fundamental equation in the atmospheric science. So, we also, many times we say that the atmosphere is a hydrous is in hydrostatic balance. We will try to see what does it mean, right.

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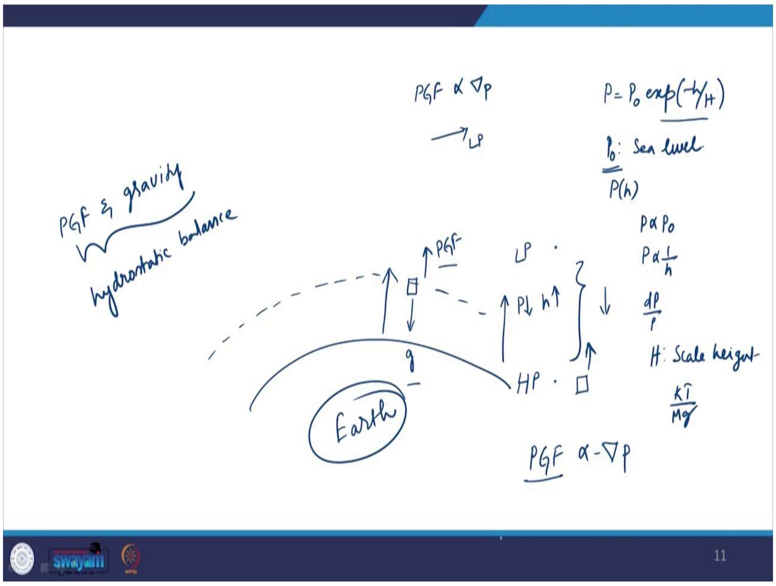

So, let us say; we have seen that let us say; we say that the pressure varies with respect to height like this, P0 exponential minus h by H. What does this mean? This means that the pressure at any given height, so P0 can be taken as the pressure at let us say sea level, ok, the standard pressure, 1000 millibar, right. So, pressure at any height P will depend on P0 the initial value and it means that P will be proportional to the initial pressure and P will decrease with respect to height. Let us say P will be inversely proportional to the height.

So, if you put up all these things you will realize the dP over P, the rate at which pressure changes with respect to the surface pressure will be exponential with respect to height. So, what is capital H? It is the scale height. We have already derived; we already discussed all this aspect. So, scale height is let us say scale height is kT by Mg.

So, what the scale height tells you? Scale height tells you that by what distance you have to travel, so that in the vertically up in the atmosphere so that the pressure at that particular height is 1 by eth of the original value that you have started with, right. So, there is a fundamental definition of the scale height in the perspective of pressure. If you want to convert it for density then density has to be decay exponentially to 1 by eth of its original value after traveling a particular distance and this distance is going to be called as the scale height, right.

Most importantly, if you consider the surface of the earth let us say it will be like this, what do you realize? You realize that if you go up the pressure will decrease. So, now in this direction height is increasing, pressure is decreasing. That means, we can imagine that there is a low pressure there is a high pressure here and there is a low pressure here, right.

Now, from our discussions on various different types of forces and the types of forces which are relevant for atmospheric science, we have seen that there is a force which is called as a pressure gradient force. We have seen that the magnitude of this pressure gradient force will depend on not on the pressure, but depend on the gradient of the pressure. So, force will be proportional to the gradient of the pressure. So, there is a difference in the pressures from here to here, right. So, the direction of the force and there is also a minus, right.

So, what this pressure gradient force tells you is that, pressure gradient force will be proportional to the difference of pressure between these two points, let us say we call this is point 1, and this point; the difference of pressures the forces' magnitude will depend on the difference or the magnitude of difference between these two points in terms of pressure and it will act in the opposite direction. So, naturally if you imagine an air parcel here at this point, the natural moment of this air parcel is to go towards the low pressure, right, but the force is in this direction; the moment is in this direction, right.

So, that means, that pressure gradient force; if the pressure gradient force alone is at is existing in this picture this is the earth, right, for reference, this is the earth. So, if the pressure gradient force is alone is existing in this picture, it would make this air parcel at high pressure to go towards the loop; that means, the entire atmosphere should just escape away from the earth, but it does not happen like that.

What happens in reality is that this air parcel that you are trying to push away to the low pressure by the pressure gradient force is again pulled by the gravitational pull. So, PGF is what trying to take the air parcel away into the low pressures and the gravity is the restoring force which is trying to bring this air parcel towards the ground. So, this balance of pressure gradient force and the gravity.

So, the balance of pressure gradient force and gravity is generally called as the is the hydrostatic balance, hydrostatic balance or the equilibrium that results when you strike this balance is called as the hydrostatic equilibrium. So, we assume that the atmosphere in general conditions is always hydrostatic in nature. So, that means, at any given height you imagine any height at any given height the pressure gradient force is always balanced by the gravitational pull.

One more very important thing is pressure gradient force is proportional to the magnitude of pressure difference, so pressure gradient force will be towards the low pressure, the moment of the air parcel should be towards the low pressure. So, the equilibrium that results from the hydrostatic balance is the hydrostatic equilibrium. Now, let us try to derive an expression for this hydrostatic equilibrium. How the equation looks like, I mean why should we have this in a differential form?

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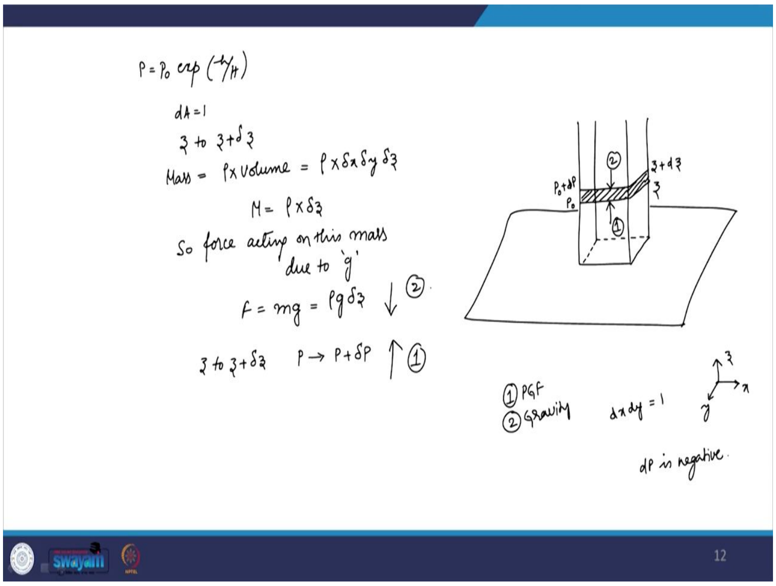

Let us say, let us consider, let us say let us say this is the column of atmosphere that you have taken for reference, ok. So, let us consider a slab of very small thickness here like this, in this column of atmosphere, let us consider this slab. Just to be able to identify this slab we shade it, ok. Now, let us say the slab is; so the what this is x, this is y and this is z. Let us consider the slab the such that dx, dy is equals to 1. So, this is the z direction that is going up. So, we take this then this point to be at z and the point the topmost point of the slab to be z plus dz.

Naturally, the pressure at the lower surface is P naught and the pressure and the upper surface is P naught plus dP. But here the dP is negative. Why? The pressure is decreasing with respect to the height, right. So, there is a force that is because of the atmosphere that is existing below the slab and there is a force by the atmosphere that is existing above the slab on the top surface. So, there is a force from the bottom which is trying to push the slab and there is a force from the top which is trying to push this slab towards the ground.

Now, we can conveniently say that the pole the force that is there is marked as 1 is the pressure gradient force which is trying to push the slab towards the lower pressures. And the force 2, marked with 2 is the gravity which is trying to pull this slab; pull this slab towards the, towards the ground, right.

So, we know that pressure P is P naught exponential minus h by H, right. Now, we take the area to be unit, if dA is equals to 1. So, slab is defined from z to z plus delta z or dz. So, the mass will now become; mass is density times the volume which is equals to rho times delta x, delta y, delta z, right. So, mass area being one, mass now becomes rho times delta z, right. So, the force acting on the mass, acting on this mass due to the gravitational pull is F; is simply mg which is equals to rho g delta z. What is the direction of this force? This force is acting downwards.

Now, let us consider the vertical force due to the pressure, the other force, right. So, it is acting from z to z plus delta z and from a pressure P to P plus delta P, but it the only difference is that it is exactly acting in the opposite direction. So, this is let us say this is a force number 1 and it is force number 2. In that case, so we can just balance these two forces the force due to the pressure gradient and the force due to the gravitational pull.

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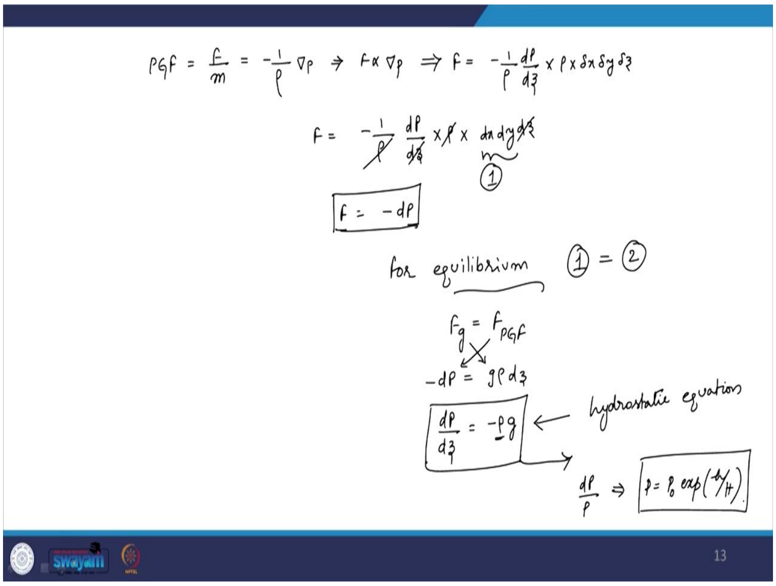

So, the pressure gradient force we have written is the force per unit mass is; we already derived this expression 1 by rho times del P. So, here P rho is the density, rho is the density, P is the pressure. Force per unit mass is proportional to the gradient in the density gradient in the pressure which implies force is simply proportional to del P or you can write the force as minus 1 by rho in one-dimension dP over dz times the mass is rho times delta x, delta y, delta z.

So, force is now minus 1 by rho dP over dz times rho times delta x, delta y, delta z. You can also write this delta x, delta y, delta z it as dx, dy, dz; dx, dy, dz. So, I can cancel this and this. So, we will write t is equals to minus dP because dx, dy being 1, right. So, force is now just the change of pressure between the surface the 1 and 2, the slab, the bottom side is the slab to the top side of the slab.

So, for the equilibrium for equilibrium to exist we expect the forces 1 to be equal to force 2, right. That is; we are trying to say that F g is F PGF. So, you can use the earlier expression minus dP is the pressure gradient force which is equal to g rho dz. So, this is the pressure gradient force and this the gravity, right. This the gravity, we have used this here minus g rho dz. So, this we can rearrange dP over dz is minus, right. So, this is called as the hydrostatic equation and or hydrostatic equilibrium.

So, we can see that we can use the kinetic gas equation, we can rearrange this equation; if you use density in terms of pressure you can realize that you will get a dP over P term which you will integrate and result in the form of P is equals to P0 exponential minus h by H. So, these two are the same. So, the rate at which the pressure changes with respect to height is equal to simply the product of density times the gravitational pull.

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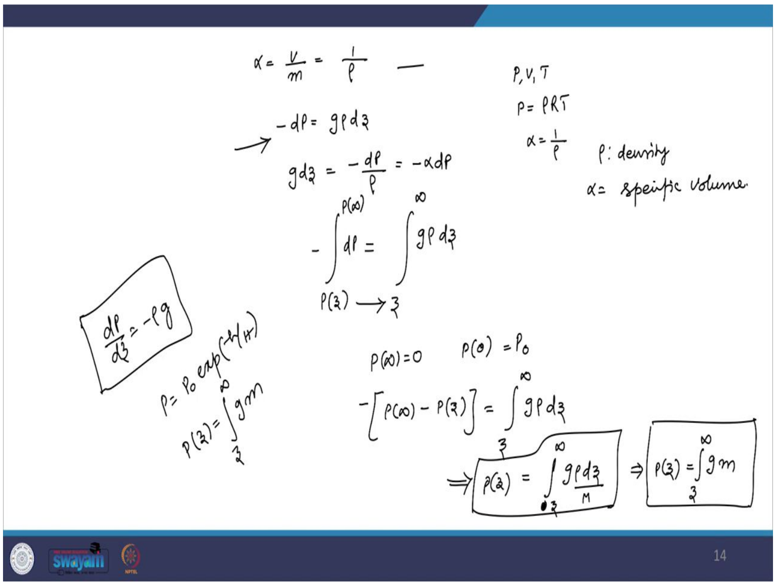

Now, let us say let us we use alpha as V by m or 1 by rho. So, we know that P in terms of P V T, we can write P is equals to rho RT or alpha is equal to 1 by rho, right. So, where rho is the density, rho is the density and alpha is called as the specific volume, right. So, you can write minus dP is equals to g rho dz or g dz equals to minus dP over rho which is equals to minus alpha dP using this. Integrating this equation; now between the pressures P0 to P at infinity; we are trying to imply trying to see the consequences of the hydrostatic equation over dP is equals to z, these two on the same limits, z to infinity g rho dz. So, we can say that the pressure at infinity is 0, right.

So, on pressure let us say at 0 let us say, 0 is P0 at z is equal to 0. So, we will write the integral as P minus infinity, P of z is equal to integral z to infinity g rho dz that implies pressure at a height z is 0 to z 0 to infinity g rho dz. So, we have taken the hydrostatic balance here; we have taken the hydrostatic balance. We have not rearranged this term, so that we get a derivative of P in terms of the density.

Then, we integrated this term to be able to see that the pressure at any given height z under hydrostatic equilibrium is equal to integral of 0 to infinity at that. So, this should this should actually be z because integral of z to infinity g times rho dz. Now, rho dz is the mass, we also we have already seen that.

So, we can write that P of z is simply integral z to infinity g m. What does this mean? This means that pressure at any given height head H or z in this particular case is essentially the weight of atmosphere that is existing above that particular point. So, pressure at any given height is the weight of atmosphere that exists above that particular point. If you take it to the if you take this point to be the surface it is a surface pressure, at any other point it is just the total weight of the atmosphere that is existing above it, right.

So, this is something about the hydrostatic balance. The hydrostatic balance is the equilibrium condition between the pressure gradient force and the gravitational pull. As long as these two are equal the there are no horizontal no there is no vertical moment in the atmosphere, as long as these two are striking a balance and this balance is called as the hydrostatic balance and the hydrostatic equation is the most fundamental equation in the atmospheric science; dP over dz is minus rho g, right.

And the most important consequence of this equation is that you write P is equal to P naught exponential minus h by H or you write that pressure at any height z is equal to integral z to infinity g m. So, pressure at any given height H is the total weight of atmosphere that is existing above that particular height, right. So, we can stop here. In the next class we will try to see what is the hypsometric equation and how hypsometric equation will allow us to calculate the difference of height between any two given pressure surfaces.