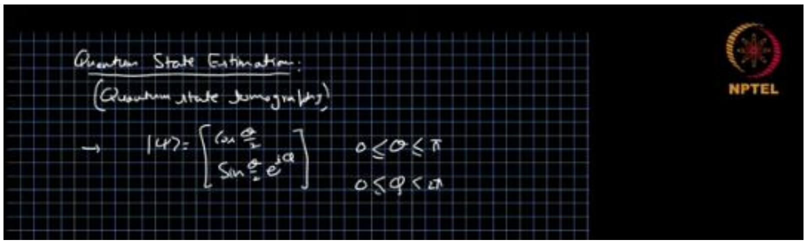
FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

Dr. Sandeep K. Goyal
Department of Physical Sciences
IISER Mohali
Week-03
Lecture-07

Qubits: State Tomography

In today's lecture we will be discussing about the quantum state estimation we are still in the qubit subspace so we will start with a state of a qubit and we will see how to estimate a state of this qubit. So, mathematically we know a state of a qubit $|\Psi\rangle$, the canonical representation we wrote last time which was $\cos(\theta/2)$ and $\sin(\theta/2)e^{i\phi}$. That's the most general representation for a state of a qubit $|\Psi\rangle$ where θ is between 0 and π and ϕ is between 0 and 2π. This is just to remove the degeneracy that we know two values of θ and ϕ should not result in the same state $|\Psi\rangle$. That's why this restriction is there.

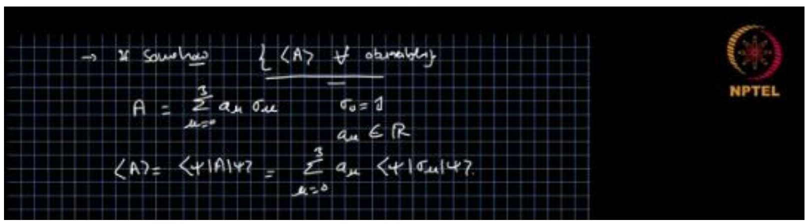
(Refer slide time: 00:49)

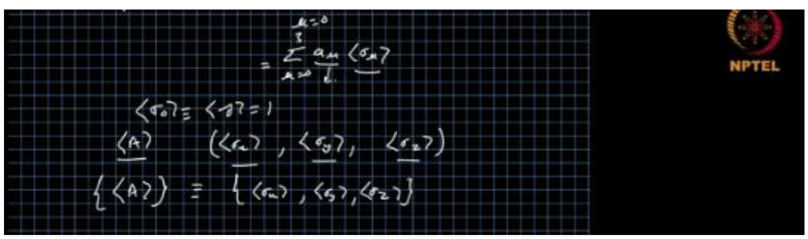


There will be evident or this will be clear later on when we discuss more about this state representation in the geometric form and in other forms it will be clearer why we are taking it in this restricted sub space. But this is also easy to find when we allow the full range of θ and full range of ϕ from 0 to 2π for both, then there will be two values of θ which will give us the same state so we don't want that because of our order phase. So, we start with this thing and now we see mathematically or the first postulate of quantum mechanics tells us that a state of a quantum system, mathematical structures, state of a quantum system should contain every possible measurable quantity in that. Okay, or we can say whatever we can measure in the lab it should be contained in this state $|\Psi\rangle$ or any representation of state we want to use, okay. So, we can also say that somehow if somehow, we can find the expectation value of all the observables, this set of the expectation values of all the possible observables of a qubit, then this stat should also be a valid representation for a state. Of course, you say it's infinite information because there are infinitely many observables.

So, it's the overkill. we have too much information we don't need so much information to represent a state this is actually neater way of writing it but this is also not wrong if it is possible to write. An observable is a Hermitian operator so we can write it as $\sum_{\mu=0}^{3} a_\mu \sigma_\mu$. where σ_0 is identity, σ_1 is σ_x, σ_2 is σ_y, σ_3 is σ_z and a_μ are real numbers for all the μ's. So, this is the more general representation of a Hermitian operator. We have discussed it earlier since expectation value is a linear operation. So, it will be, we can write it as $\sum_{\mu=0}^{3} a_\mu \langle\Psi|\sigma_\mu|\Psi\rangle$. So, we are getting a_μ and expectation value of σ_μ. Expectation value of σ_0, that is identity is just one because our state is normalized.

(Refer slide time: 03:10)





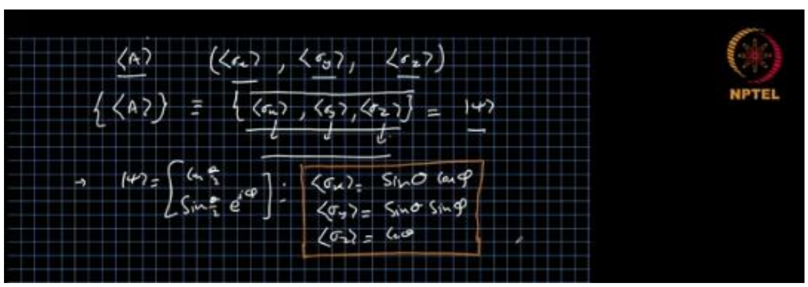
Okay, so, it means we are left with the expectation value of A which depends on σ_x expectation value, σ_y expectation value and σ_z expectation value, this set and of course it's eigenvectors or the a_μ, the coefficients of decomposition. So, it means the set of all the expectation values is not an independent set. They depend heavily and they depend only on these three expectation values. These are the quantities which we will get from the experiment and a mu are the quantity in defining the observable A itself. So, they are not the part of the experiment.

The experimental part comes in the expectation value. So, this set, we can find the minimum set which has the most information. That set will be just the expectation value, σ_x, σ_y and σ_z. So, in that way this set of three expectation value should have same amount of information as the state $|\Psi\rangle$ which we are representing or any other representation we have for the state $|\Psi\rangle$. Now, let me repeat again. The first postulate says that the state of the quantum system contains all the measurable quantity or measurable information in that mathematical structure.

We are just twisting this postulate around and saying that, if we have somehow all the measurable information which we can measure in the lab, all the measurable information if we have in some mathematical form, that should be equivalent to the state $|\Psi\rangle$. So, it means and we have reduced that whole information which is measurable in the lab to just three measurable quantities that is σ_x expectation value, σ_y expectation value and σ_z expectation value. So, these three real numbers should be equivalent to the state $|\Psi\rangle$. There should be a one-to-one relation between these measurable quantities in the lab and this state $|\Psi\rangle$.

Why is it so? Because this is all we can measure for a given quantum system. These three things. We can measure more but they are dependent on these three. Now, we have to find that, given a state $|\Psi\rangle$, again the state $|\Psi\rangle$ is given in this form cos(ɵ/2), sin(ɵ/2)$e^{i\phi}$, when we perform measurement and we get σ_x which will be sin(ɵ)cos(ϕ), σ_y will be sin(ɵ)sin(ϕ) and σ_z will be cos(ɵ).

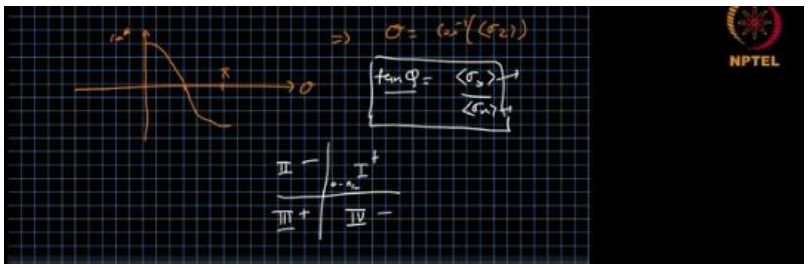
(Refer slide time: 04:08)



So, these three quantities at the state $|\Psi\rangle$ have one to one correspondence and since ɵ is from 0 to π so we can see that if we have cos(ɵ), it starts from 1, it goes to 0 at ɵ equals π/2 and it goes to -1 at ɵ equals π. So, there is no repetition in ɵ. So, from here, we can find the ɵ, which is cos inverse of σ_z. This is unique once, we measure the σ_z expectation value so we can calculate the cos(ɵ) from there directly. Now from once we have ɵ, then we can substitute in sin(ɵ) and we are left with only ϕ which is unknown. Now ϕ has value from 0 to 2 π. Okay and cos(ɵ) and minus, cos(ϕ) and sin(ϕ) can take positive or negative value for all those things, but we can have tan of ϕ which will be σ_x σ_y expectation value divided by σ_x expectation value, where can be the ambiguity now because, if we have four quadrants 1, 2, 3, 4, for the ϕ, it is 0 to pi by 2, pi by 2 to pi, pi to 3 pi by 2 and 3 pi by 2 to 2 pi. We have four quadrants. tan(ɵ) is positive here. It's negative here. It's positive here. It's negative here.

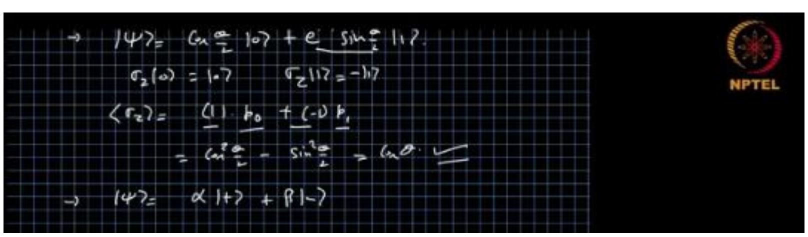
So, if we have only the value of tan ϕ, then we do not know whether it belongs to even quadrants or odd quadrants. But we can see that sin(ɵ) and cos(ɵ) have different signs in all four quadrants. So, in that way, by looking at the sign of σ_y and σ_z, σ_x, and the ratio of these two, we can determine where will the tan(ϕ) belong. And accordingly, we can assign the value of ϕ. And hence, with these three quantities, we can find ɵ and ϕ.

(Refer slide time: 07:15)



And hence, we can get the state from this experimental data. Okay so in this way by performing measurements and with the measured quantities we can estimate the state $|\Psi\rangle$, okay. So, one interesting thing i would like to make you see or it's a nice observation we have $|\Psi\rangle$ which we are writing in cos(ɵ)sin(ɵ) ta form we can also write it as cos(ɵ/2), $|0\rangle$ state and $e^{i\phi}$sin(ɵ/2), $|1\rangle$ state. Where $|0\rangle$ and $|1\rangle$ are the eigenstate of σ_z such that $|0\rangle$ belongs to the positive eigenstate, positive eigenvalue and $|1\rangle$ belongs to negative eigenvalue. So, from here, if we see what is the σ_x expectation value, that will be the eigenvalue, which is 1 times the probability of $|0\rangle$ plus the eigenvalue -1 times the probability of $|1\rangle$.

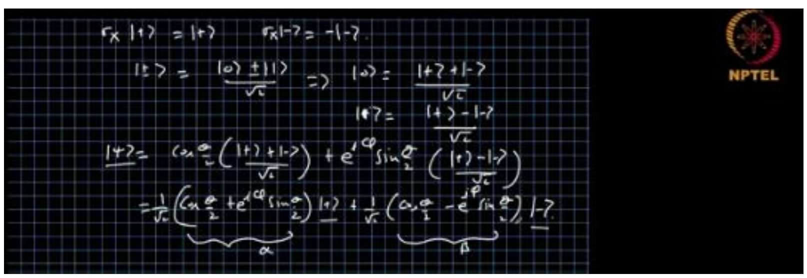
(Refer slide time: 10:09)



This is the eigenvalue times the probability, eigenvalue times the probability and sum of all those. So, probability of $|0\rangle$ is $\cos^2$(ɵ/2), and probability of $|0\rangle$ is the mod square of this element which will be $\sin^2(\theta/2)$, and minus sign comes from the eigenvalue. This is $\cos(\theta)$. This is what we saw earlier that the expectation value of σ_z is $\cos(\theta)$. Now, this is simple, can we do the similar thing for the expectation value of σ_x and σ_y. To do that, we need to find alpha and beta, okay.

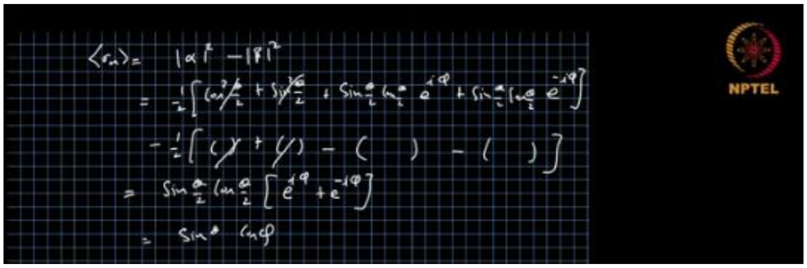
I am doing it for the σ_x expectation value now. $\alpha|+\rangle + \beta|-\rangle$, we are writing it where $|+\rangle$ is the expectation value or the eigenstate of σ_x with plus eigenvalue and $|-\rangle$ is the eigenvector of σ_x with minus eigenvalue. We can see that $|+\rangle$ and $|-\rangle$ can be represented in the $|0\rangle$ and $|1\rangle$ form. So, $|\pm\rangle$ is actually $(|0\rangle \pm |1\rangle)/\sqrt{2}$. We can also write $|0\rangle$ as $(|+\rangle + |-\rangle)/\sqrt{2}$ and $|1\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$. So, we can do this thing and we can substitute it back in the earlier equation, $\cos(\theta/2)$, we have $|0\rangle$, we write it as $(|+\rangle + |-\rangle)/\sqrt{2}$, plus $e^{i\phi}\sin(\theta/2)(|+\rangle - |-\rangle)/\sqrt{2}$. Now we can gather the coefficient of $|+\rangle$ and coefficient of $|-\rangle$ separately, $(1/\sqrt{2})(\cos(\theta/2) + e^{i\phi}\sin(\theta/2))$, these are the coefficient of $|+\rangle$, $(1/\sqrt{2})$ $(\cos(\theta/2) + e^{i\phi}\sin(\theta/2)$ is the coefficient of $|-\rangle$.

(Refer slide time: 11:48)



So, this is the decomposition. This is the representation of $|\Psi\rangle$ in the $|+\rangle$ and $|-\rangle$, $|\pm\rangle$ basis, so here this is alpha what we were saying alpha and this is our beta. Once we have this it's straightforward to write the expectation value of σ_x, which will be $|\alpha|^2 - |\beta|^2$, which will be half, we just take here the mod square of this will be $\cos^2(\theta/2)$ plus $\sin^2(\theta/2)$ plus $\sin(\theta/2)\cos(\theta/2)\, e^{i\phi}$ plus $\sin(\theta/2)\cos(\theta/2)\, e^{-i\phi}$, minus half and square of those, so we will get the same term here, plus same term minus same term minus same term okay, where the minus sign will come with just these two terms otherwise the magnitude is same. This is a negative sign when we add these cancels and they will be twice, so there is a half factor so we can cancel that so we are getting $\sin(\theta/2)\cos(\theta/2)$ $[e^{i\phi} + e^{-i\phi}]$. And that is nothing but $\sin\theta \cos\phi$.

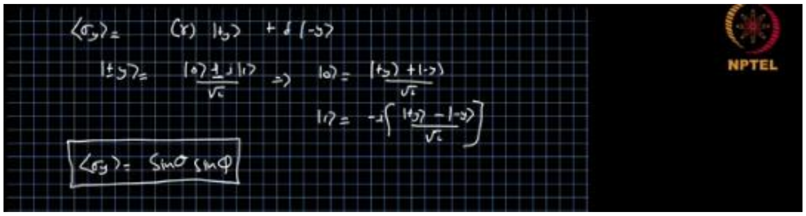
(Refer slide time: 13:37)



You see, we can write the state in the basis of the observable we want to find the expectation for. And then the coefficient mod squared and the eigenvalue product of the coefficient mod squared and the eigenvalues will give us the expectation value very easily. Similarly, we can do for σ_y. We can write it as $|+_y\rangle$, some coefficient γ, plus δ, $|-_y\rangle$, where $|\pm_y\rangle$ are the eigenvectors of σ_y with corresponding to plus and minus eigenvalues. $|\pm_y\rangle$ written as $(|0\rangle \pm i|1\rangle)/\sqrt{2}$.

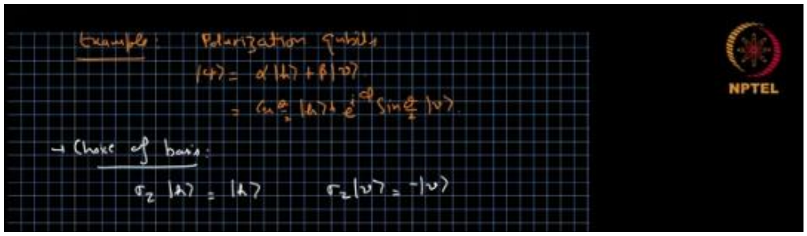
We can write it as $|0\rangle$ as $(|+_y\rangle + |-_y\rangle)/\sqrt{2}$ and $|0\rangle$ as $(|+_y\rangle - |-_y\rangle)/\sqrt{2}$. We can again substitute it in $|\Psi\rangle$ and we can write the $|\Psi\rangle$ in the basis of σ_y and then $|\gamma|^2 - |\delta|^2$ will be our expectation value and it will come out to be sinθ sinϕ. So, if we have a possibility of converting the state in the eigen basis of the measured observable, then it will be very easy and very straightforward to calculate the expectation value of the observable. So, we will be using it in a very sophisticated way, in a very nice way when we do real state tomographies. So, I would like to give an example of a state tomography.

(Refer slide time: 15:20)



So, in this example, we are taking the polarization qubits. So, here we have single photons and they are in an unknown polarization state. So, the state of the photon is some alpha times horizontal polarization plus beta times vertical polarization. Where alpha and beta are complex number, the usual practice and h and v are the horizontal and vertical polarization. We can as usual write $cos(\theta/2)|h\rangle + e^{i\phi} sin(\theta/2)|v\rangle$. Now our task is to design an experimental setup to find this alpha and beta or θ and ϕ.

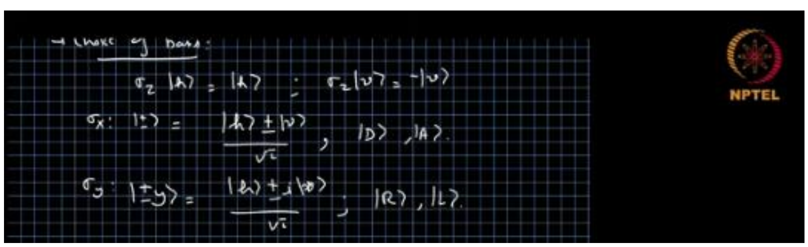
(Refer slide time: 16:54)



In order to do so, what we do is first we choose the basis. Choice of basis. We say the $|h\rangle$ and $|v\rangle$ are the eigen basis of σ_z. So, it means σ_z acting on $|h\rangle$ will be $|h\rangle$, horizontal polarization and σ_z acting on $|v\rangle$ will be $-|v\rangle$. So, $|h\rangle$ and $|v\rangle$ are the positive and negative eigenvectors of the σ_z. So, this is our choice of this. If that is the case, then the σ_x eigenvector which were $|+\rangle$ and $|-\rangle$ will be $(|h\rangle \pm |v\rangle)/\sqrt{2}$. Now, we know what is $|h\rangle$ plus $|v\rangle$ and what is $|h\rangle$ minus $|v\rangle$. $|h\rangle$+$|v\rangle$ and $|h\rangle$-$|v\rangle$ are the polarization vectors along 45 degree and 135 degree.

So, or we call it diagonal and anti-diagonal polarization. This is still linear polarization, but making 45-degree angle with the horizontal polarization and 135-degree angle with the horizontal polarization. So, by choosing $|h\rangle$ and $|v\rangle$ as the eigenvectors of σ_z, we go to the eigenvector of σ_y. And similarly, eigenvector of σ_y will be $|\pm_y\rangle$, which is $(|h\rangle + i|v\rangle)/\sqrt{2}$. And they are nothing but right and left circular polarization. So, let me repeat.

(Refer slide time: 18:28)

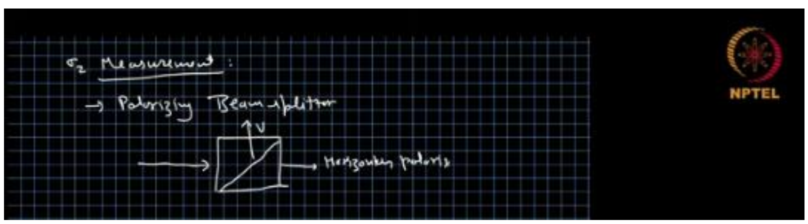


We are choosing h and v as the eigenvectors of that will give us the diagonal and anti-diagonal polarization states as the eigenvectors of σ_x and right and left circular polarization as the eigenvector of σ_y. Now, in the experiment, how do we find the expectation value? Our task is to find expectation value of σ_x, σ_y and σ_z. In an experiment, how will we do that? So, we basically sketch of the experimental setup will be a path in which single photons are coming one by one and an experimental box, experimental setup, which will give us two clicks, one corresponding to one eigenvector and other corresponding to the other eigenvector of a given observable. This experimental setup is designed for a given observable. And these observables for us are nothing but σ_x, σ_y and σ_z. So, these states $|1\rangle$ and $|2\rangle$ will be correspondingly, $|\pm\rangle$, or $|h\rangle$ $|v\rangle$.

Those will be the eigenvectors. So let us do the simplest case first that is the σ_z measurement. So, we all know what is a polarizing beam splitter. A polarizing beam splitter, just drawing it with a box with a diagonal line inside it. What it does, it's an optical element, optical gadget, such that if light comes in, the horizontal light will pass through, horizontally polarized and vertical light will be reflected. Horizontal will pass through and vertical will reflect. So, we can use this polarizing beam splitter.

(Refer slide time: 20:27)

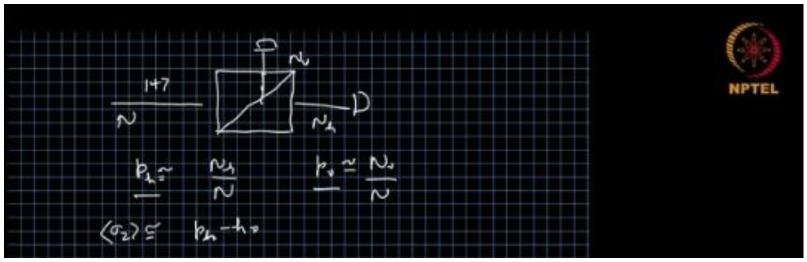
(Refer slide time: 20:58)



To do our sigma-z measurement, we have a box, polarizing beam splitter. We send photons in state $|\Psi\rangle$ and we put detectors here and here. Now, we send one by one N photons in and we get N horizontal number of photons here and N_v number of photon up there. Then the probability of horizontal polarization will be N_h, it will be not equal but almost, N_h/N and p_v will be N_v/N. So now we have probability of zero state, probability of one state, then the expectation value of σ_z will be p_h-p_v.

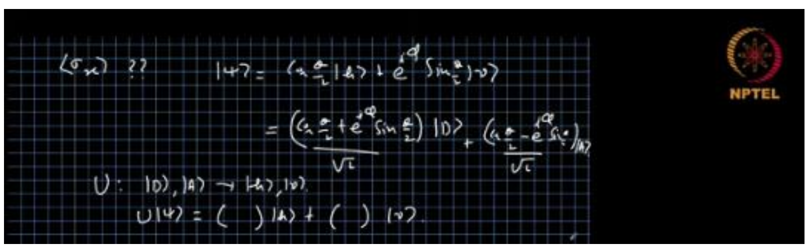
This is almost, this will be equal when N tending to infinity. So, in that way, we can calculate, we can experimentally measure the expectation value of σ_z. This was the easier part. How do we perform measurement on σ_x? How do we find the expectation value of σ_x? We can always say that we can find something similar to polarizing beam splitter which will separate diagonal and anti-diagonal states and then we find like we did for σ_z, we can do for σ_x. That is possible but that will require probably some technological development, finding or developing the gadget which is similar to polarizing beam splitter.

(Refer slide time: 22:10)



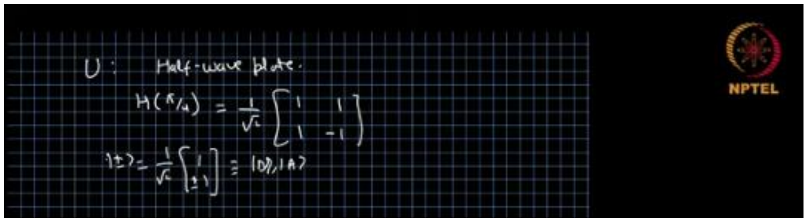
But we can do the same thing by using some smarter ways. For example, we discussed that $|\Psi\rangle$ we are writing as $\cos(\theta/2)\ |h\rangle + e^{i\phi}\sin(\theta/2)\,|v\rangle$. We can also write it as $((\cos(\theta/2)+e^{i\phi}\sin(\theta/2))/\sqrt{2})|D\rangle$, let me call it D, factor plus $((\cos(\theta/2)-\sin(\theta/2))/\sqrt{2})|A\rangle$, anti-diagonal. So, we can write $|\Psi\rangle$ in the diagonal and anti-diagonal basis. Till now we have not done anything.

(Refer slide time: 23:05)



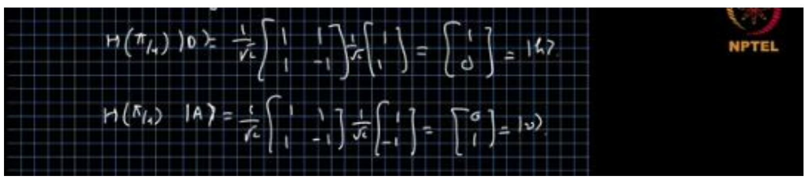
Now, what we can do is we can apply a unitary operator U which can take $|D\rangle$ and $|D\rangle$ to $|h\rangle$ and $|v\rangle$, if that is possible then we can apply U on $|\Psi\rangle$ and we get coefficient times $|h\rangle$ plus other coefficient times $|v\rangle$ instead of $|D\rangle$ and $|A\rangle$ and then we can apply the simple polarizing beam splitter, separate h and v and do what we did for σ_z. So what is this unitary which we have to apply and does it not require any additional technological development? It turns out that for polarization, this unitary can be achieved by something called half-wave plate. So, we are not going into a detail of what is the half-wave plate, but the action of a half-wave plate, when it is rotated by angle $\pi/4$, the matrix of that can be written as 1 over root 2, 1, 1, 1, -1.

(Refer slide time: 25:15)



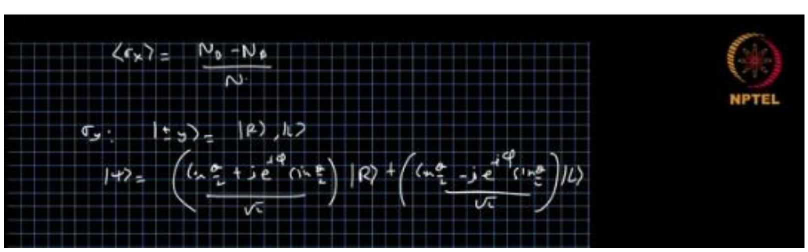
So, what is the benefit of this? We know that plus minus states are 1 over root 2, 1, plus minus 1. So, this is same as this is also the representation of $|D\rangle$ and $|A\rangle$ because we are identifying $|D\rangle$ with plus and $|A\rangle$ with minus. So, when we apply $H(\pi/4)$ on $|D\rangle$, we get 1 over root 2, 1, 1, 1, minus 1. 1 over root 2, 1, 1, we get 1, 0, which is $|h\rangle$. Similarly, if we apply $H(\pi/4)$ on $|A\rangle$, we get 1 over root 2, 1, 1, 1, minus 1, 1 over root 2, 1, minus 1, and that is 0 and 1, which is v.

(Refer slide time: 26:04)



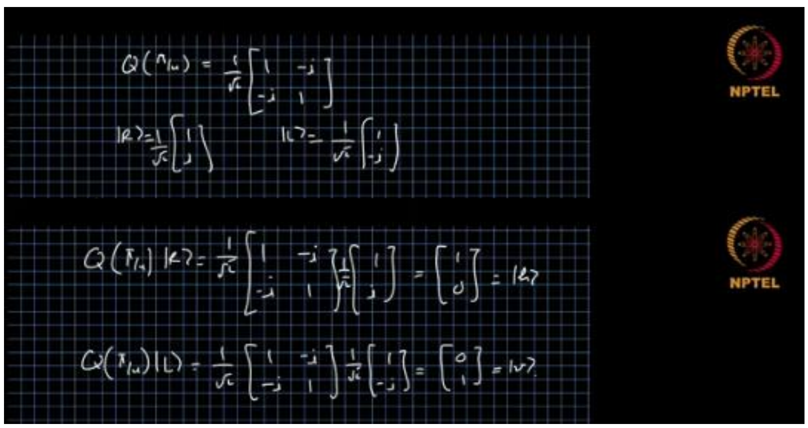
So, it seems like we can use H, half wave plate to rotate the, to transform the diagonal and anti-diagonal bases to horizontal and vertical bases and then we can use polarizing splitter to do the measurement. So, our setup for σ_x measurement will look like the following. We have state $|\Psi\rangle$ coming, N number of them. We apply a horizontal, half wave plate, sorry not horizontal, half wave plate with rotated at $\pi/4$ followed by a polarizing beam splitter followed by detectors. So, N_v or N_h number of clicks here and N_v number of clicks they will be same, this whole thing will be the setup now for σ_x and N_h will be actually N_D and N_v will be N_A in this setup and then we can calculate the expectation value as $(N_D - N_A)/N$.

(Refer slide time: 27:56)



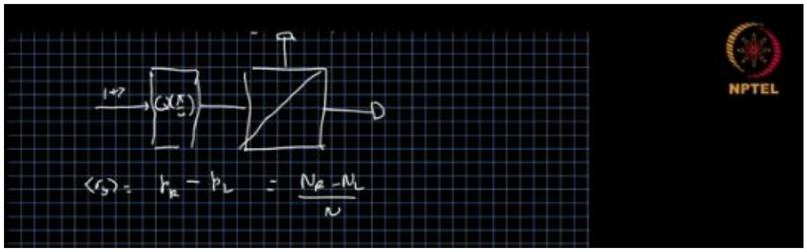
Similarly, for σ_y, we know the eigenvectors are plus minus y, which is $|R\rangle$ and $|L\rangle$. We can write the state $|\Psi\rangle$ as $((\cos(\theta/2)+e^{i\phi}\sin(\theta/2))/\sqrt{2})|R\rangle+((\cos(\theta/2)-\sin(\theta/2))/\sqrt{2})|L\rangle$,. We can apply operation unitary which takes $|R\rangle$ to $|h\rangle$ and $|L\rangle$ to $|v\rangle$ and they are nothing but our quarter wave plate. Again, we are not discussing what is the quarter wave plate, but the action of it when it is rotated by $\pi/4$ angle, the matrix representation of it will be 1, minus i, minus i, 1. The representation for $|R\rangle$ is 1 over root 2, 1, i, and representation for $|R\rangle$ is 1 over root 2, 1, minus i.

(Refer slide time: 29:20)



$Q(\pi/4)$ acting on $|R\rangle$ will be 1 over root 2 1 minus i minus i 1, 1 i, 1 over root 2 that will be 1 0 and $Q(\pi/4)$ $|L\rangle$, 1 over root 2 1 minus i minus i 1, 1 over root 2, 1 minus i is nothing but 0 1 that is horizontally polarized light, that's vertically polarized. So similar to σ_x our setup will look like the following, we have states coming then we have a quarter wave plate rotated $\pi/4$ followed by a polarizing beam splitter and detectors. And our expectation value σ_y will be p of $|h\rangle$ or $|R\rangle$ which is the same both are same here, minus p of L, this is $(N_R-N_L)/N$. So, in that way we can perform measurement over σ_x, σ_y and σ_z basis and we can calculate the probability of the clicks in the two orthogonal eigenvectors, we take the difference of that. Those probabilities difference because the eigenvalues are plus 1 and minus 1. We calculate the expectation.

(Refer slide time: 31:00)



Once we have the expectation values, $\{\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle\}$ then we can find the state $|\Psi\rangle$, which is represented by these three. And hence, we can find the value of θ and ϕ and we can estimate what state was, which was under consideration. So, this is one experimental way of finding out the state of a quantum system.