FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-02
Lecture-06
Qubits: Introduction

Qubits, these are basic blocks of quantum computation and quantum information theory. So, what is a qubit? Any two-level system called a qubit. So, it should have two levels. For example, the polarization of a single photon. Polarization can take two orthogonal directions.

Two orthogonal states like horizontal and vertical or diagonal and anti-diagonal or left and right circular polarization, so there are two orthogonal states of polarization and a single photon can exist in any superposition of those two orthogonal states then we can consider atoms with two energy levels. We can't have atom with just two energy levels. So, we just focus on two energy levels. We interact with those two and we disregard the other levels. In that way, we can treat an atom like a qubit. Then we can have quantum dots.

We can design quantum dots where the two levels are of interest and we disregard all the other levels. Again, we will have a qubit. We can have a single photon, existing in a superposition of two waveguides. The single photon can go to one waveguide or the other waveguide, so, two paths let us say. So, this is called dual rail qubit, this is also another example of qubit which is used commercially as well as in the foundational research. Other is the superconducting qubits using the Josephson junction and SQUIDs. You can realize a system where the two states can be faithfully recognized identified and that can be used as qubit. All the commercial quantum computers are based on superconducting qubit at the moment. There are qubits with neutral atoms or there's quantum computers with neutral atoms or ion traps or photons also but the most successful so far is the one with superconducting qubit.

So, mathematically, the state of a qubit $|\Psi\rangle$ belongs to a Hilbert space, which is a two-dimensional Hilbert space, two-dimensional vector space. So, it means there can exist at most two mutually orthogonal states. Like we said the polarization of a single photon, we have horizontal and vertical. So, there are only two at a given time, there are only two

orthogonal polarizations. So, the H_2 or two-dimensional Hilbert space is the defining or the vector space belonging to the qubit.

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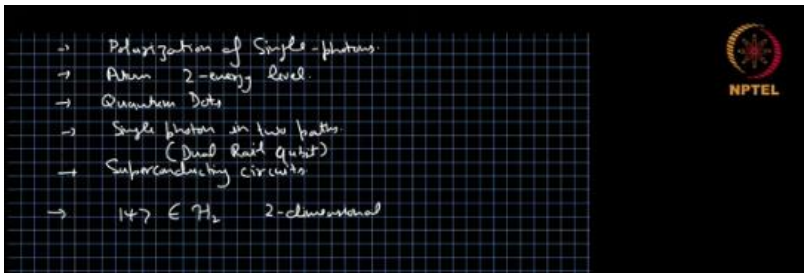

So, of course, the states are from the projective Hilbert space. So, that will be CP^1 complex projective, one dimensional. So, a state $|\Psi\rangle$ in the matrix form will be a two-dimensional vector, alpha and beta, where alpha and beta both are complex numbers. So, the states $|\Psi\rangle$ should be normalized. So, $\langle\Psi|\Psi\rangle$ should be 1, that comes out to be alpha mod square plus beta mod square equals 1. That is the normalization condition.

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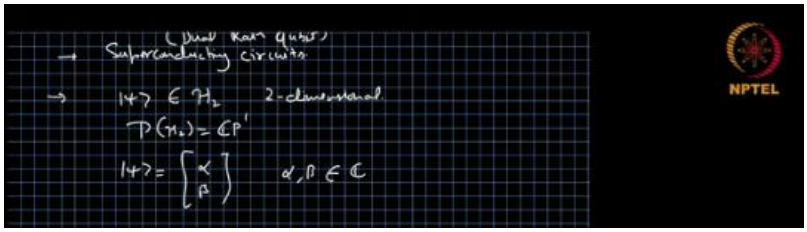

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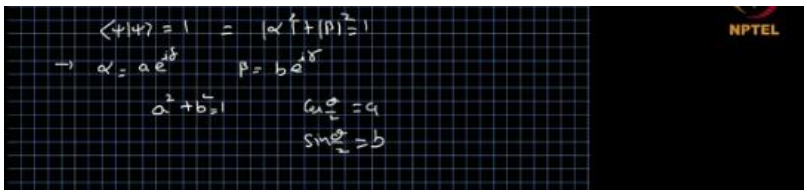

Now, both alpha and beta are complex numbers, so we can write them in the polar decomposition like alpha to be a exponential of i delta and beta equals b exponential of i gamma, where a and b are real numbers, delta and gamma are phases. So, in that way, we can write any arbitrary alpha and beta in this form. So, the normalization condition turns out to be $a^2+b^2=1$. So, instead of calling them a and b and this normalization condition, we can just call them, cos(ɵ/2) is a, and sin(ɵ/2) is b. Over 2 is more traditional and it

will be useful in some geometrical representation, but this by 2 is not really mathematically relevant, but it is good for some simplifications later on.

Once we say a and b are cosine, sine functions of the same parameter, then $a^2+b^2=1$ is automatically satisfied. Then because of the ray property of the state $|\Psi\rangle$, a state $|\Psi\rangle$ and a state $|\Psi'\rangle = e^{i\zeta}|\Psi\rangle$ represent the same physical state of a quantum system. So, it means we can first write $ae^{i\delta}$, $be^{i\gamma}$. This state is same as a, $be^{i(\gamma-\delta)}$ and it is same as $ae^{i(\delta-\gamma)}$, b. In all three what we have done in the second one we have just taken $e^{i\delta}$ common and forgotten about it and in the third one we have taken $e^{i\gamma}$ common and forgotten about it so if we say we can stick to this notation because that's again very traditional and convenient and we call $\gamma - \delta$ to be ϕ, so $|\Psi\rangle$ becomes $\cos(\theta/2)$ and $\sin(\theta/2)e^{i\phi}$. So, we get a canonical form of a state of a two-level system and that is $\cos(\theta/2)$, $\sin(\theta/2)e^{i\phi}$. So, here theta is between 0 and π and ϕ is between 0 and 2π. That just to avoid repetition. This was about state.

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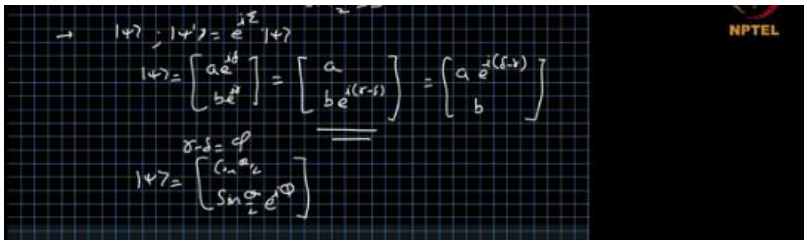

Now next is the observable, 2 by 2 Hermitian, matrices represent the set of all the observables for a two-level system. So, this is two by two set of Hermitian operators also forms a linear vector space. And the dimension of this can be calculated. We just write it dimension of this is four. The basis for it can be chosen anything, but we choose something more particular.

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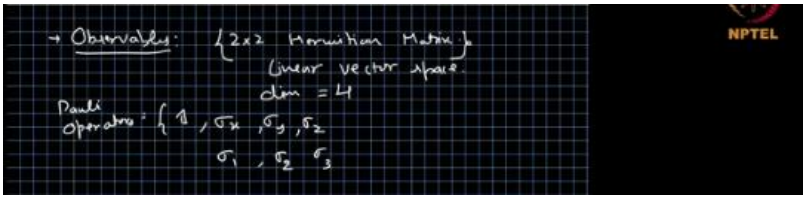

It's called poly operators. This will be used extensively in the later on studies. So, these are the four operators, identity, Sigma X, Sigma Y, Sigma Z. Sometimes we call them

Sigma 1, Sigma 2 and Sigma 3 also. The definition of these identity is just $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, σ_x is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, σ_y is $\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ and σ_z is $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$. This is how we define the four poly operators.

So, these operators being basis means any Hermitian 2 by 2 operator A, can be written as $\sum_{\mu=0}^{3} a_\mu \sigma_\mu$, where σ_0 is identity and $\sigma_{1,2,3}$ are defined up there and a's are the real vectors, a_μ is a real number. So, if we expand it, it will look like $a_0 + a_3$, $a_1 - ia_2$, $a_1 + ia_2$ and $a_0 - a_3$. We can see that once a_1, a_2, a_3 and a_0 are real, then this is a Hermitian operator. And we can also see that if we are given a Hermitian operator and arbitrary Hermitian operator, we can always find a_0, a_2, a_3 and a_3 for that operator. So, this becomes the representation of a Hermitian operator or decomposition of Hermitian operator in terms of poly metrices.

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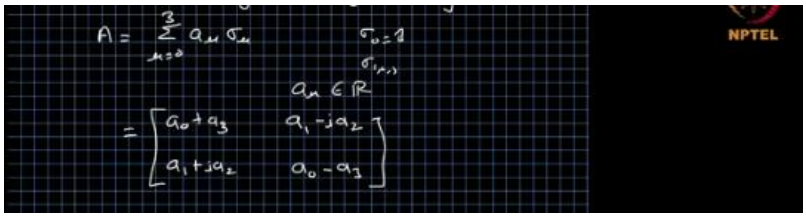

The expectation value of A is, in a given state $|\Psi\rangle$, is $\langle\Psi|A|\Psi\rangle$, which will become sum $\sum_{\mu=0}^{3} a_\mu \langle\sigma_\mu\rangle$. If we are given $|\Psi\rangle$ to be cos(ɵ/2), sin(ɵ/2)$e^{i\phi}$, then we can calculate $\langle\sigma_x\rangle$. It comes out to be sin(ɵ) cos(ϕ). σ_x expectation is sin(ɵ) sin(ϕ) and σ_z expectation to be cos(ɵ). So, in that way, for a given state $|\Psi\rangle$, we can calculate the expectation values of sigma x, sigma y, sigma z, expectation value of identity is 1. And so, from these three expectation values, we can calculate the expectation value of an arbitrary observable in lab.

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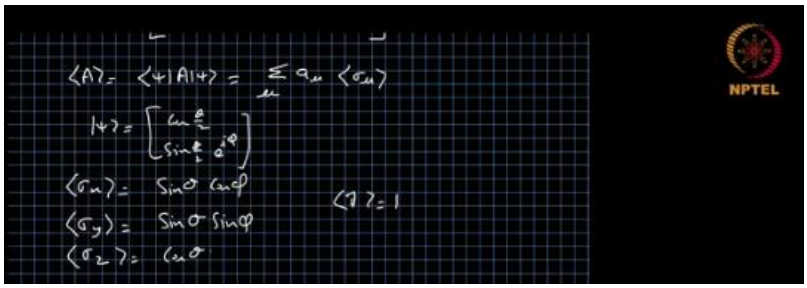

One interesting example of such operator is the Hermitian operator is the Hamiltonian. So let us represent it by H and this will be again $\sum_{\mu=0}^{3} h_\mu \sigma_\mu$. So, we expand it, it becomes

$h_0 + h_3$, $h_1 - ih_2$, $h_1 + ih_2$ and $h_0 - h_3$. So, this is the more general Hamiltonian we can encounter for a two-level system. Now, we are interested in the eigenvalues of the Hamiltonian that will tell us the energy levels of a system. So, the eigenvalues of H can be calculated and turns out to be $h_0 \pm \sqrt{h_1^2 + h_2^2 + h_3^2}$. So, it means we have two energy levels.

One is the lower one, let us call it ground state. And one is upper one, let us call it excited state. The energy of the lower one is $h_0 - \sqrt{h_1^2 + h_2^2 + h_3^2}$ and the energy of the upper one is $h_0 + \sqrt{h_1^2 + h_2^2 + h_3^2}$. So, we can shift the energy of the whole system in such a way that the center comes in the, the zero comes in between the two levels, exactly between the two levels. So, that the separation, this separation and this separation is same.

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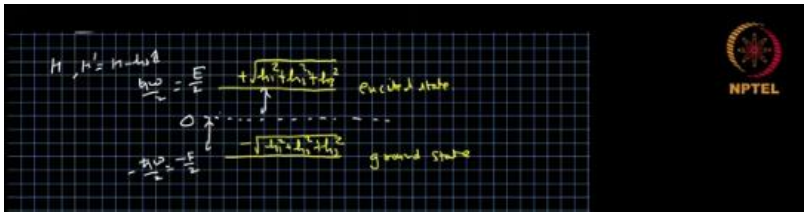

So, in that way we can get rid of h_0. So, it means we can write the Hamiltonian H and $H' = H - h_0 I$, will give you the same result, will have the same behavior. Now, we can call this energy as E/2 and this as minus E/2 or this is $\hbar\omega/2$ and this is $-\hbar\omega/2$. And we can represent excited state with $|e\rangle$ and ground state with $|g\rangle$. Then the Hamiltonian becomes the, these are the eigenvalues. So, $-\hbar\omega/2|g\rangle\langle g| + \hbar\omega/2|e\rangle\langle e|$.

This becomes the canonical form of a Hamiltonian for a two-level system, where $|g\rangle$ and $|e\rangle$ are the eigenstates, and $\hbar\omega$ is the energy separation between $|g\rangle$ and $|e\rangle$. Now, if we choose $|g\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|e\rangle = \begin{bmatrix}0\\1\end{bmatrix}$, that is the eigenvectors of σ_z, then our H becomes

$-\hbar\omega/2 \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \hbar\omega/2 \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$, which comes out to be $-(\hbar\omega/2)\sigma_z$. So, any Hamiltonian, if there is no preferred basis for a Hamiltonian, then the Hamiltonian of a system, of a qubit system can always be taken as some multiple of σ_z. That is the simplest Hamiltonian we can choose if there is no preferred basis. If there is preferred basis, then we of course have to write, we have to choose $|g\rangle$ and $|e\rangle$ in that basis.

Otherwise, we can choose it eigen state of σ_z and that sets the basis and then everything else can be represented in this basis. Next is the time evolution. So, the Schrodinger equation does not change with the dimension, where H is the Hamiltonian and if the Hamiltonian is time independent then $|\Psi(t)\rangle$ is exponential of $\exp[-iH(t-t_0)/\hbar]|\Psi(t_0)\rangle$. Now, exponential of the Hamiltonian, we can write as, this is $(e^{i\omega(t-t_0)}|g\rangle\langle g| + e^{-i\omega(t-t_0)}|e\rangle\langle e|)|\Psi(t_0)\rangle$.

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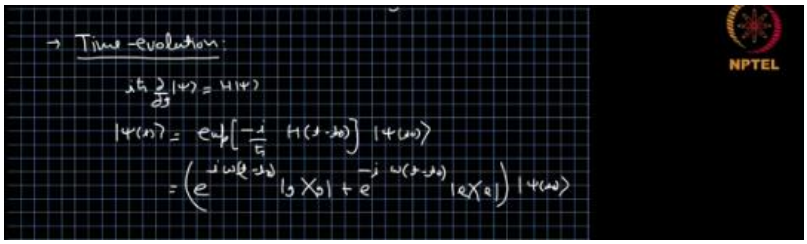

If $|\Psi(t_0)\rangle$, the initial state is $\cos(\theta/2)\,|g\rangle + e^{i\theta}\sin(\theta/2)\,|e\rangle$. Then the state at time t will be $e^{i\omega(t-t_0)}|g\rangle + e^{i\theta}e^{-i\omega(t-t_0)}\sin(\theta/2)\,|e\rangle$. In that way, we can calculate the the time evolution of two-level system.

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