FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

Dr. Sandeep K. Goyal
Department of Physical Sciences
IISER Mohali
Week-10
Lecture-27

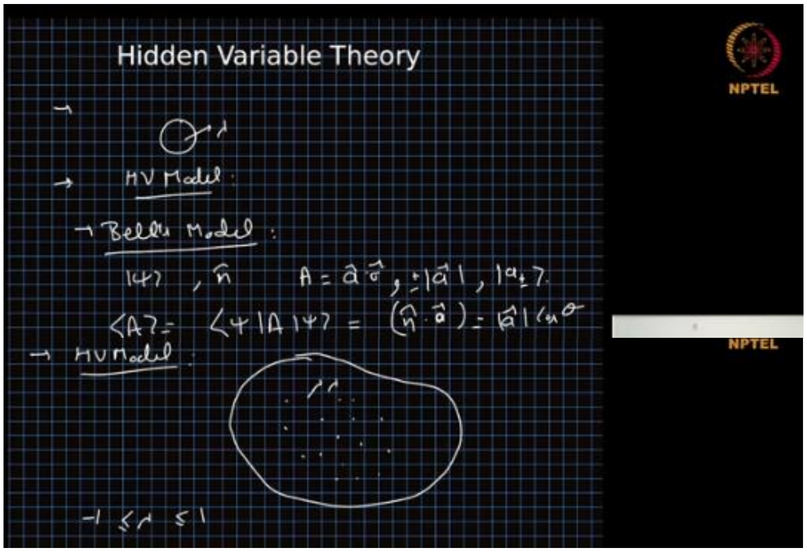
Hidden Variable Theory, Bell model, Bell Inequality, Bell CHSH Inequality

From the arguments by Einstein and Mermin and Bohm and many other people, it seems like quantum mechanics is first of all not complete and there are some issues with the quantum mechanics which need to be rectified That gave birth to a new kind of interpretation of quantum mechanics or a new theory, which is called hidden variable theory. And here the underlying assumption is that the fundamental laws of physics are deterministic and the apparent stochasticity of quantum phenomena is mainly due to our imperfect methods or imperfect measurements and the unknown variables we have in the system. In other words, quantum mechanics should be considered or should be a sub-theory of a more general theory in which there is no stochasticity. Everything is deterministic, everything is smooth.

In this theory, every quantum system has some hidden information in the hidden variable theory. Every quantum system has some hidden variables attached with them. Let us call them lambda, which, since they are unknown but physical upon measurement results in stochastic behavior in the measurement outcomes. So, what we are saying is we have every quantum system as the inherent parameter lambda which we do not know how to measure or which is hidden and that is what results in the stochasticity of the outcomes. So, what we can do is we can develop some model to understand what is going on here, hidden variable model and we will consider the Bell model here, Bell model.

So, let us consider spin half particles and the state is psi. So, the associated state vector or block vector is n cap in quantum mechanics, according to quantum mechanics. And let us say we want to measure an observable A which is given by A vector dot sigma. So, A vector mod plus minus are the eigenvalues of A and corresponding eigenvectors are A plus and A minus. So, A vector mod is the eigenvalue of A and A plus A minus are the eigenvectors.

(Refer slide time: 4:04)

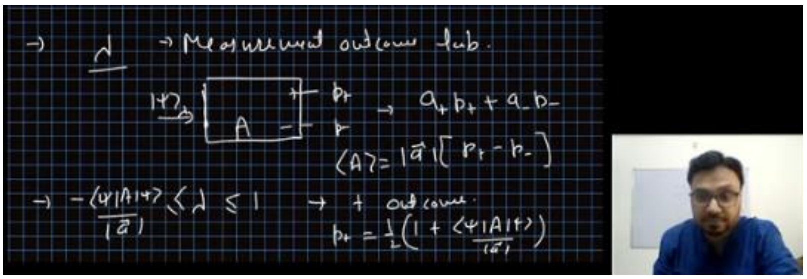


The expectation value of A, which is what we measure in the lab, given by psi A psi and which can be written as it is n cap dot A vector. That would be A vector mod times cos of theta between, theta is the angle between n cap and A, the two bloch vectors. In the hidden variable theory, hidden variable model, we assume an ensemble of particles, there are a lot of quantum particles and each one of them has some parameter in some form lambda. Lambda can be a number, a complex number, a real number, a vector, set of parameters, it can be anything. It is some hidden property of the quantum system which is not measured in the lab, which cannot be measured in any measurement setting. And let us assume for the sake of simplicity that this lambda is uniformly distributed over all these systems.

Lambda for just a single parameter lies between plus 1 and minus 1. So, that the value of lambda can take any value from minus 1 to 1 and it is uniformly distributed in this set. The Bell model, the hidden variable model given by Bell specifies that depending on the value of lambda, we get the measurement outcome in the lab. So, if we just consider the experimental setup, we have experimental setup, the state psi goes in, one particle in the state psi goes in, it will click either in the plus outcome or the minus outcome of the measurement setup A, and we calculate the probability p plus and p minus, and from here we calculate A plus, p plus, plus A minus, p minus, where A plus and A minus are the eigenvalues, and this will be A mod P plus minus P minus. This is the expectation value of A. This is from the quantum mechanics.

In the hidden variable theory, the parameter lambda associated with each of the states psi and we put, so each particle is coming with its own lambda and the lambda can take uniformly the value between minus and 1. So, the click, whether the particle will click in the output plus or minus will be determined by the inherent property lambda, hidden variable lambda. And the rules can be stated like this. If the lambda is between minus of psi A psi over magnitude of A and plus 1, then we will get the plus outcome. And the probability of this happening will be p plus, which will be given by 1 plus psi A psi over A mod times half.

(Refer slide time: 4:04)

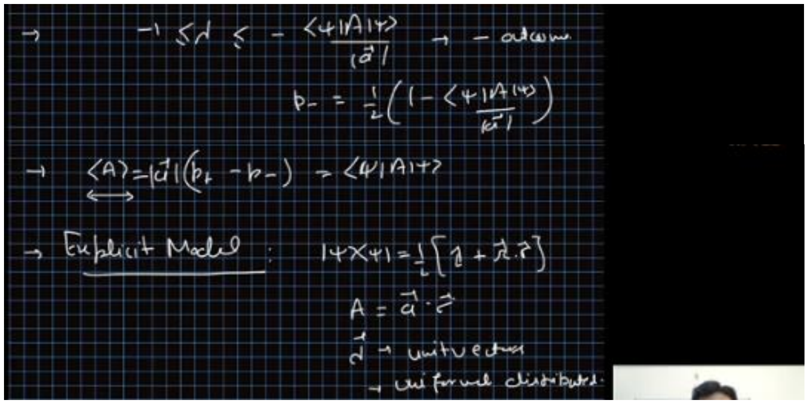


And if the lambda is between minus 1 and minus psi A psi over A mod, then we get minus outcome. And the probability of this happening is p minus, which is half times 1 minus psi A psi over A mod. Now, we can see that if this happens, then the expectation value of A will be p plus minus p minus times A vector, this satisfies. So, in this way, in this simple sense, we can have a hidden variable description of quantum mechanical outcomes. In quantum mechanics, we only know how to measure the expectation value of an observable.

If any other theory can predict this expectation value correctly for all the observables, then that can be a suitable replacement for quantum mechanics. Let us generate an explicit model for it. Again, our state psi or the density matrix corresponding to it, we write as half identity plus r dot sigma. A observable is a vector dot sigma. Lambda vectors are, lambda now we are taking as lambda vector.

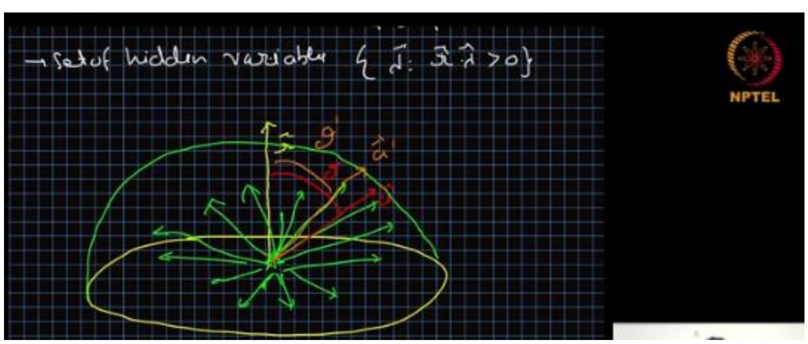
They are unit vectors. Lambda vector represents the hidden variable we have. They are unit vectors uniformly distributed over a set and that set is chosen in such a way that the set of hidden variables is such that lambda for all lambda as r dot lambda is positive. So, r is the unit vector, lambda is the unit vector for all the lambda. So, whenever lambda dot r or r dot lambda is a positive number, then we will consider those lambdas as our hidden variables.

(Refer slide time: 7:44)



So, if we see that our r vector is given by, let us say this is our r vector then there is a plane which is orthogonal to r and our lambda vectors are all the vectors in this hemisphere, upper hemisphere and because all these vectors have the inner product positive with our r vector, the set of these vectors which makes the hemisphere upper hemisphere are the set of hidden variables we have. Now, we want to perform measurement and measurement rules will be defined. So, let us say our measurement operator A vector corresponding to it, that makes an angle theta with the r vector. Now, let us consider another vector A prime, unit vector A prime vector, which has an angle theta prime with r. Reconsider this figure, we see that we have r vector and there is a plane orthogonal to this r vector.

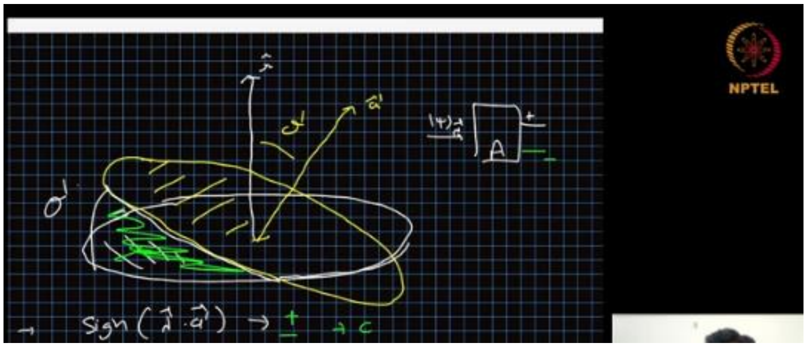
(Refer slide time: 9:46)



Then we have a prime vector and there is a plane corresponding orthogonal to this a prime vector. The angle between r and a prime is theta prime, so the angle between the plane orthogonal to r prime and a prime is also theta prime the vectors lambdas are defined as the vectors which have positive inner product with r vector. So, there are some vectors lambda which will have negative inner product with a prime vector. So the probability of outcome or measurement outcomes can be defined as sign measurement outcome is defined by sine sign of lambda dot a prime vector. Okay, so if we send a particle in the measurement setup, we have this measurement setup. We send a particle psi and this psi has lambda vector as a hidden variable and this lambda vector is somehow above this yellow plane.

And this white half is yellow and half white plane such that if lambda vector is such that the inner product of A prime and lambda is a positive number, then the click will be plus click. And if lambda and A prime is a negative number, that is if the lambda falls in this gap here, then the click will be minus click. So, sign of lambda dot A prime, whether it is plus or minus will determine the outcome of the measurement. And from this picture, we can see that the probability of getting a plus outcome is the whole of it is pi minus the part which says that part which does not fall above it. So, that is theta prime over pi, theta prime and we normalize it with pi.

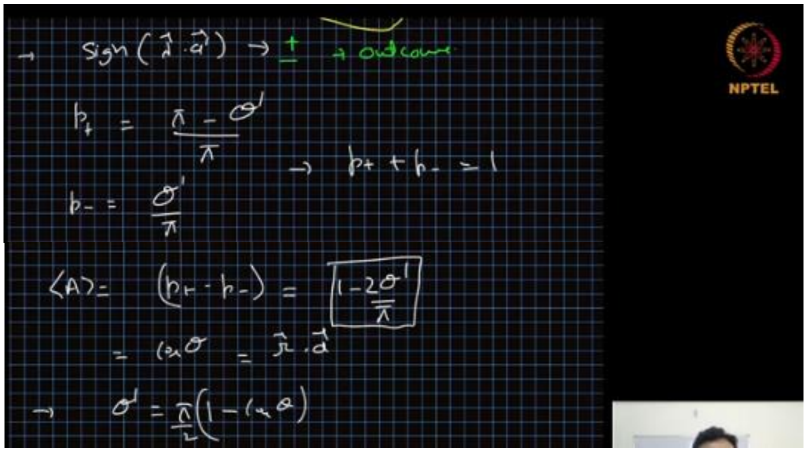
(Refer slide time: 11:53)



So, that will be the probability of getting lambda such that lambda dot a prime is positive. Generally, p minus will be theta prime over pi. So, we from here we can see p plus plus p minus is one and the expectation value of a from here it comes out to be p plus minus p minus times a vector. If we say it's a it's a unit vector a vector to be then we have just p plus minus p minus and it becomes 1 minus 2 theta prime over pi. This is the expectation value from this hidden variable model. But from quantum mechanics, A expectation value is cos of theta.

That is the r vector dot a vector and both we are assuming to be normalized. So, theta is the angle between r vector and a vector. To get the same expectation value in the hidden variable model, this particular model, as we get from the quantum mechanics, we need to choose theta prime in such a way, it is this expression is equal to cos of theta. It means theta prime comes out to be 1 minus cos of theta times pi over 2. If we choose theta prime to be pi over 2 times 1 minus cos theta, then our expectation value of the measurement of observable A will be same in quantum mechanics and in this hidden variable model.

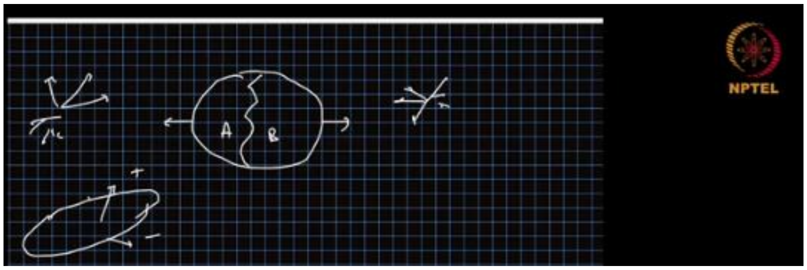
(Refer slide time: 13:56)



This is one explicit construction of the hidden variable model which yields the same result as by the quantum mechanics. Now question is have we seen such thing ever? In our life, like lambda vector associated with some system and thing. So, is there any physical example of such model? Not hidden variable model, even a physical model. Let us consider a bombshell, a sphere or some big block of explosive and it explodes into two parts.

Let us say part A and part B, they move in the opposite direction. In the beginning, it was at rest and with angular momentum zero, but due to explosion, it gained some angular momentum. Part A gained some angular momentum, part B gained some angular momentum. If initial angular momentum was zero, so the angular momentum of part A and part B are exactly opposite. But it can be in any direction, but the angular momentum of A can have any direction.

(Refer slide time: 15:47)



Part B will have exactly opposite direction to that. So, this is one example where you see a vector associated with the system A, for example, and we can measure, we can perform measurement, dichotomic measurement, which have plus and minus outcome. In the following way, we define a plane or a vector and we say if the angular momentum is above it, then we call it plus outcome and if the angular momentum is below it, then we say minus outcome. Similarly, for the B set, if the angular momentum we measure is above certain plane, then we say it is the positive outcome and if it is below it, then we say it is negative outcome. This situation is exactly same as what we have done here.

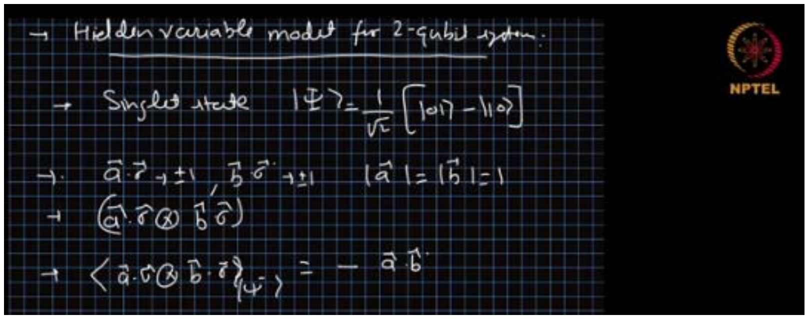
If the lambda vector associated with the quantum particle is above the plane defined by A prime vector, this vector is associated with the observable A. lambda is above this plane, we call it a plus click. And if lambda is below that plane, then we call it a minus. So, this model is inspired from some physical examples like this. From here, it seems like we can find a hidden variable model for a single particle.

But will this kind of model, local hidden variable model work for entangled particle or a composite system? So, we now discuss hidden variable model for two qubit system. And we are talking about local hidden variables. So, for each individual particle, we will have this same hidden variable model like the one we had here. And we see if it persists for two qubits or so.

So, for this, we don't want to establish a general model. We just want to show a contradiction that it does not work. And for contradiction, just one example is enough. So, we will consider that particular example. So, we consider a specific state of two qubits.

That's the singlet state. That is given by Psi 1 over root 2, 0, 1 minus 1, 0. Now we want to perform measurement of observable A dot sigma on particle A and B dot sigma on particle B. So, the joint measurement is A dot sigma tensor B dot sigma. A and B are normalized vectors. The outcome of a dot sigma can be plus minus 1, outcome of b dot sigma can be plus minus 1. So, a dot sigma times b dot sigma also it can be plus minus 1. But the expectation value of a dot sigma tensor b dot sigma for the singular state psi minus is minus a vector dot b vector.

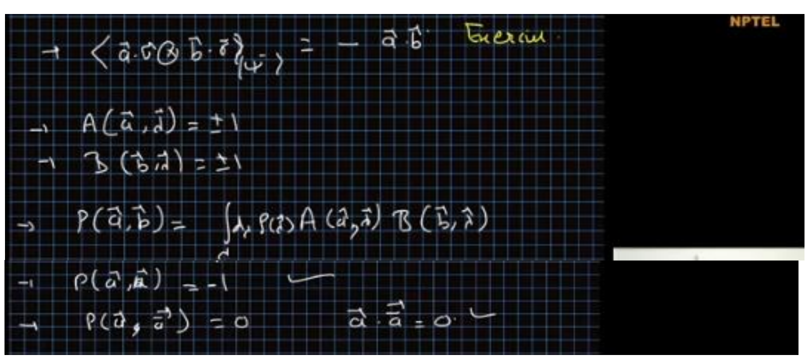
(Refer slide time: 18:30)



This is an exercise, prove that the expectation value of a dot sigma tensor b dot sigma in the state psi minus the singlet state is minus of a dot p. Now, using the hidden variable model we just described for single particle, we say the outcome of outcome a of A vector and lambda the hidden variables is, can take value plus minus 1 and we call outcome on the B system as b. This is our notation now. So, capital A with a and lambda, it can take value plus minus 1. B with b and lambda can take plus minus 1. Lambda is same, is defined for the pair of qubits now, for the entire system. So, each pair of qubit have one defined lambda.

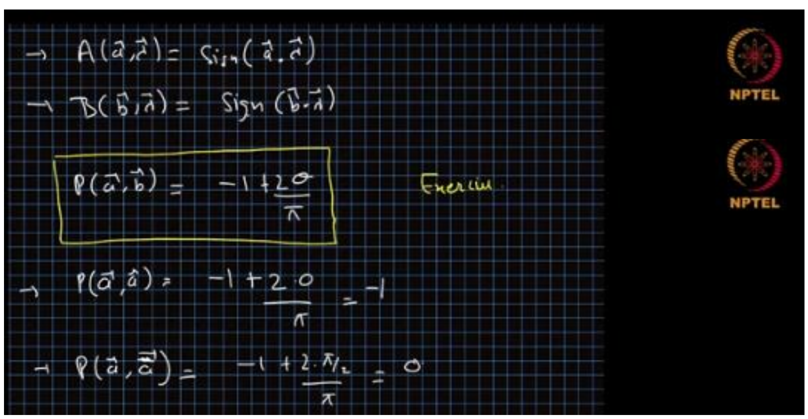
So, if you perform measurement, then lambda will be same for both the particles. This is our assumption. From here, we find the expectation value of a vector and b vector that will be a measurement of A vector a dot sigma observable on a and B vector on b. this is the expectation value and this can be written as the outcome A on the a side times outcome B on the b side, the density of the probability density of the vector lambda which we are assuming to be uniform and integration over lambda. So, this is the expectation value for this system when we are performing measurement of A and B on the two sides. We can see very easily that if a and b are same, that is we are performing measurement of a dot sigma on both sides, then the expectation value must be 1.

(Refer slide time: 21:00)



Okay, because or minus one, sorry because it is it should satisfy minus of a dot a and a is a unit vector, so, a dot a is one, it should be minus one and if we have a and a bar which is the orthogonal vector to a then it should be 0 because a dot a bar vector is 0 because these expectation value should match the expectation value given by the quantum mechanics, then these two conditions should be met by for any expression we have from the alternate theory. So, the expectation value expression is, should have a minus 1 when the measurements are same on both sides and should have 0 when the measurements are orthogonal on both sides. As we did earlier, the measurement A is given by sign of A and lambda. The measurement of B is given by sign of b and lambda, b dot lambda. So, if we use the bombshell model, and then we have the planes determined by a vector plane determined by this and b vector plane determined by this and expression used here for rho lambda to be uniform distribution over lambda, then we can calculate what is the expectation value of a, b.

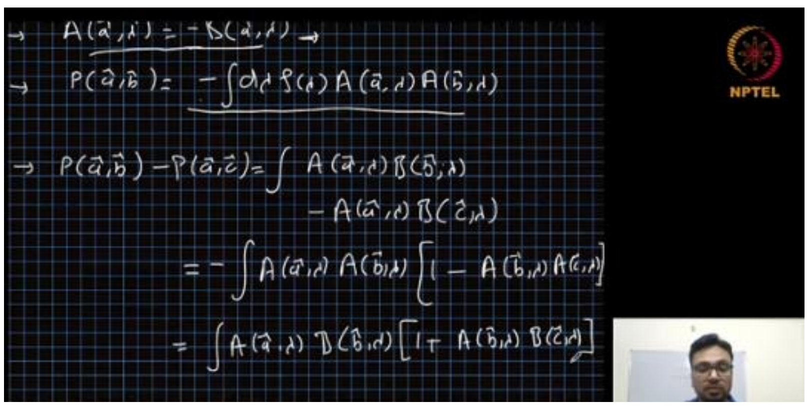
(Refer slide time: 23:12)



This remains an exercise and it is given by minus 1 plus 2 theta over pi where theta is the angle between A and B. It is not very difficult and it remains an exercise that the expectation value is minus 1 plus 2 theta over pi. From here, we can see that if a equals b, then a vector, a vector expectation value becomes minus 1 plus 2 times theta is 0 over pi, which is minus 1. And p of a, b or a bar will be minus 1 plus 2 times theta between two orthogonal vectors is pi over 2, which becomes 0. In that way, this expression satisfies the conditions placed over the expectation values by quantum mechanics has been satisfied. One interesting factor here we can see, one interesting observation is outcome A for a and lambda is minus B a and lambda.

It means that the outcome of a measurement of observable A on A qubit will be exactly opposite of the measurement, outcome of the measurement of the observable A on B qubit. They are anticorrelated in every sense. Like whatever measurement we perform on A, if we perform the same measurement on B, we will get exactly the opposite answer. Now we see that we have P of a b, which is minus d lambda rho lambda. We are assuming it to be general but for most purposes we take it as uniform distribution, probability distribution of lambda and we have A of a lambda and we had B of a lambda, b lambda but it becomes A of b lambda by using this expression and minus sign we have taken out.

(Refer slide time: 26:31)



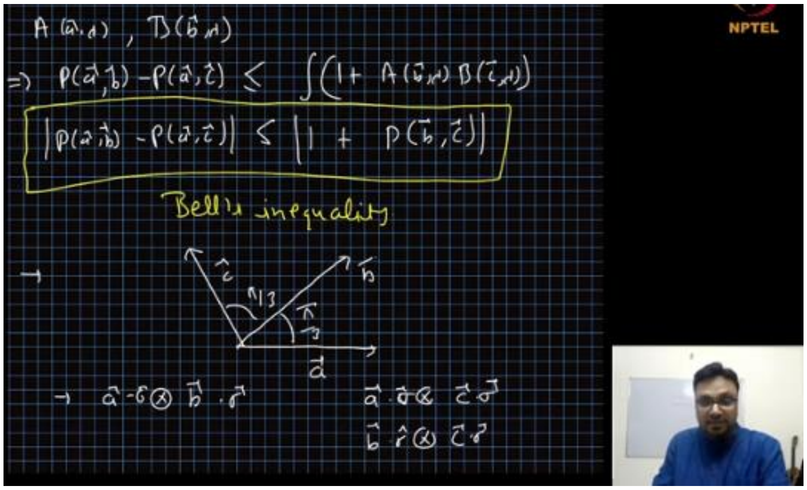
Now, consider the following. P of a b minus P of a c. So, now we have three observable, A on subsystem a, b and c on subsystem C and we are taking the difference of the two expectation values and this can be written as integration absorbing d lambda rho lambda in the sign of integration. Then we have A of a lambda B of b lambda, minus here minus A of a lambda B of c lambda. I can replace B with A with this identity and we can write it as minus sign outside A of a lambda, A of b lambda, common minus A of b lambda and A of c lambda. So, what we have done here is that we, since a is the outcome when we are performing b and b measurement on the, when the hidden variable was lambda, this a can take value plus minus 1, the square of a is 1. So, if you open this bracket, we will get the first term and the second term as such. There is no any problem.

Now, we can replace it again, A of a lambda, B of b lambda, we have absorbed this minus sign inside, we have one minus, this minus sign can be absorbed and we get A of b lambda and B of c lambda. So, by some trick or by some using some methods, some identity and that identity is precisely this, we can write the difference of the two expectation values B of a b minus B of a c as A of a lambda B of b lambda times one plus

A of b lambda B of c lambda. Now A of a lambda B of b lambda, both of them can take plus minus one value. So sometimes they are plus sometimes they are minus but if, so this whole term here will be larger if we replace both of them with 1. It means we can say A P of a lambda minus P of a b sorry P of a b and P of a c is always less than or equal to integration 1 plus A of b lambda B of c lambda. Or if it integrates, the integration over one is the average, it will be one plus P of b c.

So, we are saying P of a b minus P of a c will be less than or equal to this. So, we can also put the bars here, so that we do not worry about the plus minus sign. So, now, this is the inequality and this is called Bell's inequality. So, for any local hidden variable theory, this inequality should, so, there is no theory which can be described with local hidden variable theory, formalism and this theory violates, this inequality violates. Now, we will see whether it violates or preserves in quantum mechanics.

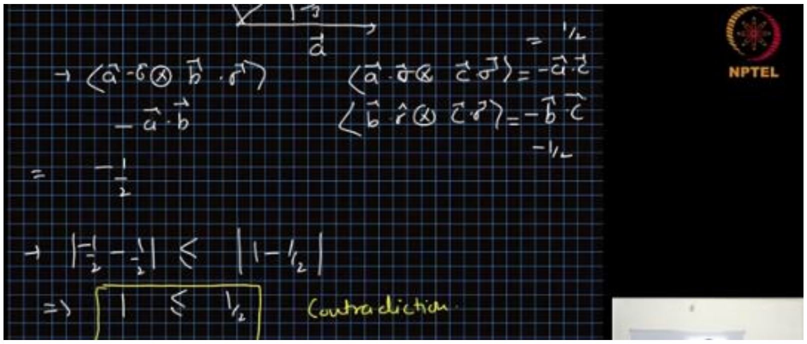
(Refer slide time: 29:32)



So, for that, let us choose the vector a to be along x axis, b at 60 degree angle at pi over 3, b vector, c at pi over 3 with b. So, we have taken these vectors. So, it means our measurements are a vector dot sigma tensor b vector dot sigma, a vector dot sigma tensor c vector dot sigma and b vector dot sigma tensor c vector dot sigma. These are the three operators we have observable. One corresponding to P a b, other corresponding to P a c, and the third one corresponding to P b c. And since we have singlet state, we know the expectation value of a dot b tensor b dot c is minus a dot b. This one expectation value is minus a dot c. And this expectation value is minus b dot c.

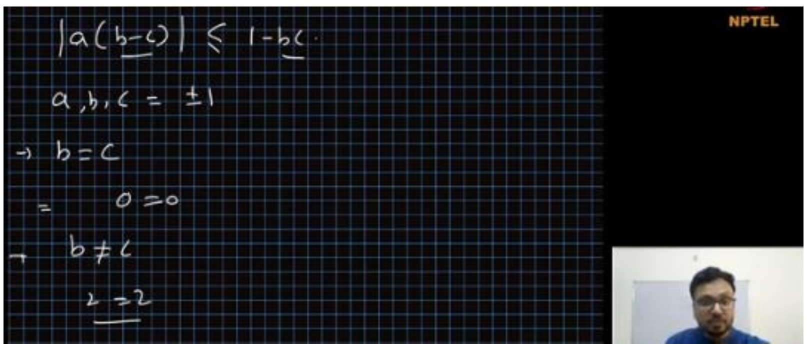
So, now when we have the pi over 3 angle between a and b then this expectation value becomes 1 over 2, this becomes a and c is pi over 2 pi over 3 so this becomes one over two and this here becomes minus one over two If we substitute it up there we get one over two minus minus one over two less than or equal to one minus one over two that is one less than or equal to half which is a clear contradiction. This points towards the conclusion that a local hidden variable model for quantum mechanics is not. So, we can write the same condition in a more simpler way. That is a times b minus c mod less than equal to 1 minus bc.

(Refer slide time: 30:52)



So, a, b and c can take plus minus 1 values. There are the outcomes of the measurement a dot sigma, b dot sigma and c dot sigma. Now, we can see that if b can be equal to c, they can be plus 1 or minus 1, it is irrelevant. But if b is equal to c, this term becomes 0. This term becomes plus one, so one minus one is zero, so zero equals zero. If b is not equal to c, so one is plus one and one is minus one then in that case, the right hand side is after taking the mode it's two and right hand side is also two, so it becomes two equals two.

(Refer slide time: 32:00)

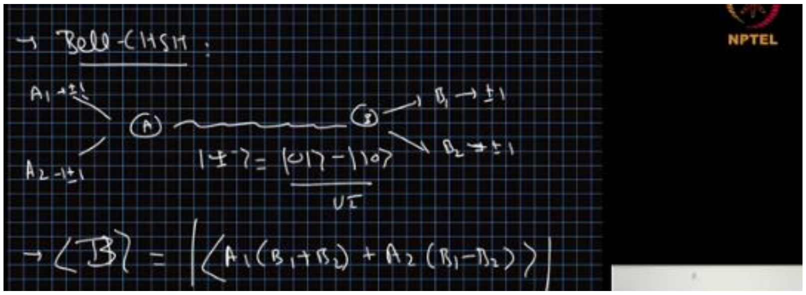


So, this inequality or this equality holds through, when we take the average over all the outcomes, we will get less than equal to, so x equal to 2 or 0. So this inequality holds for the local hidden variable models but it does not hold for quantum system as we shown earlier. Now we will move on to the bell CHSH model inequality. So, in that we take two particles. They are entangled. Let us take the state singlet state again, 0, 1, minus 1, 0 over root 2. On particle A, we can perform two measurements randomly.

Let us say A1 measurement or A2 measurement. And both of them can get us plus minus 1 eigenvector outcomes. Similarly, B1 can get us plus minus one outcome and B2 can get us plus minus one outcome. So, randomly we choose A1 or A2 measurement on particle A and B1 or B2 measurement on particle B and their outcomes are plus minus one, either one of them. So, now we define quantity capital, let us call it Cal B, which is A1 outcome B1 plus B2 plus A2 B1 minus B2.

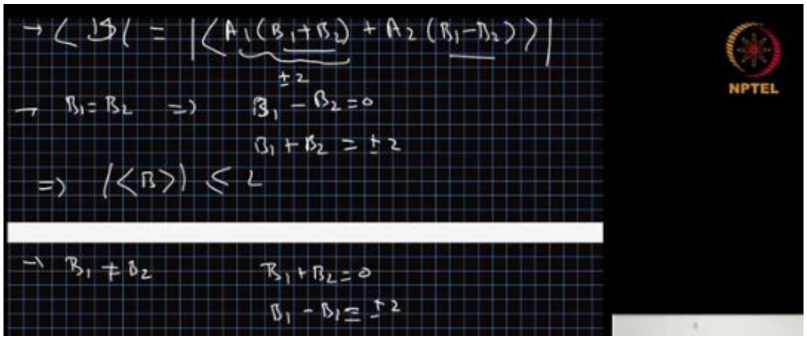
So, whatever outcome we get for A1, A2, B1, B2, we plug it in this, then we get the value here and then we take the average of it and mod of it. Now, we see that if B1 equals B2, then this term will be 0. But B1 plus B2 will be plus minus 2. And A1 can be plus 1 or minus 1. So, this quantity is plus minus 2.

(Refer slide time: 33:48)



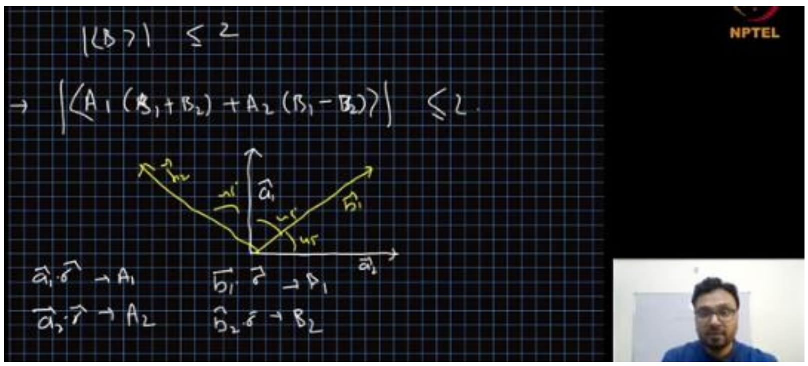
When we take average over it, it will be less than plus minus 2. Then we will take mod, it will be less than plus 2. So, this implies that B mod is less than or equal to 2. If B1 is not equal to B2, then the only choice is its opposite. So, either B1 or B2 is plus 1 and the other one is minus.

(Refer slide time: 34:47)



In that case, the B1 plus b2 is 0. But B1 minus B2 is plus minus 2. Hence, again, B expectation value mod is less than or equal to 2. So, in this scenario A1 B1 plus B2 plus A2 B1 minus B2 average and mod is less than or equal to and this is the case whenever we have a local hidden variable model for the theory for any theory any classical theory or classical statistical theory will give us this result. Now, the contradiction here will be if we take the vector a1, so we are performing measurement of a1 dot sigma which is our which will give us a1 and a2 dot sigma which will give us a2.

(Refer slide time: 36:13)

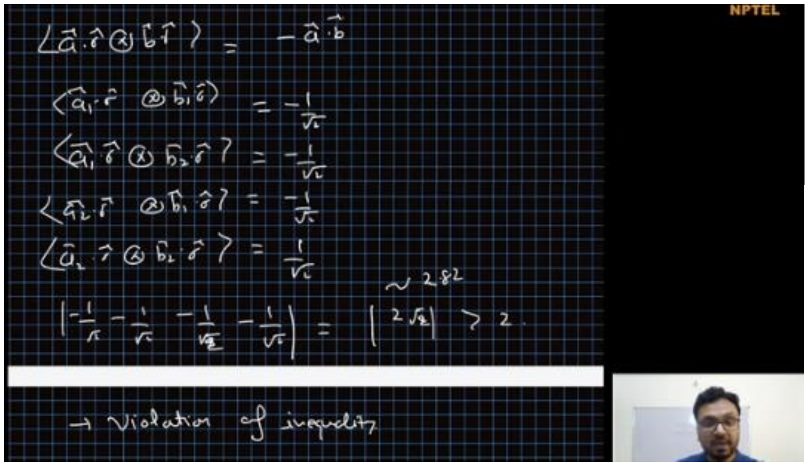


Similarly, b1 dot sigma which will give us b1 and b2 dot sigma which will give us b2. Now we have take them as a1 a2 b1 all are normalized vector and b2 and all of them have 45 degree angle between them. And we know for singlet state that a dot sigma tensor b dot sigma expectation value for singlet state minus of a dot b. So, here we have four terms a1 b1, a1 b2, a2 b1 and a2 b2. So, it means we have a1 dot sigma tensor b1 dot sigma, a1 dot sigma tensor b2 dot sigma expectation value, a2 dot sigma tensor b1 dot sigma and a2 dot sigma tensor b2 dot sigma. There will be minus of a1 dot b1. a1 and b1 have 45 degree angle between them.

So, it is cos of 45 degree which is 1 over root 2. a1 and b2 have 45 degree angle. So, again 1 over root 2. a2 and b1 has 45 degree between them. It is minus of 1 over root 2.

But a2 and b2 have 135 degree angle between them. So, it is minus of minus of 1 over root 2 which is 1 over root 2. And our expression requires the first term plus the second term plus the third term minus the fourth term and mod, which is mod of 2 root 2, which is greater than 2. Hence, it is a violation of the inequality. Let me repeat, by the local hidden variable theory, this quantity should always be less than or equal to 2.

(Refer slide time: 37:50)



But if we choose the vector scale for quantum mechanics, the observable, then this quantity can reach up to 2 root 2, which is close to 2.82, which is much larger than 2. Hence, it violates the inequality and this inequality is called Bell-CHSH inequality.