FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-09
Lecture-24

W9L24_Entanglement Measures

Our next topic is entanglement measures and in this topic we will discuss how to quantify entanglement in a given bipartite quantum system. Once we are given a bipartite quantum system Rho AB, now we have established how to tell whether it's a separable state or an entangled state. Next obvious question is given two entangled states, how can we say which one is more entangled and which one is less entangled or if it is entangled at all or not? So, for that we need to develop the entanglement measures method to quantify entanglement in a given quantum state. So, for a quantity to be an entanglement measure or measure of entanglement, it should satisfy two conditions.

One is that it should be monotonic under local operations and classical communication. Whatever measure we come up with for entanglement in a given quantum state, then that should be monotonic under local operations and classical communication. What does it mean is the entanglement can, since it is a correlation between two parties, we should not be able to create or increase the amount of entanglement in a bipartite system by applying local operations on the two subsystems and by classical communication. Entanglement is a purely quantum resource, quantum correlation. We should not be able to create it with classical correlations and we should not be able to create it with local operations.

This is what it means monotonicity under LOCC, local operations and classical communication. Second, whatever measure we have for entanglement, it should result zero. The entanglement measure for a state rho should be zero for every rho separable. This is an obvious statement, but this must be included for any measure of entanglement that a bona fide entanglement measure should yield zero answer for every separable state. So, any quantity which we use to quantify entanglement should satisfy these two conditions.

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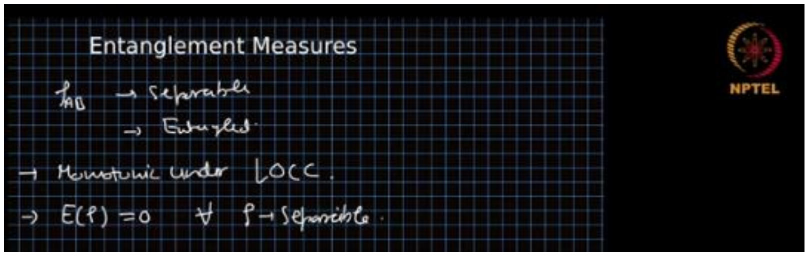

Now, as usual, everything is easy for pure states, so we will start with pure states. So, a pure state psi AB, we can write it as sum over ij, alpha ij, i tensor j. And we can write it in the Schmidt decomposition, sum over n, dn, en, fn, where dn are the Schmidt coefficient. en is an orthonormal basis in HA, the first subsystem and fn is an orthonormal basis in HB, the second subsystem. So, this is Schmidt decomposition. Any bipartite pure state can be brought to this form.

We have proved it and we have used it on several occasions. So, I am not going to go in details about this. Now, from Schmidt decomposition, we can tell whether a given state is entangled or not. That is, if we arrange dn's in the descending order, so that d1 is greater than or equal to d2 is greater than or equal to and so on. Then if d1 is 1 and all other di's where i is not equal to 1 is 0, then the state psi AB can be written as e1 tensor f1 and hence it is a product state or separable state.

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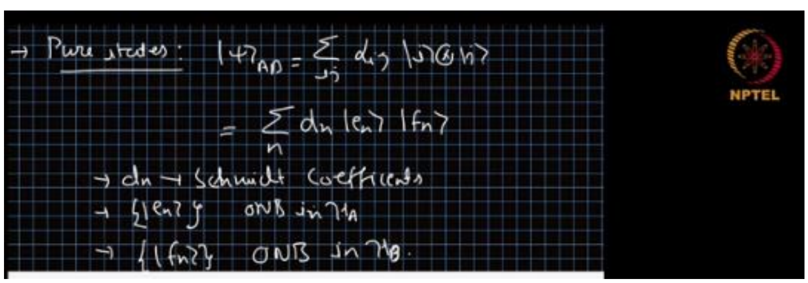

But if more than one d Schmidt coefficient is known zero then we cannot write the state psi as a product of two states of the subsystem, so psi cannot be written in this form. So, it must be entangled. Now let us take a scenario where d1 is 0.99 and d2 is 0.0, d2 is close to 0.01, but it will be 1 minus d1 square square root because d1 square plus d2 square equals 1. So, if that is the case. Then, we have one case. Consider a case where d1 is 0.99 and d2 is square root of 1 minus d1 square and other case where d1 is 0.5 and d2 is not 0.5, let us say 1 over root 2 and d2 is also 1 over root 2.

So, in these two scenarios, which one is more entangled and which one is less entangled? And let us, for the sake of simplicity, let us talk about the qubit system. So, we have only two qubits. So, there are two Schmidt coefficients for a qubit system and those are d1 and d2. Now we can see that when d1 is 1, then it is a separable state.

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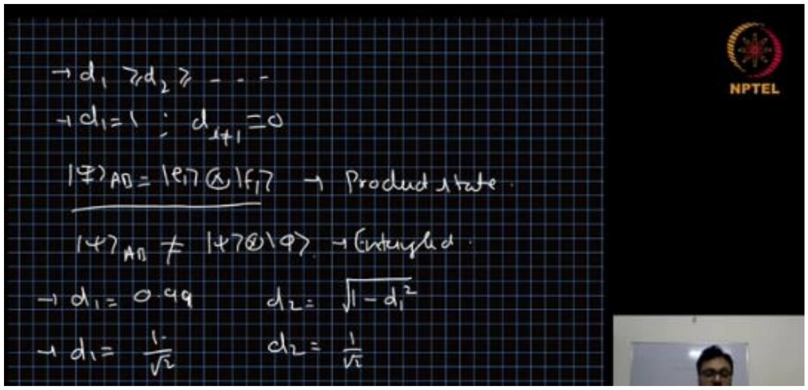

So, if the entanglement is a continuous function and smooth function and all those properties, if it is satisfied, then if you reduce d1 very slightly and increase d2 very slightly, then it is slightly entangled. Just let me repeat the argument here. If d1 is 1 and d2 is 0, then it's separable state. That has been established from the decomposition. Then if d1 is reduced little bit, epsilon, epsilon tending to 0 and d2 is close to epsilon, which is again 0, so then it should be very close to a separable state.

So, it should be a weakly entangled state. On the other hand, when d1 and d2 are same, then it should be very, very entangled state. From this argument we can say that this second state where d1 is equal to d2 is more entangled than the first state. But till now we are only using arguments to understand how we see the quantity of the measure of entanglement or how to quantify entanglement. We recall that from the Schmidt composition, we can write the reduced density matrix rho A, which is d n square e n e n sum over n and rho B, which is sum over n d n square f n. This is the reduced density matrix of subsystem A and this is a reduced rest matrix of subsystem B. So, these are the states of subsystem A and subsystem B when we disregard the other subsystems.

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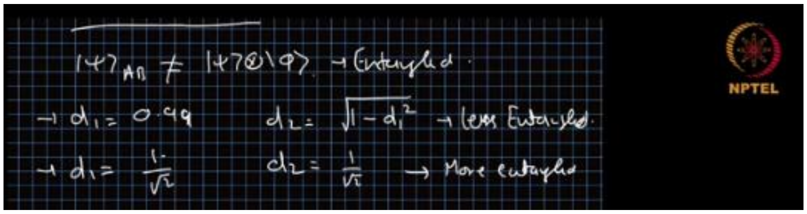

So, now from the Schmitt decomposition, we made a claim earlier that for separable state, if psi AB is a separable state, then rho A and rho B will be pure states. If psi is an entangled state, then rho A and rho B will not be pure states. So, it means rho A square is not equal to rho B, rho A square is not equal to rho A, and rho B square is not equal to rho B. So, it seems like the purity of a state of the reduced density matrix can be a good measure of entanglement. And for qubits, it should be reasonably straightforward to quantify the state rho A for a single qubit can be written as 1 over 2 identity plus r dot sigma, where r vector is the vector of expectation value of sigma matrices, poly metrics. Now, if we recall the concept of the Bloch sphere, then we know that when r vector equals 1, that implies pure state.

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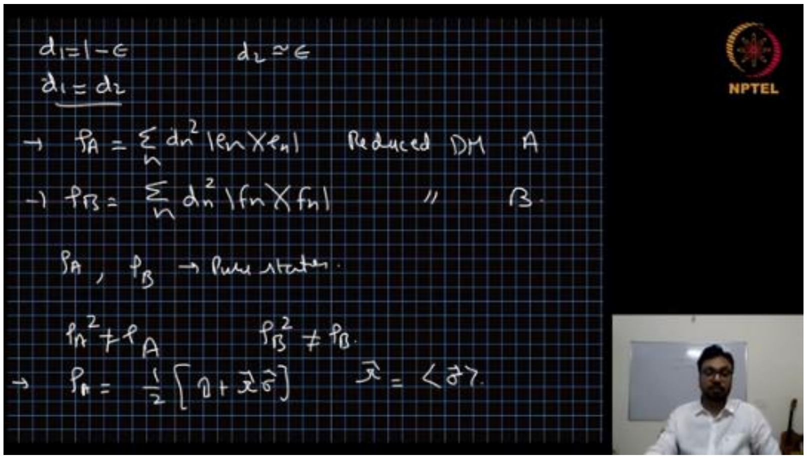

When r vector mode is less than 1, then it's a mixed state. And when r vector mode is 0, then it's a maximally mixed state. So, in that way, the r vector mode, the length of the vector r, can be a measure of purity. If it is 0, then purity is 0. If it is 1, then the purity is 1.

So, in that way, this r vector can also be a valid measure of entanglement. There is another measure which is more prevalent, more used measure. And that measure is called the Von Neumann entropy. In this measure, we have a reduced density matrix rho A and let us say the eigenvalues of this are lambda 1 and lambda 2. Lambda 1 and lambda 2 are the eigenvalues.

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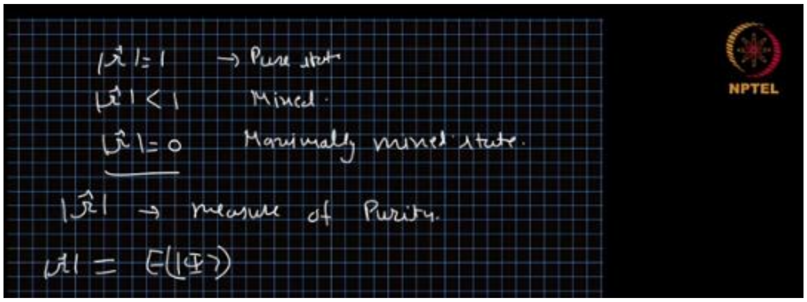

Actually, if we see, if we look at the arithmetic composition, then lambda 1 is d1 square and lambda 2 is d2 square and d1 square plus d2 square is 1. Then the Von Neumann entropy for rho A is defined as d1 square log base 2 d1 square with negative sign minus d2 square log base 2 d2 square. Or it is written as minus rho A log base 2 rho A and trace of that, trace of the rho log rho. In a way, this is also a measure of the purity of the state, rho A, the resistance matrix, because if our d1 is 1 and d0 is 0, then these terms go to 0 because log of 1 is 0 and log of 0 is minus infinity, but it is slowly diverging as compared to 0. So, it's 0 times log of 0, which is 0.

So, for pure state, the Von Neumann entropy as rho A when it is pure is zero. It is S of rho A is maximum when you have d1 square equals d2 square equals half, so when d1 square equals d2 square equals half and in that case our phenomenon entropy as of rho A will be minus half times log of half log base half minus half times log of two base two half and that will become. So, the maximum Von Neumann entropy entropy for a two qubit system is 1 and minimum fundamental entropy is 0. So, in that way, it is a measure of purity, but in a reverse sense. For purity, when the measure is 1, then it is maximally pure. When the measure is 0, then it is minimally pure.

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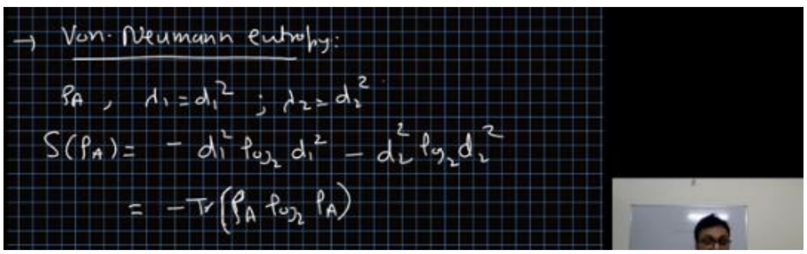

It is the other way. For maximally mixed state, the Von Neumann entropy gives us 1 and minimally mixed state, it gives us 0. So, in that way, S of rho A is a valid measure of entanglement. Not just S of rho A, the spectrum of the two reduced matrices rho A and rho B are the same. They are same d1 square and d2 square.

So, the Von Neumann entropy entropy of rho A is same as the polynomial entropy of rho B when they are the reduced density matrix of a single pure state. So, we will be using the von Neumann entropy as the entanglement measure for pure states. Now we move to the mixed state. Now entanglement measure in mixed state. So, let us say we are given a bipartite entangled state rho and we want to find the entanglement in this state rho.

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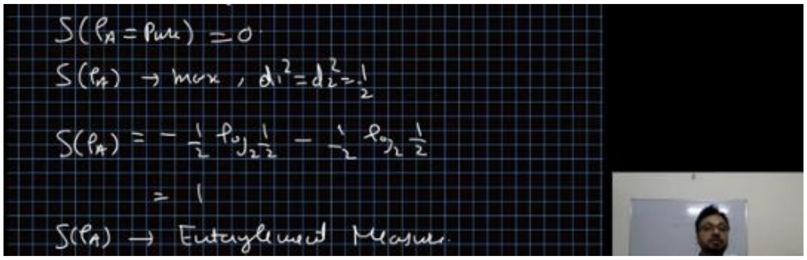

Obvious approach will be, we write, we find a preparation scheme for rho that is probability distribution p i and states rho i pure states psi i psi i where i is from one to some number n. So, these two together the probability distribution p i which is defined as all the elements of this distribution p i should be non-zero non-negative numbers and they should add up to one and normalized state psi i these two together form a preparation scheme for a density matrix rho of an ensemble. Now, we know the entanglement measure in a pure state which is the phoneme entropy in the reduced density matrix of the the corresponding state then we can write the entanglement in rho as sum over i from 1 to n p i and the entanglement in psi i because if we prepare the state of an ensemble in the state rho using the pure state psi i with probability p i then in each preparing each of the psi i will cost us e psi amount of entanglement. So, in that way but in creating this state rho with this particular preparation scheme we have used this much entanglement per quantum system. So, this is entanglement required to make the to prepare the state's rho using the preparation basis p i psi i .

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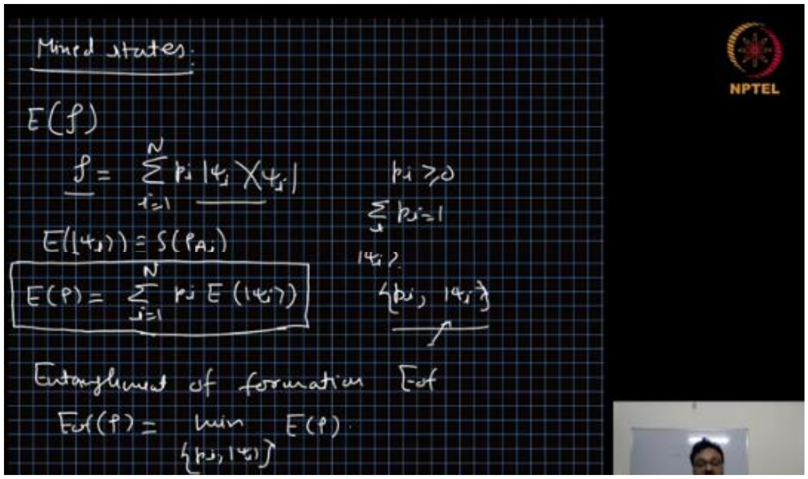

Let me repeat. If we prepare a bipartite mixed state rho with the preparation scheme using p i as the probability distribution and psi i as the pure state, then the average entanglement in rho used to prepare this state is given by this expression. This is the amount of entanglement we have used to prepare the state rho using the preparation method p i. But this is the entanglement cost for this particular preparation scheme. we can have pure we can have separable states also in which the the preparation scheme has many entangled states in that way it seems like the entanglement measure in this session depends on the preparation scheme we are using not the absolute amount of entanglement in the state rho.

So, the entanglement of formation or EOF can be defined as the minimum of entanglement over the distribution p i psi i over all the distribution over all the preparation schemes p i psi i in rho. That is, if we try all the possible preparation schemes for a given density matrix rho and then we find the average entanglement of preparation or entanglement course in each of that preparation scheme, then we minimize over those schemes. If the minimum of that entanglement, that average entanglement will be the entanglement of formation. It means we cannot prepare the state rho in lower entanglement using lower entanglement than the entanglement of formation. So, this is a definition.

This will be the definition for the measure of entanglement in a density matrix rho. This is open-ended definition. We have to try all the preparation schemes and there are infinitely many preparation schemes in order to arrive at the entanglement of formation to find the minimum of the entanglement but for two qubit systems, uh people have proven we are not going to prove that but we will give you the final results for two qubit systems, people have achieved a closed form relation for the entanglement of formation and we will be presenting that next for two qubit system. So, before giving you the expression, let me motivate you a little bit here. If we have a state psi for two qubit system and let us say it can be written as eta tensor phi.

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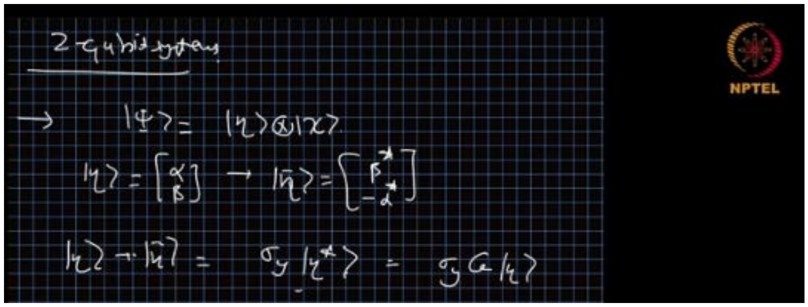

That is, it is a product state or it is a non-entangled state. Eta, it is a qubit state, it can be written as alpha beta. And from here, if there is a unique of the overall phase, there is a unique state eta bar, which is orthogonal to eta. And that we can write as beta star minus alpha star or minus beta star alpha star. So, you can check that eta bar and eta are orthogonal to each other and all the orthogonal state you can find orthogonal to eta, they will be related to eta bar by over by an overall phase that's all. So, if we see how to go from eta to eta bar it will be sigma y times eta sigma y times eta star.

So, if we take the complex conjugate of eta and then we apply sigma y we get eta bar up to our overall phase i think it will be i times, it doesn't matter this is the overall phase i, so it doesn't matter. So, up to overall phase eta bar is the same as sigma y eta star. So, we can see the complex conjugate operation C followed by sigma y will give you a spin flip, okay, the eta is a state of a qubit. We can think of it as a state of a spin half system, then if you want to flip the spin or get the orthogonal state, we take first complex conjugate and then sigma y operation on it so sigma the complex conjugation followed by sigma y operation is a spin flip operation. So, it means if we go to psi tilde state, which is complex conjugate followed by sigma y on first qubit and complex conjugate followed by sigma y on the second qubit, then psi tilde will be orthogonal to psi if psi is a separable state, for separable states or product state. If Psi tilde is not orthogonal to Psi then the state must be entangled. In fact, how far they are from each other or how non-orthogonal they are that can serve as a measure of entanglement.

So, if we define as quantity C, which is Psi Sigma y tends to Sigma y Psi star, then this can be a measure of entanglement. And this quantity is called concurrence. Now, we extend the same idea to density matrices. So, how to calculate concurrence in a density matrix? So, if we have a density matrix rho, from here we find a matrix R defined as rho times sigma y tensor sigma y, rho star sigma y tensor sigma y.

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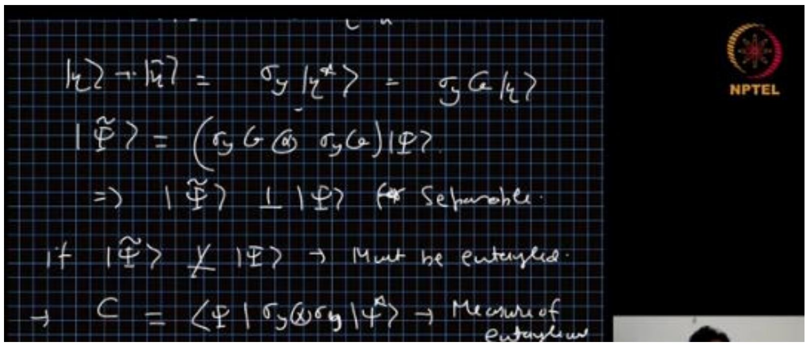

We start from rho, we calculate a matrix R, which is defined as rho times sigma y tensor sigma y times rho star times sigma y tensor sigma y. It has been proven that this particular matrix, if rho is a valid density matrix, then R has eigenvalues given by lambda 1 square, which is greater than lambda 2 square, lambda 3 square, lambda 4 square. It means all the eigenvalues of R are real and positive. So, we are putting them just in ascending order so that we know which one is which because that is required here. Then the concurrence of rho is defined as maximum of 0 or lambda 1 minus lambda 2 minus lambda 3 minus lambda 4. Just be careful that here we had squares and here there is no square.

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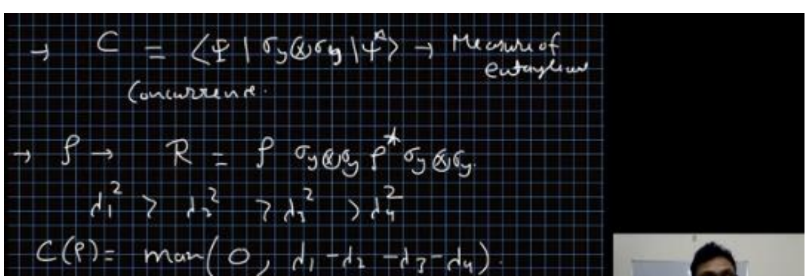

It's not a typo. It is intentional. That's why it was important that we have the eigenvalues of R to be real and positive, so that we can take square root of it. So, lambda 1 is the square root of the largest eigenvalue. Lambda 2 is the square root of second largest, third largest, fourth largest.

And this is how we calculate the concurrence. This can be a valid measure of entanglement because the concurrence satisf the conditions we put forward in the beginning that is the concurrence is monotonic under local operations and classical communications and concurrence is zero for all the separable states. Now, this is not the entanglement of formation. The entanglement of formation in rho is defined by a function of 1 plus square root of 1 minus C square over 2 where this function H is called Shannon entropy. It is defined as H of x as minus x log x minus 1 minus x log of 1 minus x. Log is always base two here.

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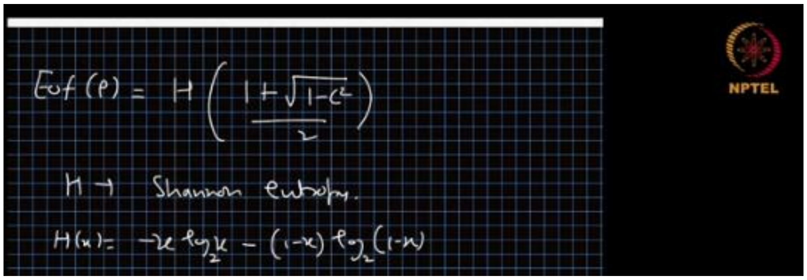

And in that way, we can calculate for a 2 qubit system, the entanglement of formation in a closed form. This is entanglement of formation, but if we don't want to calculate one step forward, this concurrence itself is a valid measure of entanglement. And most often, this is what is used as a measure of entanglement for two-qubit system.