FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-08
Lecture-23

Entanglement: Introduction

In this lecture, we will talk about entanglement and we'll start with the definition. Although we have mentioned entanglement and we have discussed them in some detail earlier when we were talking about the states and the operations and maps, here it will be more formal and more systematic approach to entanglement. So, we start with pure states. So, a given state of a bipartite system psi AB, which is sum over ij, alpha ij, i tensor j, where ij is computational basis. This will be called an entangled, this state if it can be written as some eta tensor chi, where eta is a state from the first Hilbert space and chi is a state from the second Hilbert space, then this is called a product state.

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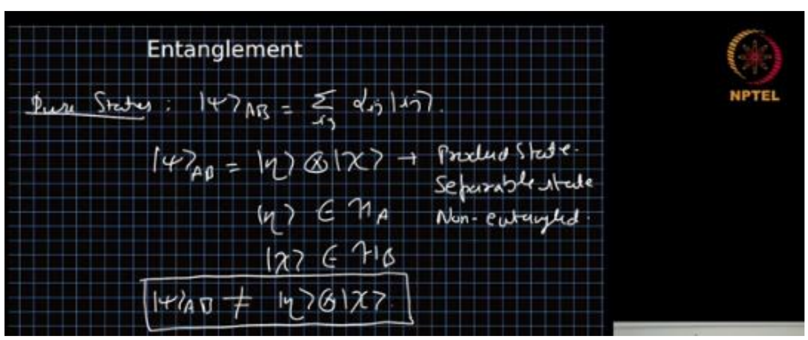

It has other names also. It is a separable state and it is non-entangled state. All these words are synonyms. So, this is the definition of entanglement that if a state cannot be written in this form, then it must be entangled. This is the entanglement definition for a pure state.

But when the states are mixed, then this definition is slightly more general, more elaborate for the mixed states. Rho, where rho is the state of a composite system H a tensor the H b. If rho can be written as rho A times rho B then it's called product state, it is separable, it is non-entangled as usual. If it can be written as sum over i p i rho A i times rho B i where pi's are positive semi-definite and sum over i pi is 1 and rho A i is a valid state for subsystem A and rho B i is a valid state for subsystem B for all i's. Then it is called a separable state. So, we have two things here, one rho being rho A and rho B, which is a product state, another rho A, rho being a convex mixture of rho A and rho B. So that's the separable state. So, by default, a product state is a separable state but a separable state may not be a product state means a state which is written in this form for multiple non-zero p's may not be we may not be able to find a decomposition of that rho in terms of rho A tensor rho B, okay, so this is the definition of a separable state then if a state cannot be written in the separable form then it must be entangled.

So, the definition is that if we can find a separable decomposition of a given state rho which is a state of a bipartite system and the separable form is the convex mixture of the product states rho A and rho B, then it must be an entangled state. Now, what are the applications of these entangled states? So, they are used whenever we talk about quantum supremacy, whenever we talk about quantum advantage in quantum communication and quantum computation. Entanglement plays a very significant role there. So, some of the few of the applications of entanglement are as follows.

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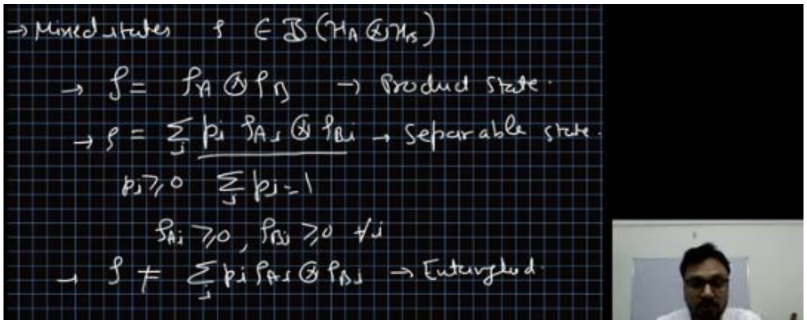

Entanglement is used for quantum key distribution, which is the protocol for sharing a key to communicate securely between two parties. So, there are several protocols which depend heavily on entanglement. Then we can have teleportation, And super dense coding. We have entanglement swapping. I'm just mentioning this application without going in detail and quantum computation and communication.

We have discussed communication so we just write here computation. The quantum advantage in quantum computation comes generally from the entanglement properties of the states of composite systems. Out of all these applications, I will go in detail for the teleportation scheme. We will discuss teleportation next in detail. So, teleportation is a process in which we can send the state of a quantum system from one physical location to other physical location without sending the quantum particle through the space.

So why it is important is because there is a theorem in quantum mechanics which we did not discuss in this course but that is a part of a, very much a part of quantum information and computation course that is called the no cloning theorem in which it has been proven that if you are given a quantum system in an unknown quantum state then there is no way we can find out that state for from the single quantum system and replicate it So we cannot start with a with single quantum system in an unknown state psi and we get two quantum systems in the same state psi and psi this is not possible. This is called no cloning theorem. So, it means if we are given a quantum system in an unknown state psi and we want to send it to some other distant location and we are not allowed to send the quantum system itself to the space, then there is no way of sending it in by any classical means. We cannot copy the state of the quantum system, we cannot fax it, we cannot do email, we can cannot send the state via email. Then the quantum teleportation becomes very important. Quantum teleportation does exactly what is required, that is sending the state of the quantum system from location A to location B. So, the protocol starts like this.

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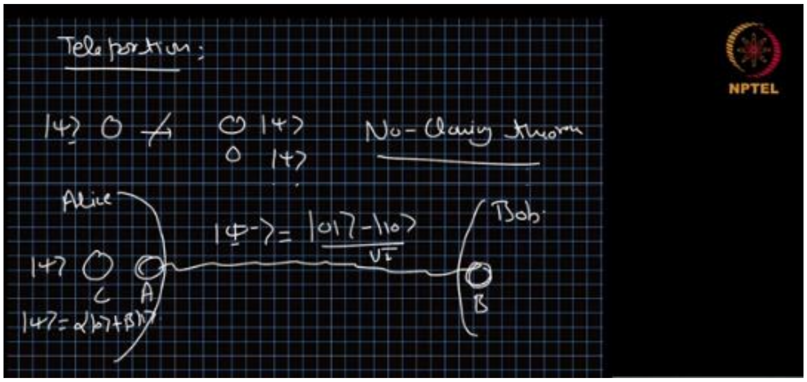

We are given a quantum system in the state psi. Let us call it C. It will be clear why we are calling it C, and then we want to send it from the lab A or Alice lab to lab B, that is Bob's lab. So, in order to send this information from Alice lab to Bob lab, we need a quantum channel between the two labs. We need a network between the two labs which can assist us to send this information from Alice lab to Bob lab. And this cannot be done with classical means, we cannot copy, because we cannot copy the state of the system C if it is unknown state.

Then this quantum channel is realized with a two qubit entangled state. We are talking about qubit system here, so this channel is realized with two qubit entangled state. For the sake of concreteness we start with the state psi minus, it's called, and that is zero one minus one zero over two. So, this is the state of the two qubit, qubit A and qubit B. And the state of these two qubits is 0, 1, minus 1, 0 and over root 2. This is an entangled state. And this is what is used to teleport the state of C, qubit C to qubit B. Let us say the unknown state was psi alpha 0 plus beta 1. Then we have to send this alpha and beta to Bob's lab without knowing what are alpha and beta.

So, the protocol goes like this. The initial state of the total three-qubit system CAB is psi, that is the state of C, tensor, the state of A and B, that is psi minus. So, we expand it, we get alpha 0, 0, 1 minus alpha 0, 1, 0 plus beta 1, 0, 1 minus beta 1, 1, 0 over 2. This is the state of the three qubit system with the total three qubit we have with us in this protocol. See in Alice lab we have two qubits A and C. So, what are the bases we can have in that lab, one is the zero zero zero one one zero and one one basis this is the computational basis. Other which is more practical, more useful for us for this particular protocol is the following basis.

Phi plus minus, that is defined as 00 plus minus 11 over root 2. And psi plus minus which is defined as 01 plus minus 10 over root 2 and this basis is called Bell basis. And this is the basis in which all the four states are highly entangled states. Why we are saying this thing is because we want to perform the measurement on A and C, these two qubits together in the bell basis. So, now let me write it again. Psi CAB, the three-qubit state we had, that is alpha 001 minus alpha 010 plus beta 101 minus beta 110 over root 2.

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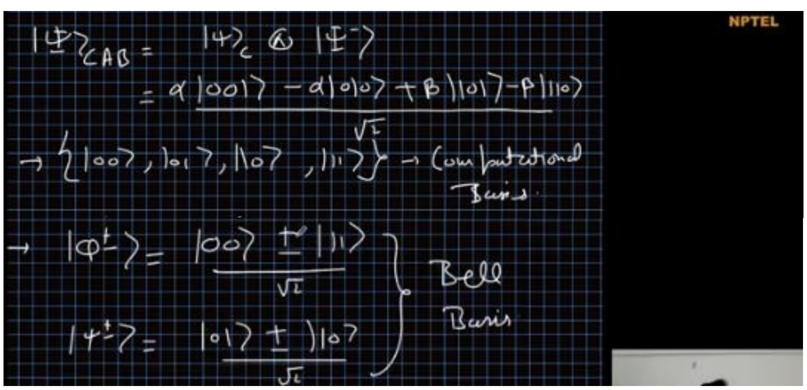

Out of these, the first two qubits, first two qubits are the state of the C and A. And last one is the state of the B. So, now we convert the state of first two. We write the state of first two in the Bell basis. So, in terms of Bell basis, that is phi plus plus phi minus over root 2. And 0, 1 can be written as psi plus plus psi minus over root 2. 1, 0 can be written as psi plus minus psi minus over root 2.

And 1, 1 can be written as phi plus minus phi minus over root 2. So, we can write the computational basis in terms of the bell basis. I'll bring it here, so that it's a little easier to work with. Now we substitute 0 0 with the bell basis and 0 1 and 1 0 and 1 1 and we get psi CAB as phi plus plus phi minus over root 2 and there was already over root 2. So, I just combine them and I get over 2, this times 1. This 1 is the state of the Bob, we keep it. There was alpha also. Let me not forget that.

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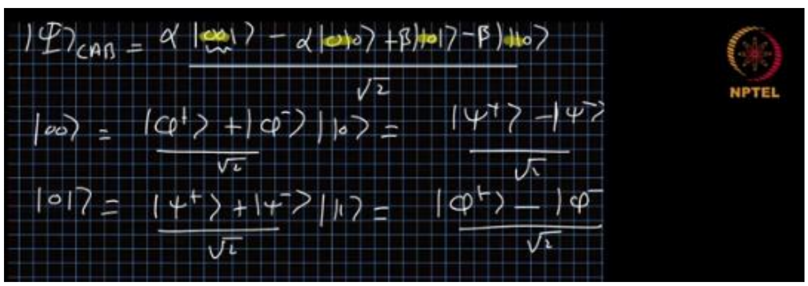

One over two alpha and minus alpha times zero one zero one is psi plus plus psi minus over root two and root two we have taken that out tensor zero state of the Bob plus beta times one zero one zero is this one, it becomes psi plus minus psi minus over root two the root two has gone out this times the state 1 of Bob minus beta times phi plus minus pfi minus over root 2 times 0 state. So, now what we have done is we have just instead of using computational methods for C and A qubit, we have used the Bell basis. Collect the terms properly and we get phi plus we get with alpha 1 minus beta 0 plus phi minus term comes with alpha 1 plus beta 0 plus alpha plus psi plus minus alpha 0 plus beta 1. And minus psi minus becomes alpha 0 plus beta 1. So, this is our state, where the first two qubits that is C and A is written in the, is represented in the bell basis and the third one is in the computational basis. So, now if you perform measurement on the bell basis that is phi plus minus and psi plus minus for C and A qubit, then outcome will be either phi plus or phi minus or psi plus or psi minus.

If this is the outcome for CA qubit C and A qubit then the corresponding state of the Bob after the measurement will collapse to alpha 1 minus beta 0 that is what we get with phi plus, so this is the state of Bob after we get a projection of phi plus in CA, if we get phi minus, then the state on the other side is alpha 1 plus beta 0. For psi plus, we get minus alpha zero plus beta one and for psi minus we get alpha zero plus beta one. Now, from here, it seems like we have managed to send the alphas and betas from Bob's side by just choosing to perform measurement in the Bell basis on C and A. So, the information which we wanted to send to Bob's side has been sent, that is alpha and beta. But the state psi is only when we get psi minus in the measurement. This is alpha 0 plus beta 1, that is psi.

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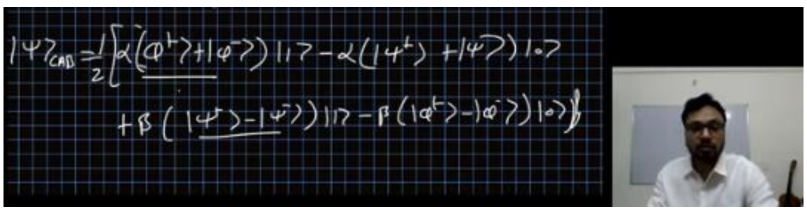

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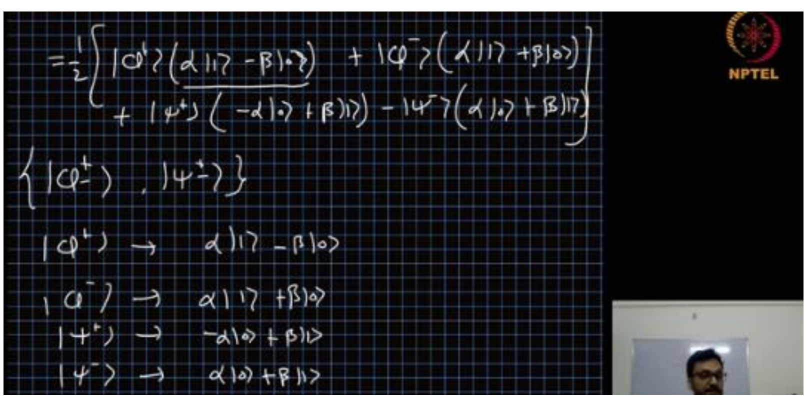

All other states are not psi. But let us see if they are related to psi or not. For example, this state can be written as minus i sigma y times psi. This state is related to psi state by sigma y transformation. This state is related to sigma y with sigma x transformation, psi state.

And this state is related to sigma y minus sigma z transformation. So, what we have seen so far is Alice performs measurement on C and A qubit in the bell basis and depending on the outcome she gets that is either phi plus or phi minus or psi plus or psi minus the state of the Bob side will transform to psi followed by an operation. And fortunately, this operation is unitary so it's invertible, all these operations i times minus i times sigma y sigma x minus sigma z all of them are unitary so the protocol dictates now that if Alice gets phi plus state, then she will communicate. We can say classically communicate, with to Bob that he should perform the sigma y transformation inverse sigma y transformation. So sigma y square is identity, he just need to perform sigma y on his state minus i sigma y psi and he will get psi back with some phase that overall does not matter. If Alice gets a Phi minus state, she communicates to Bob that it's Phi minus, then Bob will apply Sigma X operation and he will get the state Psi.

If Alice gets Psi plus, then he will apply Sigma Z operation. Then he will get the state psi back and if she gets psi minus then he does not need to implement any operation and he will get psi back. In that way by performing the Bell measurement on C and A and communicating the results of the outcome they can send the unknown state of a quantum system C to quantum system B. This is how the teleportation works but some people have a confusion or some people think that this may mean that we are communicating faster than the speed of light. But this is not true because unless Alice performs the measurement, send the, informs Bob's about the outcome of the measurement. The state of Bob's is Sigma Y Psi outer product Psi Sigma Y plus Sigma X Psi Psi Sigma X plus Sigma Z Psi Psi Sigma Z plus Psi.

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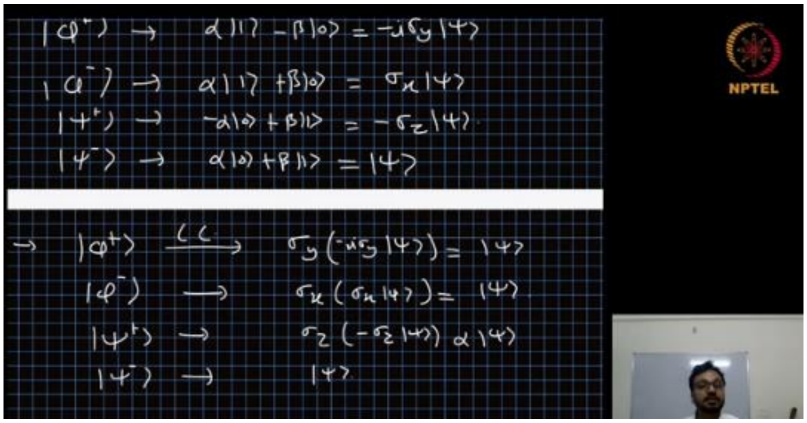

The probability of collapse of all of these was 1 by 4. So, on Bob's side, he gets all these states with equal probability, 1 by 4, and equal mixture of them. So, his average state on Bob's side, rho B, Bob's average state will be this. And this is nothing but half times identity. Unless Alice conveys classically the outcome of her measurement to Bob, in Bob's lab, the average state of the quantum system is just half times identity.

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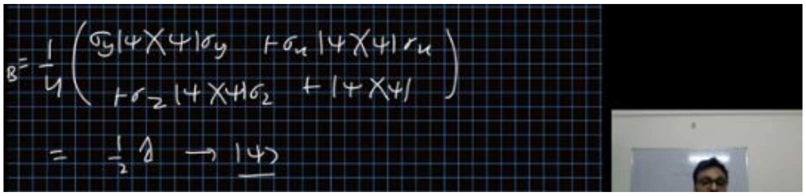

But upon the classical communication Bob can transform the state of individual quantum systems by their corresponding unitary and can get this state to be psi. So, the teleportation does not work without classical communication and classically nothing can travel faster than speed of light. So, in that way teleportation does not allow the communication faster than the speed of light but it can help to send an unknown state of a quantum system from one place to other. Another thing to remember here is we are sending the state of quantum system from C to B, from one location to other location. But in this process, the state of C has been demolished, has been removed completely, erased completely, and the information has been transferred to Bob.

It is not that after the teleportation protocol, we have two copies of the same state, because that is the violation of No cloning theorem. So, here we can see very clearly that after the measurement, the qubit C has lost its identity and it exists in an entangled state with qubit A and that entangled state is either phi plus or phi minus or psi plus or psi minus depending on the measurement outcome. So, this is the teleportation protocol and here we can see that without entanglement between Alice and Bob, we cannot perform the teleportation. In fact, if this is not maximally entangled state then we will not be able to reproduce the state psi completely. So, this can be a exercise for motivated students to start with a state which is not maximally entangled and see what outcome we get on the Bob side after the teleportation protocol and then they try to find the fidelity between the input state of the C qubit and the output state on Bob side and that fidelity or average over all the input states is called the teleportation fidelity. Next, we will talk about the optical implementation of the teleportation.

So, here we will talk about single photons with polarization. These are our qubits so there is a process called spontaneous parametric down conversion. This is a non-linear optical process in which, and this is realized by non-linear material in which we have a non-linear crystal, generally it is a BBO crystal, so in this this is non-linear crystal. So the property of this non-linear crystal is in right conditions, if we pump it with some frequency omega tau the outcome will be two photons in frequency omega over two and if we and the conversion depends on the polarization and many other factors so in certain situations we can start with a single photon of frequency omega and we get two photon out. And these two photons can be made in an entangled state and generally the entangled state is HV minus VH over root 2. So, with SPDC in this spontaneous parametric down conversion process, we can start with a single photon and get two entangled photonss out.

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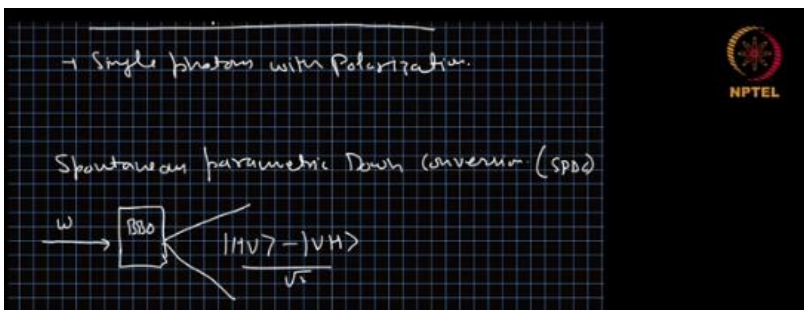

And they are entangled in the polarization basis, people have realized other kind of entanglement also not in just polarization but here for the sake of our teleportation protocol and for this discussion we will stick to the entanglement in the polarization. So, this is how we can have the entangled qubit to perform, which is needed to perform the teleportation protocol. So, we have a source, we just call it a source and it is producing the entangled pairs of photons. They send state psi minus which is HV minus VH over root 2. This is Alice side, this is bob side so our task is that we are given another photon in an unknown state alpha H plus beta V and we want to teleport the state of this photon to both side so according to teleportation protocol we need to have a measurement setup here which will project the outcome of these two photons to the Bell measurement. And once we get the result here, Alice will convey to Bob that the given photon was, this measurement result was phi plus or phi minus or psi plus or psi minus. And Bob will get the state of the given photon C that is alpha H beta B in this process.

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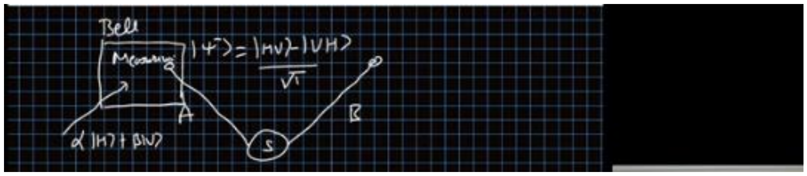

So, the whole task now is to design this measurement setup. This measurement setup or Bell measurement works as follows. So, we use a beam splitter, this is balanced beam splitter, we use a beam splitter which is balanced beam splitter and the beam splitter operation can be written as one one one minus one one over root two. works as follows so we use a beam splitter this is balanced beam splitter we use a beam splitter which is balanced beam splitter and the beam splitter operation can be written as one one one minus one one over root two So, this beam splitter is independent of the polarization. So, the action of this on the polarization can be studied like this.

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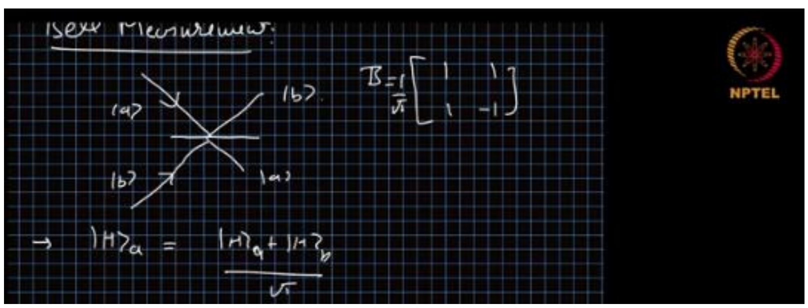

So, we represent there are two modes, mode A and mode B input and mode A and mode B output. So, H in A will go as H in A plus H in B over root 2. Similarly, for V, this is true so let me write just P here for polarization. Any polarization we have. Similarly, polarization in B will be polarization in A minus polarization in V same state over root to simplify the calculation we will introduce field operators or photon creation operators. So, we say HA dagger acting on vacuum is proton with polarization H in mode A. Similarly, we have HB dagger and we have VA dagger and we have VB dagger. And when they act on vacuum, they will give us a single proton in polarization H in mode B, single proton in polarization vertical in mode A and single proton with polarization V in mode B.

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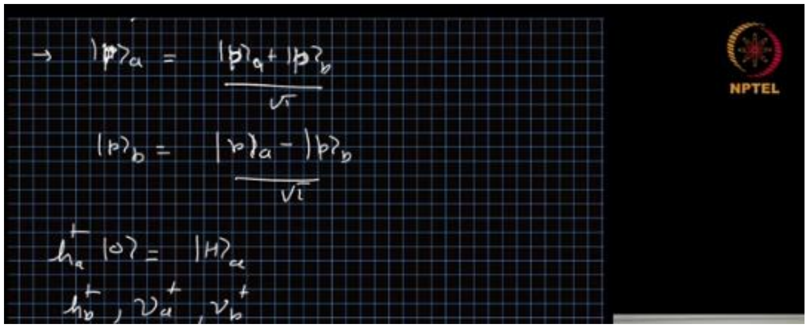

So, now action of our beam splitter can be written as HA dagger goes to HA dagger plus HB dagger over root 2, HB dagger, H or V, whatever, same polarization. HA dagger minus HB dagger over root 2 and the same is true for the vertical. So, what we want is that we want to perform Bell measurement and Bell measurement requires to perform measurement on HV plus minus VH over root 2 and HV HH plus minus VV over root 2. So, one photon is in H, other is in V or first one is in V, other one is H and we perform measurement. So, if we are given two photons, let us say it is in HH plus VV over root 2.

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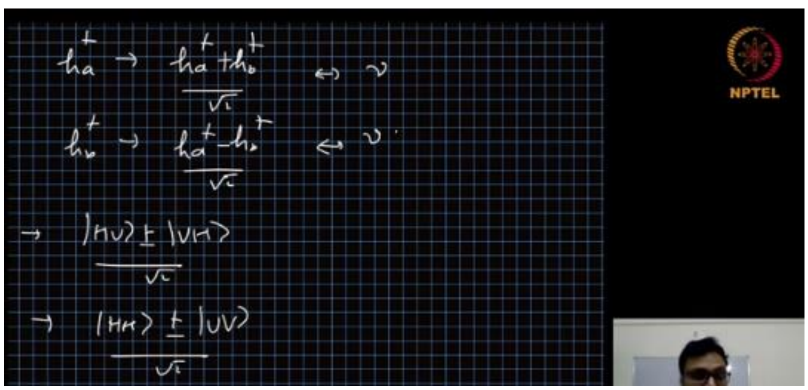

This can be written as HA dagger, HB dagger plus VA dagger VB dagger over root 2 acting on vacuum. This is the input state and then we put it on the beam splitter, after beam splitter the state will be HA will transform to HA dagger plus HB dagger over root 2 and there is a root 2 outside so we can put 1 over 2 times HB dagger which goes to HA dagger minus HB dagger plus VA dagger plus VB dagger VA dagger minus VB dagger acting on vacuum. So, when we put this state on a beam splitter, we get this after the beam splitter, we take the product it becomes HA square dagger plus HB dagger square minus HB dagger square plus VA dagger square minus VB dagger square even vacuum and there is a root factor additional here. Now ha dagger square is one over root two, two photons in h polarization on in mod a and zero in the other one so the whole state can be written as one over two, two photons in h and zero minus two zero two photon in h in v mod minus two photons in v in a mod and zero in the other plus minus 0, 2 v, photon in v mode. So, this is the state we have.

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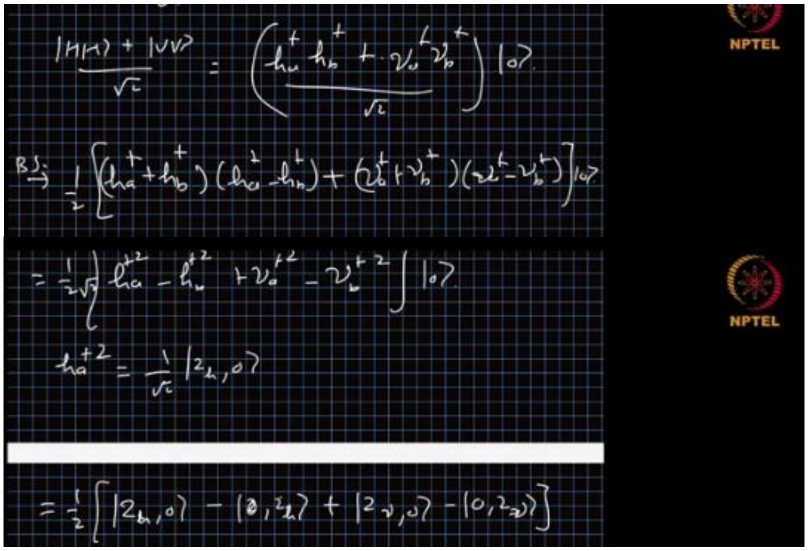

Now, what do we see here is we have a beam splitter. We put two photons. They are in the phi plus state and the outcome is either two photons in horizontal in one path and 0 in the other. Or two photons in v, another is 0 or 2H, 0 and 2B, 0 are the four states we are getting as the outcome. So, it means if we put detectors in the two outcomes, it will never click simultaneously, either this will register photons or this detector will register they will never give a simultaneous click. So, it means if we have this detector, this setup that a beam splitter followed by two detectors and we wait for simultaneous click, then Phi plus will not give simultaneous click.

In fact, if you do repeat the same calculation, Phi plus will not give simultaneous click, Phi minus will not give simultaneous click and Psi plus will not give any simultaneous click. We will get simultaneous click coincidence count only for Psi minus. All these three will give you two zero or zero two clicks and psi minus will give you one one. There will be exactly one photon in one outcome and other photon in other outcome. So, it means in the whole teleportation protocol we are trying to do with photons, we put a beam splitter.

We have the C photon here. We put two detectors. And we wait for the one one click. Whenever we have one-one click, one here and one here, we know the state has collapsed to psi minus. When we have two-zero or zero-two clicks, then we do not know whether it was phi plus or phi minus or psi plus.

So, in that way, one in four cases, we will be successful and we will know exactly what measurement we have performed. In other three cases, we will not be successful. So, whenever we are successful, we communicate to Bob that the measurement was successful and Bob does not need to do anything because we saw earlier that if it is psi minus, then the state Bob achieved is the original state of the C qubit. This is how the first demonstration of teleportation protocol in photonic system was performed. This probability of success which is 25 percent, one in four at the moment can be improved a little bit by implementing some more complicated setups but at the moment this is the simplest one and this is how one can achieve a teleportation as a proof of principle in photonic systems. Since now we have established the importance of entanglement it is important to tell whether a given state is entangled or not.

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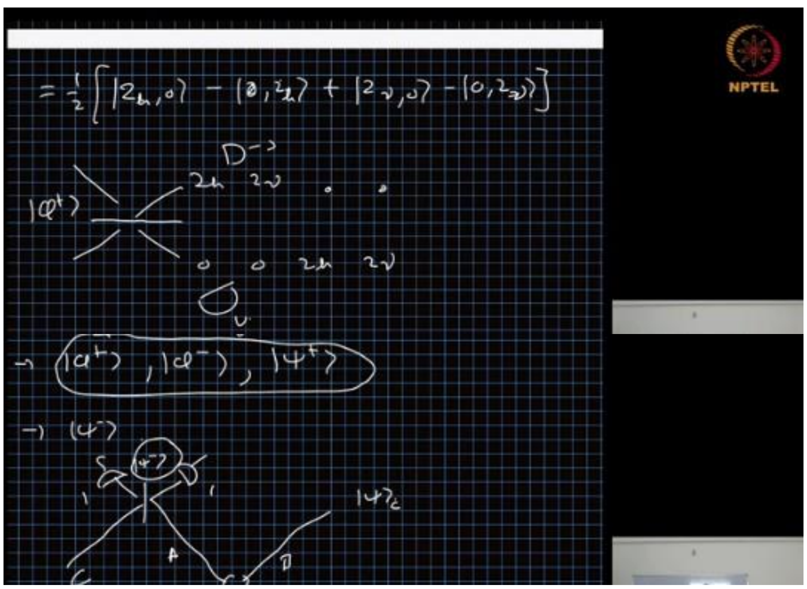

So, if the question is, if I give you an arbitrary state, pure state or mixed state, so how can we say whether this is an entangled state or a mixed or a separable state? So for pure cases, let pure state case start with pure state. So, we are given a state Psi AB, which is sum over ij, alpha ij, ij, where ij is the computational basis. From here, we know we have discussed Schmidt decomposition. So, we can write the state Psi as sum over n dn en fn where dn are called Schmidt Schmidt coefficient. en is an orthonormal basis in the first qubit Hilbert space, first system in Hilbert space.

fn is an orthonormal basis in the Hilbert space of the second system. In this way, we can represent any state of a bipartite system in the Schmidt decomposition. If only one of the Schmidt coefficients is non-zero and all other are zero, then it is a separable state. If more than one Schmidt coefficient is non-zero, then it is an entangled state. So, this is the criteria for testing a pure state whether it is separable or entangled.

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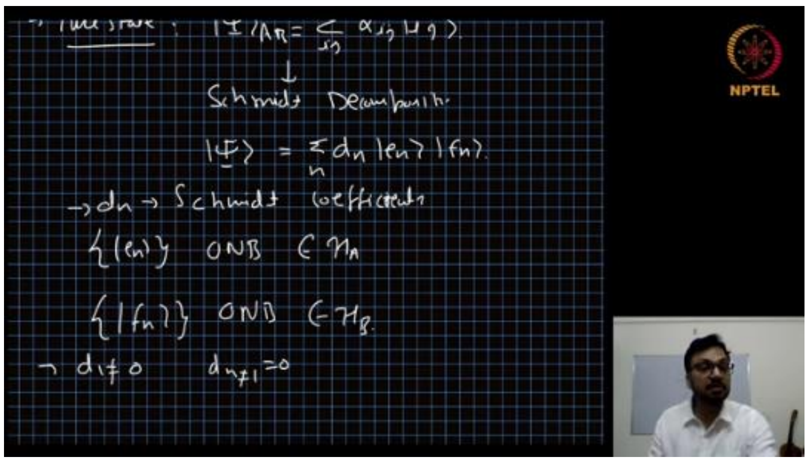

This is a very straightforward way. There is a well-defined algorithm to calculate the Schmidt coefficient and Schmidt decomposition. Other criteria, other method to tell whether a given state, pure state is an entangled state or separable state is to find the reduced density matrices rho A, which is trace over B, psi, psi. If rho A square is not equal to rho A, this implies that psi is entangled. It means if the reduced density matrix of the bipartite system is a mixed state, then the pure state was entangled. Similarly, it goes for B also, the trace over A psi psi, if rho B square is not equal to rho B, then it is an entangled state.

Rho B square, if rho B square is equal to rho B, then it is a rank one projector, it means it is a pure state so rho B square not equal to rho B implies that rho B or rho was mixed state. So, in that way witnessing entanglement in pure state is reasonably straightforward and very simple but this becomes a challenge when it comes to mixed state, so witnessing entanglement in mixed state. There are few tricks to find out whether a given state is entangled or not. For example, if our state given state rho can be written in the form pi, rho Ai, tensor rho Bi, that is, if the given state is separable state, then if we take the transposition over one of the subsystems in this state, this will also be a valid state of the quantum system, of the bipartite system. It means the partial transposition on a separable state will always yield a valid state of the bipartite system. It is rho TB will always be positive, semi-definite, and it will have trace one.

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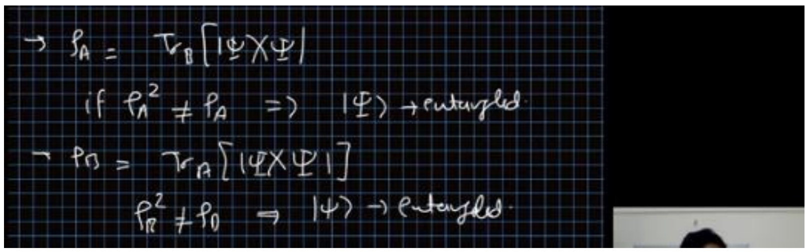

But sometimes some entangled state will map to non-valid state upon partial transposition. This is a signature that the underlying state is an entangled state. So, this whole criteria is called positivity under partial transposition. So PPT, it stands for. So if a state remains positive under partial transposition, then it might be a separable state. But if it violates PPT, then it must be an entangled state.

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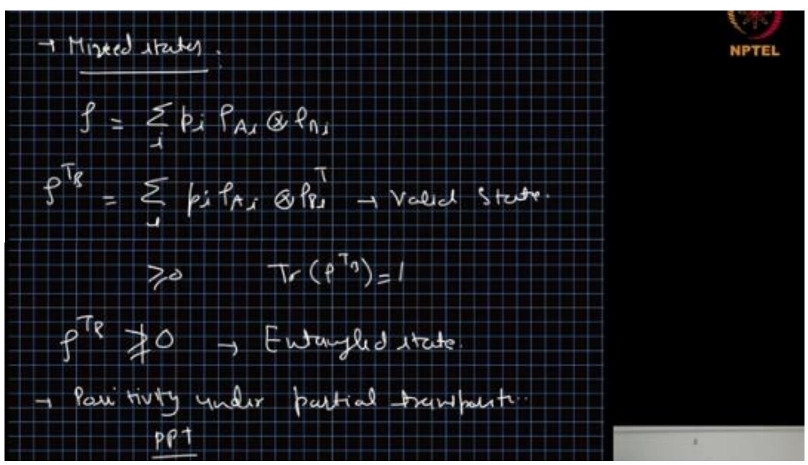

So, the state which changes sign or which maps to non-state, they are entangled state upon partial transposition. So, we can see the whole graph as we have the set of all the states. Then we have set up all the entangled states and inside that there is a set of all the states which violates PPT criteria, so they map to non-states upon taking the partial transposition. So, these states are negative under partial transposition. These are entangled, but they do not change sign over partial transposition and they are separate.

For two qubit system, this area vanishes. So, for two qubit system, negativity under partial transposition or positivity under partial transposition is necessary and sufficient for separability. It means all the entangled states upon partial transposition will map to norm state and all the separable states upon partial transposition will remain valid state. So, partial transposition is the only criteria we need to check to give it entanglement. There is other more general criteria.

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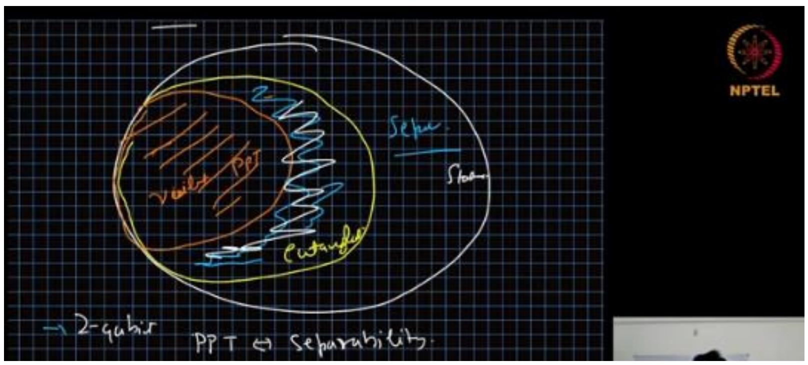

Partial transposition works because transposition is a positive map, but it is not a completely positive map. So, in general, to witness entanglement, we need positive, but not completely positive map. So, that if we apply this map, let us call it omega map. If we apply this kind of map which is positive but not completely positive on a subsystem or a composite system, then all the separable states will map to separable states. But there might be some entangled state which will not map to any physical state.

In that way, these kinds of maps can help witnessing the entanglement. Third is the entanglement witness operator. Integral witness operators are defined, operator W is defined in the following way. That this is the operator W, which acts, which belongs to the set of operators acting on HA tensor HB. It is the operator acting on the two Hilbert spaces.

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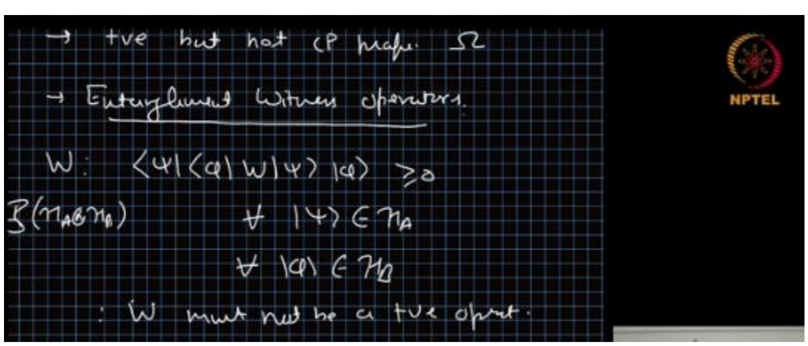

Then it must be positive for all the product states, for all the psi in HA and for all the phi in HB. Not just that, the expectation value of W operator should be negative for at least some states or in other sense the W operator, W must not be a positive operator. So, we have an operator which is not a positive operator, but it is positive for all the separable states. So, in that way, whenever we have a state for which it is not positive, the expectation value of W is not positive, then that state must be an entangled state. Let me repeat, an entanglement witness operator is an operator which is not a positive operator, that is, it has at least one non-positive eigenvalue, but it is positive for all the separable states.

That is, the expectation value of W for all the separable states is positive. So, these operators can be used to witness entanglement. One example of such operator is the swap operator. So, swap operator W is defined like this, sum over ij, ij outer product ji. So, what it does is, if we apply it on some psi tensor phi state, we get phi tensor sub state.

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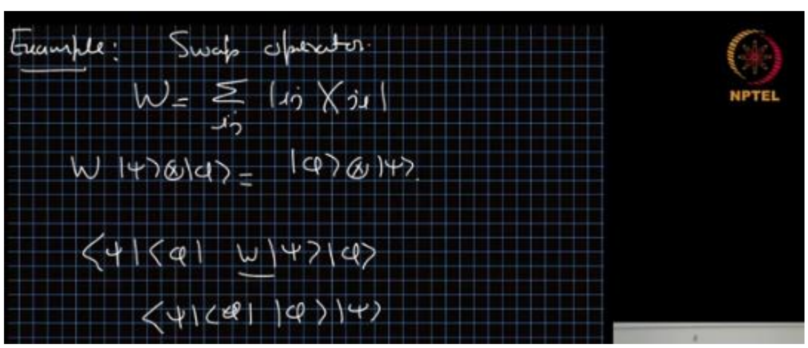

So, the expectation value of W in the product state, psi tensor phi, W psi tensor phi. Now, W acting on psi tensor phi will give us phi tensor phi. And on the other side, we have psi tensor phi. And the inner product will be psi phi mod square which is always positive and the swap operator has always a positive expectation value for all the separable states, for all the product states. If we write the operator W for two qubits, it will be a four by four matrix and it is 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1.

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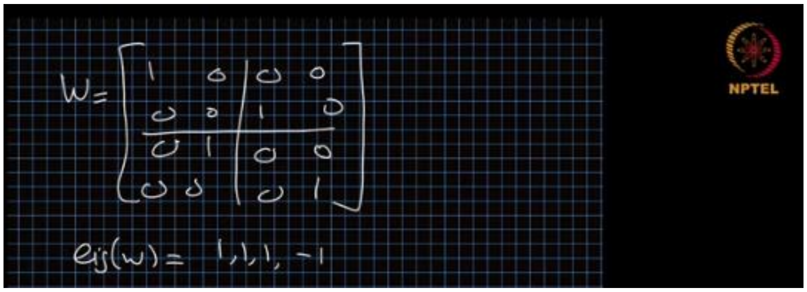

It will be precisely this matrix. And the eigenvalues of W are plus 1, there are three 1s, 1, 1, 1 and minus 1. This implies that W is not a positive operator. So this operator can be used to witness the entanglement in a system. We need to perform measurement for W and we find the expectation value and if the expectation value is non-positive, then we know the given state was entangled.

In that way, in a very physical way, we can witness the entanglement using the witness, the entanglement witness.