FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Measurements: Weak Measurements

Consider observable O, which is n cap dot sigma. We are talking about the qubits, the sigma vector. Sigma vector is the vector of Pauli matrices and n cap is a three-dimensional real vector such that the norm of n is 1. So, it represents the direction in the R3. We know that in this situation eigenvalues of O operator are plus minus 1 and let us say O and O bar are the two orthogonal eigenvectors of O. So, O acting on O will give us O and O acting on O bar will give us minus O bar.

Now, if we define the projectors or operators P plus minus, which is half identity plus minus O, then they represent projective measurements. In that case, the expectation value of operator O is the expectation value of P plus minus expectation value of P minus. But now let us define a parameter lambda, which is between 0 and 1. Then we define E plus minus as half identity plus minus lambda O. Lambda we will call the strength of the measurement. When lambda tending to 0, we have the weakest measurement.

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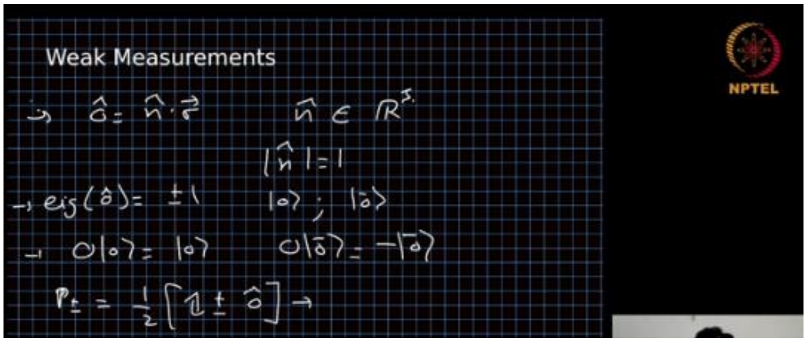

And when lambda tending to 1, we have strongest or projective measurement. We can see that P plus minus of projectors we defined just above are same as E plus minus when lambda equals 1. So, in this case, now the expectation value of O is related to expectation value of E plus minus expectation value of E minus with a slight modification that the lambda will come in the denominator. So, if we have lambda very small, then we need to have much larger data and we need to find the expectation value of E plus and minus and divide it by lambda. If lambda is very large, then it is equivalent to the projective measurements.

These kinds of measurements are important from the point of view of the following scenarios. These kinds of measurements are important in the scenarios where we want to get some information about the system, but we do not want to disturb the system too much. So, in that way, if we perform measurement in E plus and E minus, then the system is deviated from the original states slightly. And that deviation is characterized by the number lambda. That is why it is called the strength of the measurement.

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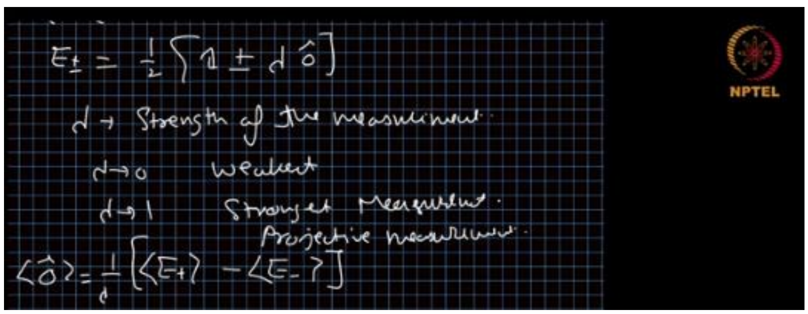

When lambda is 0 or tending to 0, then the system is undisturbed. So, the state will remain undisturbed. But when lambda is 1, then the state is maximally disturbed and it will be collapsed to one of the eigenstate of operator. One very interesting application of these weak measurements comes in the form of reversible measurements. Let us say the observable O is nothing but sigma z. Then our E plus minus are half identity plus minus lambda sigma z.

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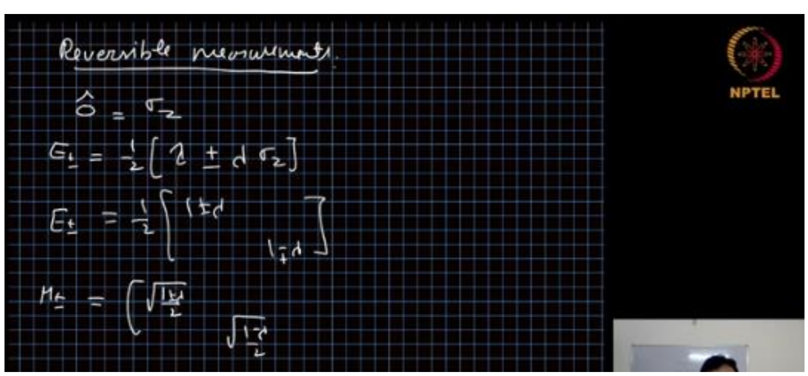

That will be half 1 plus lambda and 1 minus lambda. So, these are the two effects we have which constitute the POVM. Now, let us define the measurement operators M plus minus and they are easy to define in this case. We just say square root of these. That is plus minus, minus plus, n plus minus, n minus plus.

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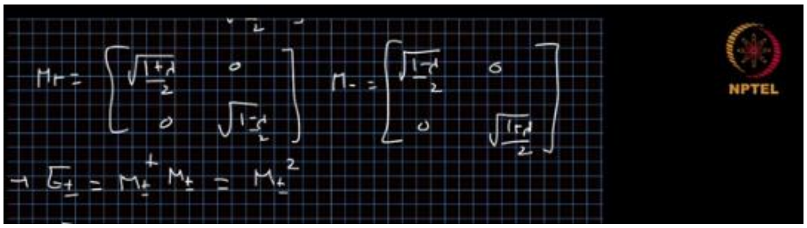

So, our M plus measurement operator is 1 plus lambda over 2, 1 minus lambda over 2 and M minus is square root of 1 minus lambda over 2 and 1 plus lambda over 2 and rest is 0. Let us recall that e plus minus is M plus minus dagger M plus minus. Since M plus minus are the Hermitian operators, so M dagger is same as M, so it is same as M plus minus square, so E plus minus are M plus minus square. Another interesting thing here is M plus and M minus both being the diagonal matrices, they commute. So, it does not matter whether we have M plus M minus or M minus M plus, they will give us the same result. Now, let us consider the measurement setup.

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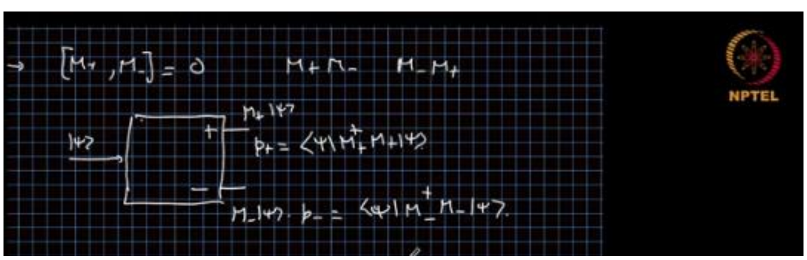

We have a measurement setup box. The input state comes here. Output goes here, this is plus and this is minus. The output state here is M plus psi proportional to M plus psi and here it is M minus psi. The probability of this p plus is the expectation value of M plus dagger M plus psi and similarly p minus is psi M minus dagger M minus, right. We can see another interesting thing, we can see is M plus M minus is 1 minus lambda square over 2 times identity and that is same as M minus M plus.

So what does it mean that if we have a system going in the experimental setup? And the system comes out after giving a click. And then we perform the same measurement setup. This was the measurement setup M. Let us call it M. The same measurement setup we pass it through again. Then again it will have plus, minus, plus, minus.

The state here was M plus psi a normalized state and here it was M minus psi, state here is M plus square psi M minus M plus psi M plus M minus psi and M minus squares. But just now we saw that M plus M minus and M minus M plus they are proportional to identity so this acting on psi will give us the state psi back. So, these are the state psi. This will happen with the probability one minus lambda square over two and here it will happen with one minus lambda square over two so total probability of getting the state back psi is one minus lambda square. So, probability of retrieval is 1 minus lambda square. So, we start the state psi, doesn't matter what state it is, as long as it is a qubit state. We perform measurement in the chosen measurement basis, measurement operators, then we get either plus click or minus click.

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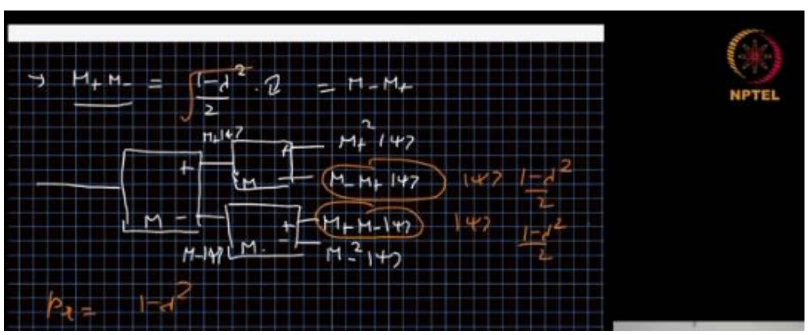

We expose the system again to the same measurement and then we get M plus square, M plus M minus, M minus M plus and M minus square, operators acting on the initial state. Two of them will give us the state back, the original state psi. And the probability of this happening is 1 minus lambda square. So, in that way, we can reverse the measurement and we can retrieve the system back. Now, can we use it in some way?

Now, first of all, let us say that if we simplify it, we have a state going in. We have measurement setup we get, in the same arm, we get either plus or minus outcome, we send another measurement setup and another measurement setup and let us say we do it few number of times, then the output state is proportional to product over i M i from 1 to q acting on psi, that will be the output state and then i Mis, let us say where si where si is from the set plus or minus, so is measurement outcome is either plus or minus. And the probability of retrieving the state in this case because all the M plus and M minus commute. So, it is just a matter of being the number of M plus equal to M minus. Number of M plus equals number of M minus in this set will imply that we have retrieved the state psi back.

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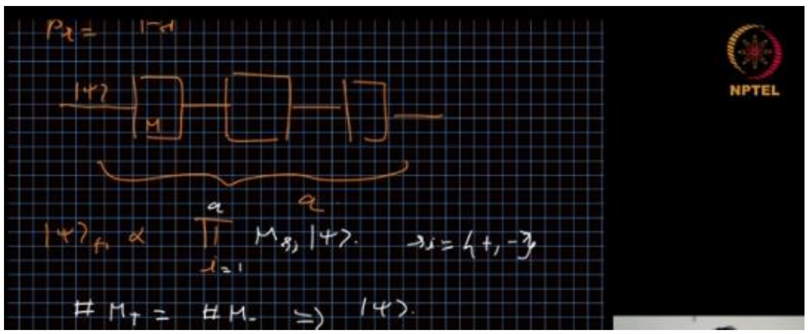

Probability of that happening is the number of times this thing can happen that is the q choose q over 2 times the probability of this thing happening which is 1 minus lambda square to the power q over 2. So, total will be q factorial over q over 2 factorial square times 1 minus lambda square to the power q over 2. So, with this probability if it repeats the experiment q number of times then this is the probability that we will retrieve our original state back psi. Now how can we use it in any useful way. Let us say we have the measurement setup. And we say it goes up for plus measurement it goes down for minus measurement down up. We discard this, we go down, we go up, we discard the down one and this one is the state psi back. Now we can send this in the loop, so, we had psi here we perform measurement m we perform the same measurement again and with certain probability that is one minus lambda square we get the state psi back and we can put it in the loop again.

If we can manage to do for some lucky particles this whole measurement n times with the same particle, then this must be equal to having one particle going in the measurement setup getting us two outcomes and another particle going in getting the two outcomes, another particle going in getting the two outcomes and with n particle we repeat this measurement. So, one particle going through this setup should be equivalent to n particle going through the measurement setup. Of course, we can only consider the result at these points. So, we calculate the number of clicks in the N plus and number of clicks in the N minus at this point and then the second phase of this measurement is to retrieve the state again. So, the probability of getting plus should be N plus over N and probability of getting minus should be N minus over N and these are the expectation value of E plus and E minus.

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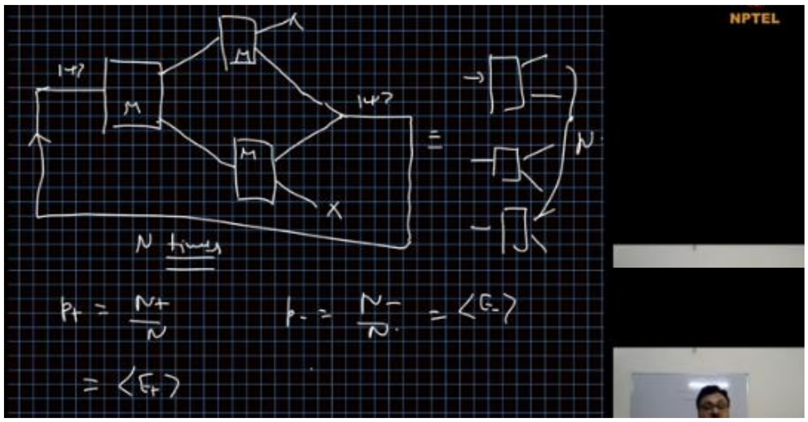

And from here we can calculate all the expectation values and stuff like that of the operators, we can estimate the state. Whatever we want to do, we need the expectation value of the POBM elements, the effects. And this click n plus and n minus will give us that expectation value. If we are given let us say an ensemble of quantum system and they are in the state identity over 2. And we have been given additional information that either they are prepared in 01 basis or in plus minus basis.

That is either in the eigen basis of sigma z or in the eigen basis of sigma x. So, either the state identity over 2 is 0 0 plus 1 1 over 2 or it is plus plus plus minus minus over 2. Now can we figure out whether the preparation basis was 0 1 or plus minus. What we need to do is we take particles from this ensemble and send through this setup, we set one particle we see N plus clicks N minus clicks, we retrieve the state we again do it so some particle will just go through this setup once and they will come out by getting in these two arms. Some particle will go in the second round, some particle will go in third round and with probability given by the number we calculated earlier, we will get some particle going through this circle n number of times. When N is large enough, then we calculate N plus and N minus probabilities, numbers and from there the p plus and p minus probabilities and from there we can calculate the expectation value of E plus and E minus. Expectation value of E plus minus in the state 0 that is 0 E plus minus 0 will come out to be 1 plus minus lambda over 2.

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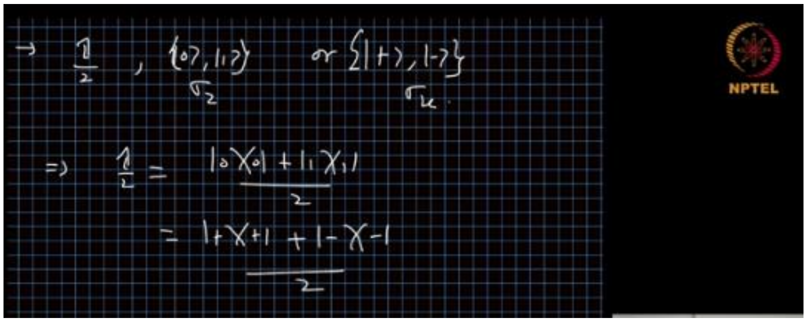

So, for E plus the expectation value of E plus in a 0 state is 1 plus lambda over 2 and expectation value of E minus is 1 minus lambda over 2. E plus minus in state 1 is also the same just with the different signs. But E plus minus for the state plus minus is actually half. So, the expectation value does not matter whether we are in the states plus or minus, the expectation value is always half. So, these are the expectation value, these are the probabilities of plus minus in the state 0, probability of plus minus for the state 1 and probability of plus minus for the state plus minus.

But we are doing a stochastic process here, we have n number of events happening and we get N plus and N minus clicks. So, when N is very, very large, then N plus minus over N will tend to p plus minus but for small N it will not be exactly equal to p plus minus but it will be a Gaussian distribution around this mean value of p plus minus. It means for 0 and 1 state the probability distribution or this expectation values will look like this for this is half this is half plus lambda and this is half minus lambda over 2. So, now, if we had very, very large N, then we will get exactly here the expectation values. But if we have small N, then it will fall on this curve, for 0 and 1.

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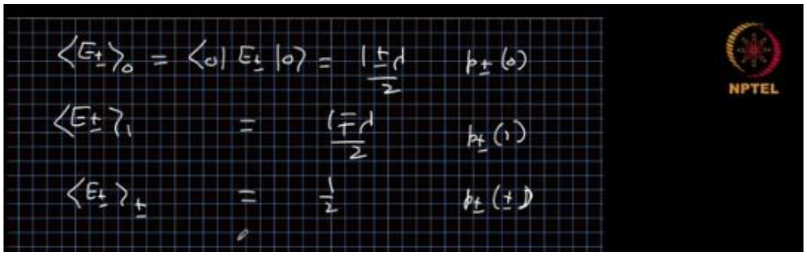

We do not care whether it was for 0 or 1, but this one will be for 0 and this one will be for 1 when we are doing E plus and the other way when it is E minus. So, that is fine. For plus minus state, this graph looks entirely different. It will be centered around half. So, this striking difference between the probability distribution of 0, 1 and plus minus can be used to estimate the preparation basis of the input state, which was i by 2.

So, what we have to do is we send it to send the particles of the ensemble one by one through the experimental setup. And then we wait for some particles to go through the loop N number of times where N is large enough, we take the probabilities or N plus minus from there, and we divide them with N and we repeat this experiment many, many times. We will get a set of N plus, we will get a set of N minus and then we plot the frequency of those N plus and N minus divided by N normalized with N on a graph. If the graph looks like this bimodal distribution, then we know the preparation basis was 0, 1. When it is a unimodal distribution, then it was plus minus state.

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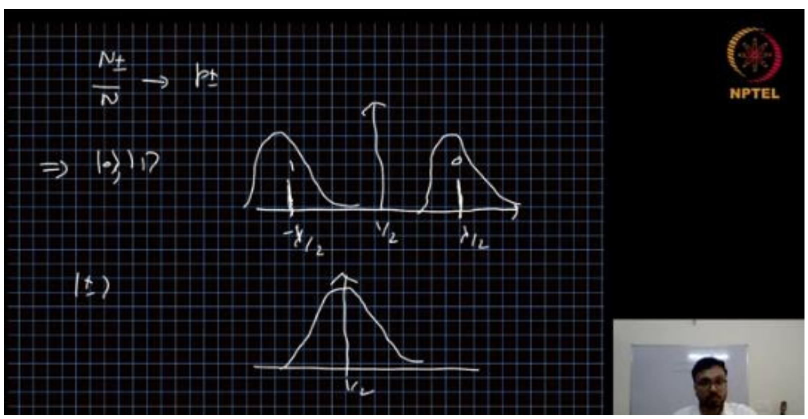

In that way, it seems like we can distinguish the preparation basis. But quantum mechanics says that once we prepare a quantum system, we cannot distinguish the preparation basis. Density matrix is all the information we have. We cannot have the information about the preparation basis. We discuss this point when we are discussing about the preparation basis and decomposition of a mixed state into a mixture of pure states.

Once the density matrix is constructed there is no way to retrieve the information about the quantum system but here it seems like we have a method to retrieve the information so let us see if it holds. Let us consider another scenario, we have two qubits they are far apart and they are in entangled state, the one qubit is in Alice's lab and other qubit is in

Bob's lab. Let us say the state of these two qubits is 0, 0 plus 1, 1 over root 2. Interesting thing is if Alice performs measurement in 0, 1 basis in her lab on this, on her qubit and she gets 0 state, 0 outcome, Bob will also get 0 outcome if he performs measurement in sigma, in 0, 1 basis. If Alice get 1, Bob will also get 1. If Alice perform measurement in plus minus basis and she gets plus, if Bob also performs in plus minus basis and Bob gets, then Bob will also get plus.

If Alice gets minus, Bob will also get minus. This is the property of this entangled state, that if you are performing measurement in 0, 1 basis in both the labs, then the results will be identical. If you perform measurement in plus minus basis, then also results will be identical. So, this is the eigen basis of sigma z, this is the eigen basis of sigma x. If both perform measurement in sigma z, the results are identical. If both perform measurement in sigma x, results are identical.

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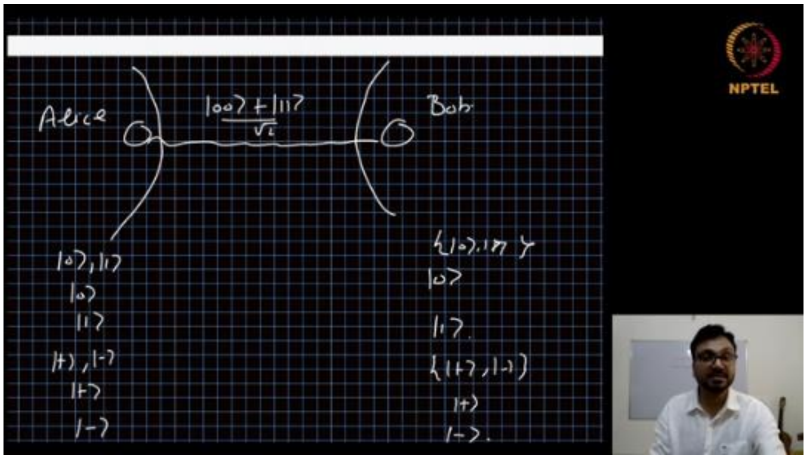

They are far apart, they can be as far apart as possible, entanglement will survive, if there is a way to distribute entanglement between these two parties. So, if let us say Alex wants to send some information to Bob without communicating over the classical network then what they can do is let us see if she can send one bit of information then that one bit can be logical bit zero or logical bit one. It can be outcome of a match, cricket match, or it can be an outcome of a result of a voting of some type. So the protocol they can follow, which is pre-decided, that if the measurement is done in sigma z, then it would mean that the logical bit Alice is trying to transmit to Bob is zero. If the measurement is done in sigma x, then the logical bit Alice is trying to transmit to Bob is one. The measurement in sigma z implies zero, measurement in sigma x implies one.

When the measurement is performed in sigma z, Alice will get answer randomly, the outcomes will be randomly zeros and ones. So, her average state on Alice side will be identity over two. This is in the Alice lab, and Bob's slab also it will be identity over two but this identity will be zero outer product zero, plus one out of product one over two. So, is in the case of Bob's lab. If the measurement is performed in sigma x basis then the identity will be plus plus plus minus minus over two and Bob will also have the same state without even Bob knowing. Just because Alice performed measurement in sigma x, Bob state will be a mixture of plus and minus.

If the Alice performed measurement in sigma z, then the Bob state will be a mixture of 0 and 1. So, now after Alice has performed measurement, Bob has an ensemble of particles, quantum systems, which where the average state is identity over 2, but it can be either a mixture of 0 and 1 or a mixture of plus and minus. This is exactly the scenario we discussed just now in the previous section, where the Bob's task is to find whether the preparation basis was 0, 1 or plus minus. And Bob can perform this whole algorithm, whole protocol and see whether the distribution he gets is the bimodal or ordinary model and accordingly he can infer whether the measurement was performed in sigma z or sigma x and the bit was transmitted was 0 or 1, the value of the bit. In that way, Alice can instantaneously transmit one bit of information to Bob over arbitrary distances.

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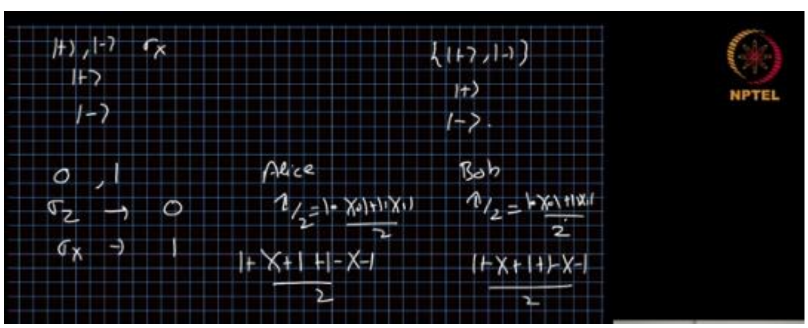

This is in conflict with one very fundamental principle and that is the superluminal communication. Nothing can travel, no message can travel faster than the speed of light. But it seems like from this protocol that is violated. There are certain things which we cannot violate and superluminal communication or causality is one such thing. So, there must be something wrong with this protocol.

The reason for this scheme to show this kind of contradicting, this kind of worrisome results can be explained as follows. Let us consider the experimental setup again.

Operators M plus minus, this is plus and minus and we send the state psi in and the output is M plus psi and M minus psi. So, let us recall that our effects E plus minus 1 over 2 identity plus minus lambda sigma z. Our aim in this experimental setup is to get the information of 0 state and 1 state, which correspond to lambda tending to 1.

So when lambda tends to 1, when lambda is equal to 1, then E plus minus are 00 and 11 projectors. In that case, we get this is the strongest measurement, this is the most perfect measurement we can have. And when we do not have such strong lambda, when we do not want to have very large lambda, that is lambda tending to 1, in that case, we still want some information about the 0 and 1. But not exactly, but some approximate information. What I mean by information is the probability of 0 and probability of 1 or probability of something close to 0 and something close to 1, which is possible when lambda is 1.

But when lambda is not 0, then we do not get 0 and 1, we get 0 tilde state and 1 tilde state, where 0 tilde is defined as M plus acting on psi or let me say proportionally because it is not normalized and 1 tilde state is defined as M minus. Since lambda is not, if this lambda is less than 1, so it is not equal to 1, then most likely 0 and 1 inner product is not 0. So, they are not orthogonal. Unlike in the earlier case where 0 and 1 were orthogonal. So, we can write 0 tilde as some coefficient a times 0 plus some coefficient b times 1.

Similarly, we can write 1 tilde as they are not normalized, so a and b did not add up to 1. c times 0 plus d times 1. Since they are not orthogonal and they are two-dimensional states, so we can decompose them in, we can expand them in 0 and 1 basis and a, b, c and d are some coefficients yet to be determined. We want information about 0 from 0 tilde state, but 0 contains one state also. So, this is what will give us false positive. So, sometimes we get a click in zero but there was no zero it was one actually and similarly this term here will give us the false positive and these are the terms which make possible that when we in for the first measurement we have tilde and the second measurement we have one zero tilde and we have one tilde. So these are the states which gives a false positive to start with and these are the states which makes the reversal of the measurement possible.

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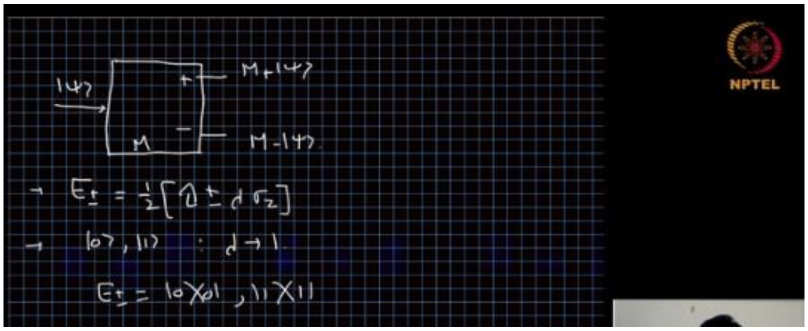

So, these are the terms like the 1 in 0 and 0 in 1 tilde. These are the terms which are making the reversal of the measurement possible. So, that we can use the same system again because we are getting the state psi back. But these are also the terms which makes which dilutes our measurement outcome. Why do we need large number of measurements to get the same amount of information with weak measurement as we need for stronger measurements because these kinds of false positives dilute our results.

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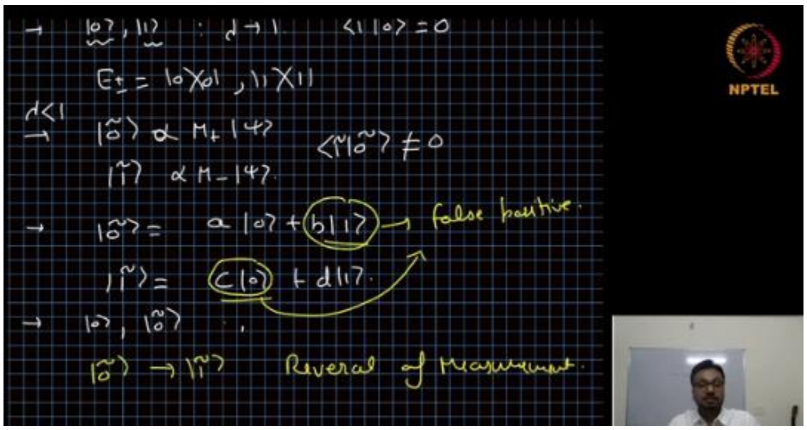

And these scenarios where the measurement reversal happen contains no information about the system at all. But let me repeat it again. Since our measurement is weak, the outcome we have 0 tilde and 1 tilde. They are not orthogonal. They should be orthogonal and they will be orthogonal when lambda tends to 1.

Then 0 tilde will converge to 0 and 1 tilde will converge to 1. But when it is weak, then 0 tilde has 0 and 1 and 1 tilde has 0 and 1. So the presence of state 1 in 0 tilde is what is making the measurement reversal possible. Similarly, presence of 0 in 1 tilde is making the measurement reversal possible. But also 0 in 1 and 1 in 0 tilde is what is causing the false positive, is causing a click when there should not be any click.

Hence, we can claim that the scenarios in which the measurement reversal happened contains no information about the system or the state of the system. So, those cases cannot be used to infer any information about the system.