FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

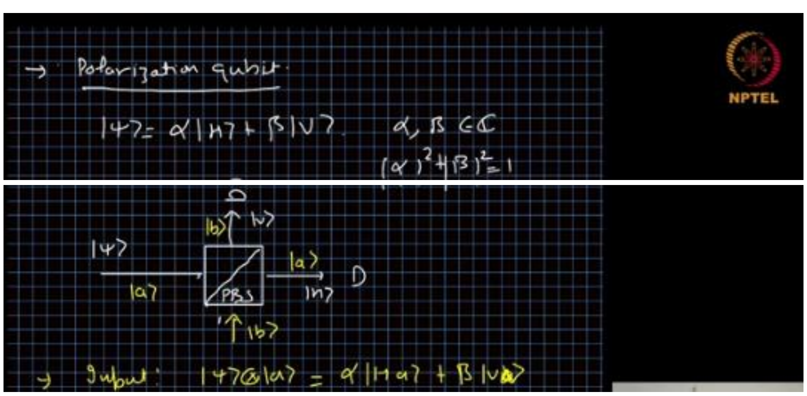
Dr. Sandeep K. Goyal
Department of Physical Sciences
IISER Mohali
Week-07
Lecture-20

Measurement: Optical Scheme for POVM

The topic today is the optical scheme to implement POVM measurement. So, here, as usual, we will consider the polarization qubits. And we will also consider the path degree of freedom of the photon. So, that will be dual rail qubit or some other form of qubit. So, we'll see when it is revealed.

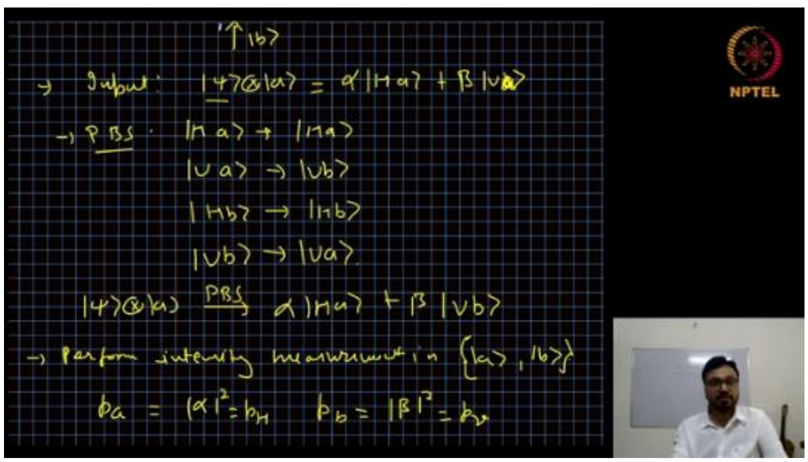
The state psi of the qubit can be written as alpha horizontal plus beta vertical where alpha and beta are complex numbers and h and v are horizontal and vertical polarizations alpha mod square plus beta mod square equals one that is for the normalization now we take a beam splitter. And this is a polarizing beam splitter. So, what it does is if the horizontal polarization is coming in it will let it go and if vertical polarization is coming in then it will reflect it so vertical vertical will go vertically up and horizontal will go horizontally but horizontal will go straight, so, if we send state psi here then h will go here and we will go there and we perform measurement and we get the alpha mod squared and beta mod squared. This whole thing can be thought of in the following way. We have one path of input path of the beam splitter. Let us call it state A. If the photon is coming in this path, then we are calling it in the state A and there is another path which we are not using for the time being, but there was a path.

(Refer slide time: 2:50)



So, that's state b. These are input. The output we are naming the straight path as same path and the reflected path as a different path. So, output a and b we have. So, the initial state can be thought of as state of a photon in two degrees of freedom, one is the polarization other is the position basis. So the total state, initial state input is psi tensor a, so psi is the polarization state of the photon and a is the spatial state path information, that is equal to alpha Ha plus beta Va. Now, when we pass it through the polarizing insulator, then Ha goes to Ha, it will remain in the same part, but Va will go to Va. Hb will remain same, Hb and Vb will become Va.

(Refer slide time: 4:24)



Now, what is the action of the polarizing beam splitter on psi tensor a? So, psi tensor a will go to alpha, Ha remains Ha plus Va becomes Vb. After that, we perform measurement, perform intensity measurements in a and b That is, we put detectors which do not discriminate between the polarization, but they just see whether the photon was in a or b. So, we are performing measurement in the spatial modes. So, whether the photon was in a or b, we perform measurement that, and accordingly, we know that the probability of getting the photon in a is alpha mod square and probability of getting in b is beta mod square.

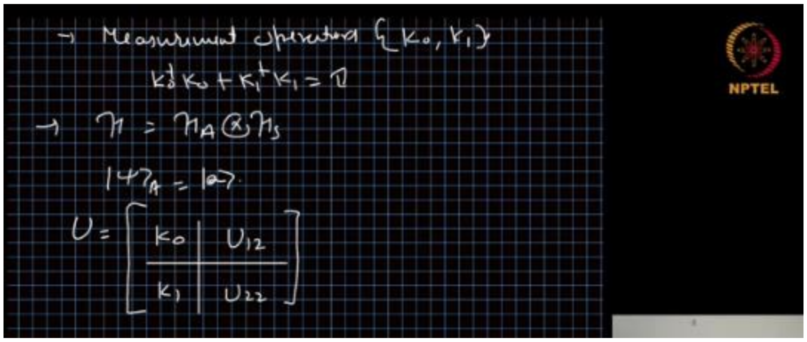
And which is same as probability of getting in H and probability of getting in V. So, what we have done here is we had a system that is the polarization. We have an ancilla that is a special degree of freedom, bar degree of freedom. And we take these two systems together. We use a unitary transformation to couple those two and then we perform measurement on the ancilla.

And in that fashion, we are performing measurement on the polarization indirectly. But here the interaction was very very strong between the system and the ancilla. That's why this effectively became a projective measurement. Now how do we perform POVM on such system? On the same system, if we want to perform POVM, how will it work?

For simplicity, we take a POVM, where the measurement operators are K0 and K1, such that K0 dagger K0 plus K1 dagger K1 equals identity. These are measurement operators, these are not effects. In fact, for these given measurement operators, K0 dagger K0 will be one effect and K1 dagger K1 will be another effect. So, for given measurement operator it is easy to find the effects. Now we want to find an optical setup which can implement this measurement operator.

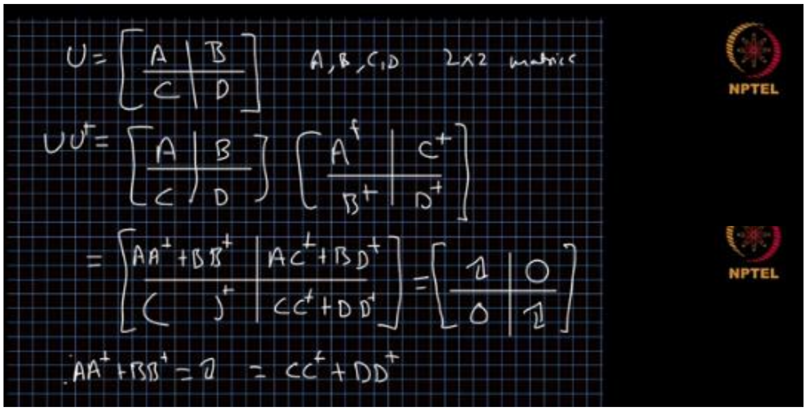
So, from our earlier treatment when we are given measurement operators we know that we have to consider system and ancilla, the total Hilbert space will be the Hilbert space of the ancilla tensor Hilbert space of the system. We have to choose the state of the ancilla psi a, initial state to be zero state, in this case it will be just path a and we have to have a unitary acting on the two and the unitary is chosen in such a way that the first block is k0 second block is k1 and then there are two blocks let us call them u12 and u22. All of these four are 2 by 2 matrices K0 and K1 are given to us. U12 and U22 are arbitrary in such a way that the total matrix U becomes unitary. So, let us say we have U12 and U22.

(Refer slide time: 6:50)



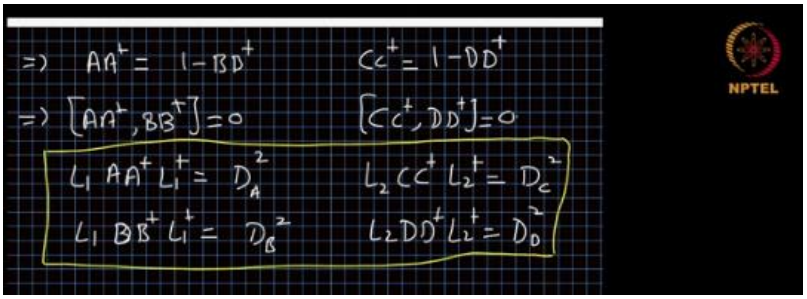
Now, how to implement this arbitrary unitary on the polarization and path degree of freedom in a combined fashion is a challenge. To let us tackle that problem first, let us say we are given a unitary U, which is of the form A B C D where A B C and D are two by two matrices complex matrices such that the total U is a unitary so if U is a unitary then U times U dagger which is A. Sorry, my notations are slightly different. I want to stick to them. BC, BCD, A dagger, B dagger, C dagger and D dagger. The product will be AA dagger plus BB dagger. A C dagger plus B D dagger the complex, the Hermitian conjugate of that and C C dagger plus D D dagger.

(Refer slide time: 9:04)



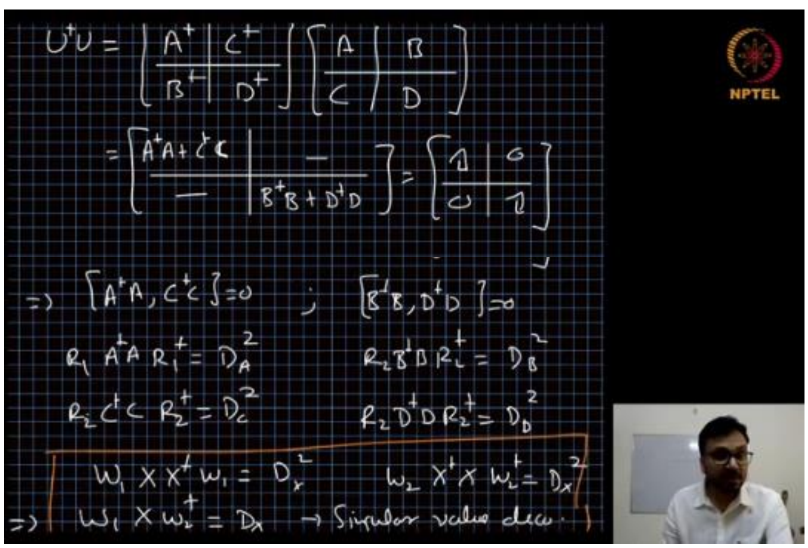
So, this is the matrix we get when we multiply U with U dagger in terms of a b c and d matrices and we know this must be equal to identity and identity and zero and zero two by two identity two by two zero two by two zero and two by two identity, so, from here we get a information that A A dagger plus B B dagger must be identity and same as C C dagger plus D D dagger. This implies that A A dagger equals 1 minus B B dagger and C C dagger equals 1 minus D D dagger. So, it means A A dagger commute with B B dagger. Similarly, CC dagger commutes with DD dagger. If two operators commute, then there must be a common unitary which can diagonalize both of them, so it means we have L1 A A dagger L1 dagger. Since A times A dagger is a positive and matrix positive operator so the eigenvalue will be real so we are writing diagonal matrices as Da square so it can be real and positive matrix.

(Refer slide time: 10:28)



Similarly, L1 B B dagger L1 dagger is Db square. Here we can have difference L2 C C dagger L2 dagger as Dc square and L2 D D dagger L2 dagger as Dd square. For the time being they just look mathematical but trust me this will become very very interesting result soon. Till now, we just use U U dagger, if we use U dagger U equals identity that will be A dagger C dagger B dagger D dagger A B C D. It becomes A dagger A plus C dagger C. We don't care about the off diagonal terms, we did not use it in the previous one, so we will not use it here B dagger B plus D dagger D, something here, something here and this must be equal to identity, identity is zero zero. From here we can see that A dagger A commute with D dagger C and D dagger B commute with D dagger D. It means we can have R2, some unitary R2, A dagger A equals R2 dagger equals DA square, the spectrum of A dagger A and A dagger will remain same in this case. And R2 C dagger C, R2 dagger as DC square, R2 B dagger B, R2, that was R1, sorry.

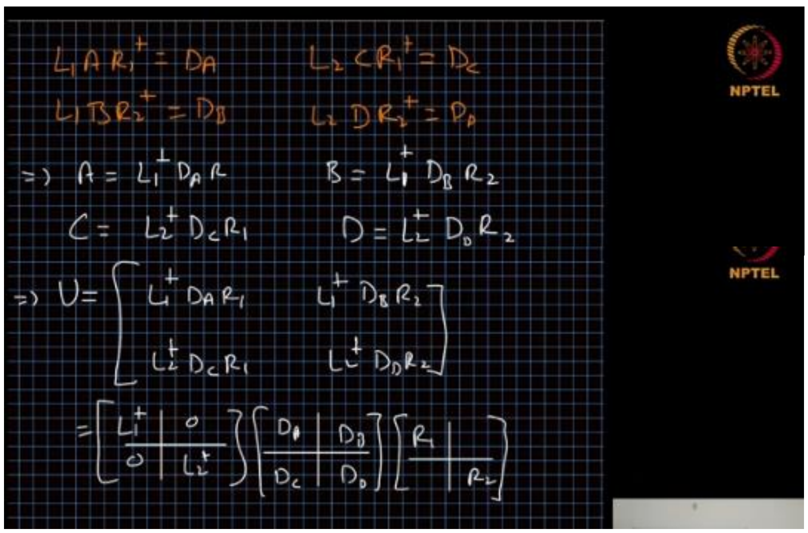
(Refer slide time: 13:56)



The naming does not matter, but I just want to be consistent with the not so that later on if we use them, we don't make a mistake. Db squared and R2 D dagger D R2 dagger equals D D squared. Now if let us say if you have a unitary W such that W1 let us say, such that X X dagger X1 is DX square and W2 X dagger X W2 dagger equals DX square DX square, then we can write W1 X W2 dagger as DX. This is called singular value decomposition. One should read the proof of this, this is very beautiful theorem so if we have two unitary operators W1 and w2 which diagonalizes X X dagger and X dagger X. Then W1 and W2 are the left and right eigenvectors of X and we can diagonalize X in this fashion and we get Dx which is the matrix of single value diagonal matrix and this theorem holds for any dimension of X, X need not be one square matrix it can be rectangular matrix it can be anything so why we are measuring this thing because, because from our analysis so far, we have L1 which diagonalizes AA dagger and we have R1 which diagonalizes A dagger.

So it means we have L1 A R1 dagger to be DA. Similarly, we have L1 diagonalizing B and R2 diagonalizing B dagger B. So, L1, B, R2 dagger equals DB. L2, C, R1 dagger equals BC and L2, D, R2 dagger equals DD. So, D was R2 here and L2 and D2.

(Refer slide time: 16:06)

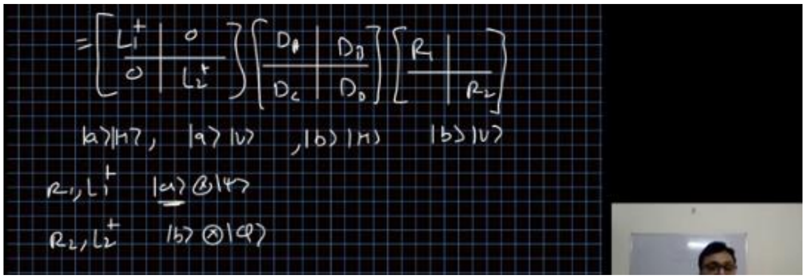


So, what we have is A can be written as L1 dagger, DA, R1 dagger. B can be written as L1, B, DB, R2. C is L2 dagger, DC, R1 and D to be L2 dagger, DD, R2. This implies that U can be written as L1 dagger DA R, L1 dagger DB R2, L2 dagger DC R1 and L2 dagger DD R2. Or we can write it as protocol C matrices, that is L1 dagger, L2 dagger, 0, 0, DA, DB, DB, DC, and DD, R1, R2.

Now, for a moment, we should consider what we have done so far. We have a two-dimensional Hilbert space of the polarization qubit and two-dimensional Hilbert space of the path states. So, the base series we have chosen is HA tensor HB, is the total Hilbert space. So, the basis is aH, aV, bH, bV. So, it means the L1 vector is acting on state A tensor psi, where psi is a polarization state, but the path is fixed A. So, L1 is acting only in the A subspace.

Similarly, L2 is acting only in the B subspace. Similarly, R1 is acting only on A subspace, the subspace corresponding to A and R2 is acting on the state with B sub space. So these L1, L2, R1, R2 are the unitary operators acting on the polarization state of light but when they are in specific path. So R1 and L1 will act on polarization when they are in path A and R2 and L2 will act on polarization when they are in path B. So these are operations acting only on the polarization qubits but in a specific path. And this is a unitary.

(Refer slide time: 17:02)



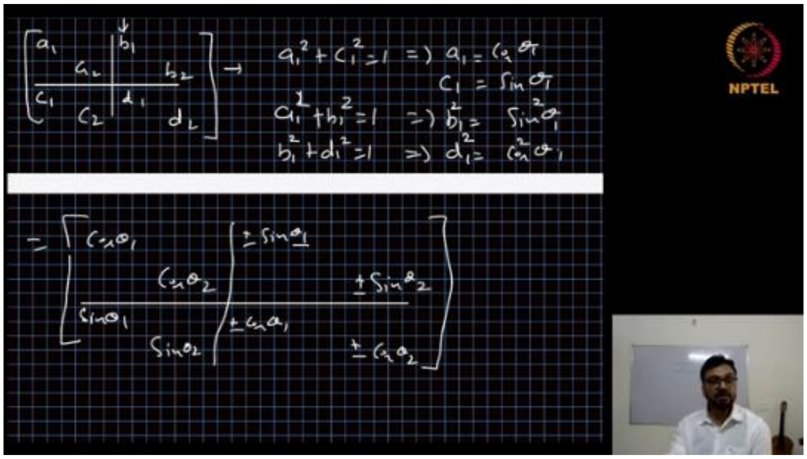
L1 and R2 are unitary. R1 and R2 are unitary. So, the total matrix is also unitary. Total matrix is also unitary. This one and this one is also unitary.

And U is unitary. So, this must be unitary in the middle. Now, let us consider, let us analyze the matrix of diagonals. Matrix of diagonals is DA. Let me, DA was the matrix of eigen, singular values of matrix A. Let us call them A1, A2, 2 by 2 matrix and wherever I do not write an element, they are 0.

Then we have B1, B2, we have C1, C2 and D1, D2. And it is a unitary matrix. First condition of a unitary matrix is every column and every row must be normalized. And A, B, C, D, all of the elements are real here. It means we have a1 square plus c1 square equals identity 1.

This implies that we can assume a1 is cos of theta and b1 is, or c1 is sine of theta. Okay, we are fixing the sign here then this is what was about the first column now first row says a1 square plus b1 square is 1 then we know a1 is cos theta this implies that b1 must be sine theta but it could be minus sin theta also so instead of that we just say b1 square is sine square theta. Now, if we consider this the third column it becomes b1 square plus d1 square is 1. This implies that d1 it can be d1 square is cos theta cos square theta similarly we can do with a2 c2 b2 d2 and instead of theta we can say theta 1 here and theta 2 for the other one, so the matrix becomes theta 1 and cos of theta 2 plus minus sine of theta 1 we have to choose whether it is plus or minus and plus minus sine of theta 2. Then sine of theta 1 sine of theta 2 and cos of theta 1 plus minus and plus minus cos of theta 2 this is the form of the matrix we have the first one we can choose on our wish but other times we cannot choose arbitrarily. Now, if we analyze it further, it is very straightforward, but it might take a lot of time in the lecture to prove it.

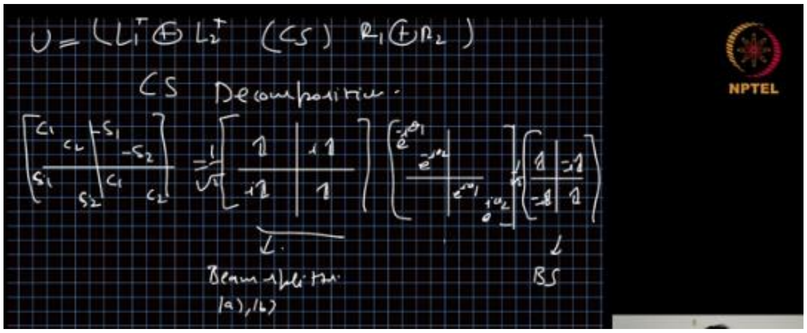
(Refer slide time: 20:14)



But if we look at it very carefully, we will see that if these have to be plus sign, then they must be negative sign here. So, these two should have opposite signs. And if we want the whole thing to be unitary, then and this vector to be orthogonal to this vector, then this must be plus. So, our matrix of diagonals will take this form cos of theta 1, cos of theta 2 in the first block, minus sin of theta 1, minus sin of theta 2, sin of theta 1, sin of theta 2 and cos of theta 1, cos of theta 2. This becomes our canonical form for the matrix of diagonals.

Since it is a matrix of cosines and sines, so it is called CS matrix. And with this, our unitary, which is L1 dagger direct some L2 dagger and then CS matrix and then R1 direct some R2. This whole thing is called CS decomposition. Analyzing further this matrix, CS matrix and I will just write C1, C2 for cos theta 1, cos theta 2, S1, S2, S1, S2 here and C1, C2 and this can be written as identity I times identity, I times identity and identity. This is 2 by 2 identities.

(Refer slide time: 23:22)



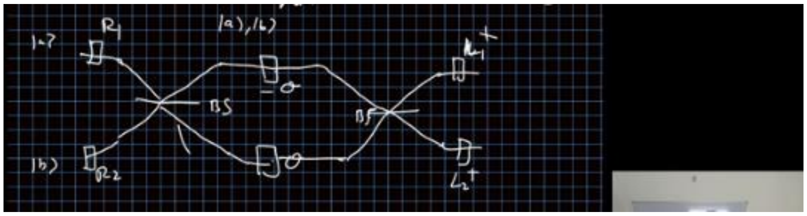
And then we have exponential of minus I theta 1, exponential of minus I theta 2, exponential of I theta 1, exponential of I theta 2. Identity minus I times identity, minus I times identity and identity. There is 1 over root 2 here and 1 over root 2 here. So we can decompose this CS matrix into a matrix without thetas and a phase matrix and matrix without thetas. What we have done here so far is we have decomposed this matrix of diagonals, this CS matrix into a matrix without any phase information theta and A matrix with diagonal matrix having the information about phases.

This matrix is a beam splitter acting only on the path information A and B irrespective of the polarization. It does not depend on polarization. Similarly, this matrix is the beam splitter matrix and it does not depend on the polarization state. But this is a phase matrix which gives you phase of theta 1 and theta 2 to horizontal and vertical polarizations in the two different paths. So the CS matrix can be represented as a beam splitter where the path A and path B are coming, then they merge and they go.

Here is the minus theta and this is the theta. This is a wave plate which will give us theta 1 and theta 2 phase width for horizontal and vertical polarizations. And here it will have minus sign in the phase and this will have plus sign. So, these two will give you the matrix, diagonal matrix of phasor. And then we have another beam splitter, beam splitter and beam splitter.

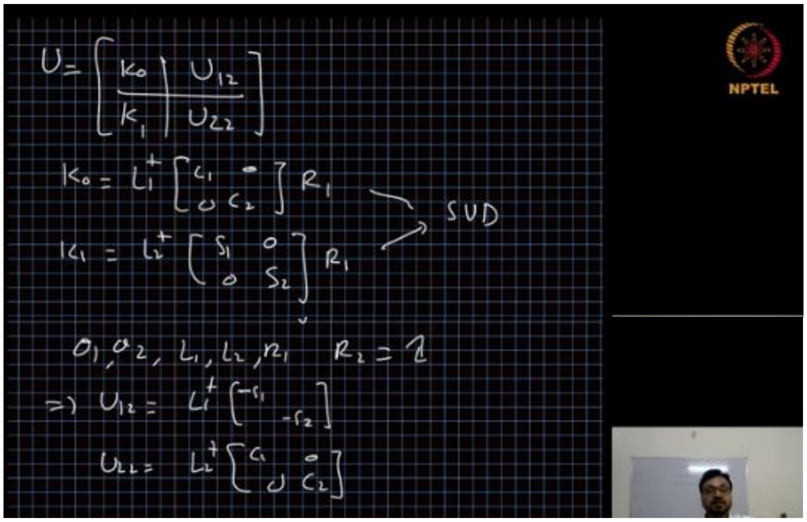
In this way, we can implement this cosine and sine matrix. So, now what is left is the R1 R2 L1 L2 matrices in this representation. R1 and R2 acts on the state first, then the diagonal matrices and then L1 l2 in the diagram, state goes from here, this is path a this is path b. So first we apply R1 and R2, then we have our CS matrix and then we apply L1 dagger and L2 dagger. In that way, we can implement an arbitrary four-dimensional unitary on polarization and power degree of freedom of a photon. Now, one can ask the question, how do we implement L1, L2, R1 and R2 matrices on the unitary on the polarization? This can be achieved by using two half-wave plates and one quarter-wave plate or two quarter-wave plates and one half-wave plate.

(Refer slide time: 24:50)



That scheme is pretty well known and one can find it online. This L1, L2, R1, R2 can be implemented easily using quarter wave plates and half wave plates and we are not going to discuss that in detail in this lecture. Now once we know that any arbitrary unitary can be implemented by decomposing it into actions on polarization and action on the paths alone. Then we consider our U which contains the information about the measurement operators K0 and K1 and arbitrary matrices U12 and U22. By comparing this thing with the earlier discussion, we can see that K0 is L1 dagger DA or let me write cos theta 1, cos theta 2, R1 and K1 is L2 dagger DA, sine theta 1, sine theta 2, R1.

(Refer slide time: 27:51)



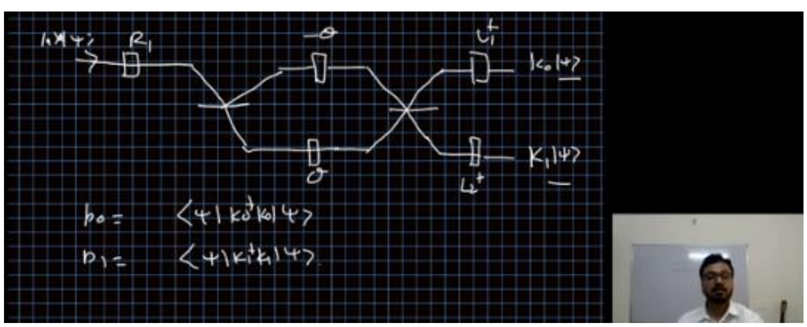
So, K0 and K1 are given to us, L1 and R1 are unitary matrices, L2 and R1 are unitary matrices and there is a diagonal matrix. So, this automatically becomes the singular value decomposition of matrix K0 and K1. So, for a given K0, K1, we can always find L1, R1 and C1, C2. Similarly, for K1, we can find L2, R1 and S1, S2. From here, we have information about theta 1, theta 2, L1, L2, R1 and R2 is unknown.

For simplicity, we can choose R1, R2 to be anything and we can choose it to be identity also. Identity is a valid unitary matrix. Then our U12 will become L1 dagger minus S1, sine 1 minus sine 2 sine theta 2 and times identity. So, identity we do not need to write and U22 becomes L2 dagger cos theta 1, cos theta 2, 0. So, in that way, by choosing R2 to be identity, we have made the choice of U12 and U22 to be easy.

It is no longer arbitrary. It is the simplest we can find in our setup. Maybe if we are using different scheme to implement the unitary, we can choose R2 to be different so that the U12 and U22 are more easy to implement. And the whole scheme will look like the following. There is R1 and this is path A. So, if the photon enters in the path A, then there is a beam splitter.

There is a phase theta, minus theta here and plus theta. So, I'm just writing theta but it contains theta 1 and theta 2 corresponding to h and v and this can be implemented with the wave plates there is another beam splitter and then we have L1 dagger L2 dagger and outcomes. So, we send the photon in this path. We perform the unitary U, which contains the information about K0 and K1, and after passing through all these things, the polarization state here will be proportional to K0 psi. And this is the psi state coming in. So, it is A tensor psi, and K0 psi and this will be K1 psi state coming in.

(Refer slide time: 29:30)



So, if we just detect the number of photons coming in the upper mode or the lower mode that will give us the probability P0 which is Psi k0 dagger k0 Psi and P1 which is Psi k1 dagger k1 Psi. So, in that way we can implement these two effects POVM when the measurement operators are given to us. We can generalize this scheme to any number of POVM elements in a set. So, in that way, this is very general scheme.