FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-07
Lecture-18

Measurements: Introduction

Our topic in this lecture is the measurement. We have been mentioning measurements in some of the other contexts in several lectures earlier but this will be one full formal discussion about measurements. So, here we will be, there are several kind of measurements one which we are familiar with which is in the fourth postulate of quantum mechanics that's the projective measurements. They are also called von Neumann measurements. So, these are the measurements we have been talking about which gives you the Born rule, which gives you the state collapse postulate and the state after the measurement, those kind of things. This is the measurement which we are familiar with which comes in the fourth axiom of the quantum mechanics. This talks about the born probability rule of the measurement and the collapse of the wave function after the measurement.

So, all the other kind of measurements which we will be discussing will be related to the projective measurement in some other context, but they are more general, so they can be treated in a slightly more general way. So, the second one is like generalized measurement. So, in these generalized measurements, we will talk about many measurements like indirect measurement where we will not perform the measurement directly on the quantum system, but we will make it interact with another quantum system and perform measurement on the other one. So, in that way we are not directly measuring the quantum system under consideration, but we are measuring another quantum system which we have coupled with the desired quantum system. Then there are POVMs, they are called positive operator valued measurements and they contain mathematical structure of all the possible measurements on a quantum system.

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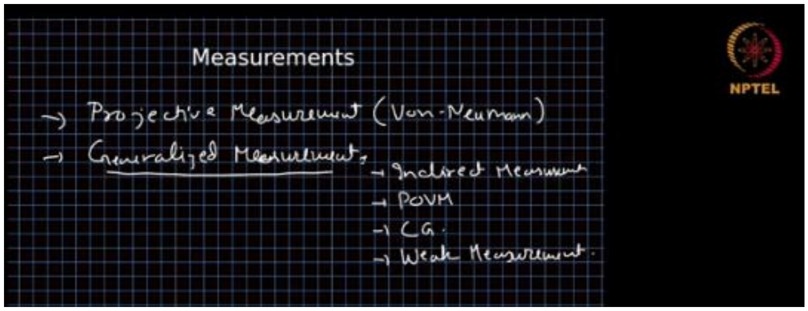

Then we will have coarse grain measurements and then we will have weak measurements. We may not discuss all this kind of measurement, but they will be surely touched upon in the lecture. So, let us start with the von-Neumann measurement. So, we have an observable A, which is a Hermitian operator. And it has eigenvalues ai and eigenvector ai, such that A ai equals ai, ai.

This is its eigenvalue equation. Also, observable must be the Hermitian operators. The property about Hermitian operators is their eigenvalues are always real and their eigenvectors form an orthonormal basis. We are choosing the eigenvector to be normalized, although the eigenvalue equation does not care about the normalizations but we choose them to have norm one, then, one, if we have a quantum system in the state psi then the probability of getting the is outcome ai, the probability of getting ai is given by the Born probability rule which is psi ai mod square. This is the Born rule. Second, once we have ai outcome then the state psi collapses to the eigen state ai, this is the collapse postulate.

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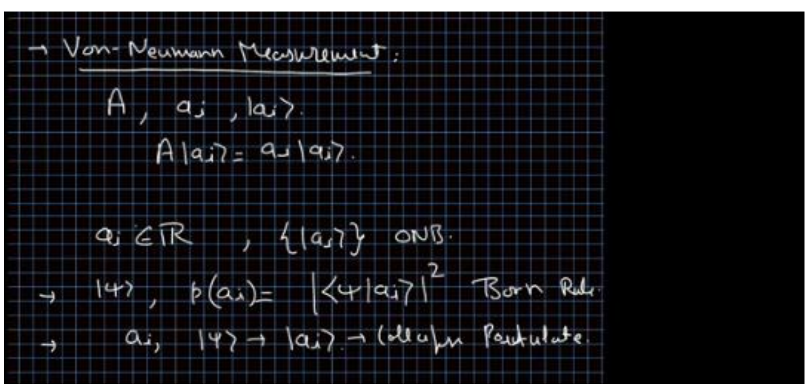

So, the projective measurement is that if we want to observe if you want to measure an observable A where a small ai's are the eigenvalues and vector ai's are the eigenvectors of the observable A, then the probability of the i-th outcome is given by the Born rule which is psi inner product ai mod square. And after the measurement, the state of the quantum system collapses to the outcome eigenstate ai. We can generalize this treatment in the following way. Let us define a projector Pn. So, a projector is an operator such that Pn square equals Pn.

This is the only definition of a projector. So, any operator P, if it satisfies P square equals P, then it must be a projector. So, for example, we are defining Pn to be an outer product an. It means Pn square, which is an, an, an, an, this is a scalar here, which is inner product of an with itself. And since an is normalized, this is 1.

We get an outer product an and this is Pn. Therefore, Pn equals an outer product an is a valid projector. Now, we can reformulate the two statements about the projective measurement that is Born rule that p of an psi an mod square. This can be written now as psi Pn psi. That becomes the new Born rule. Both are the same statement.

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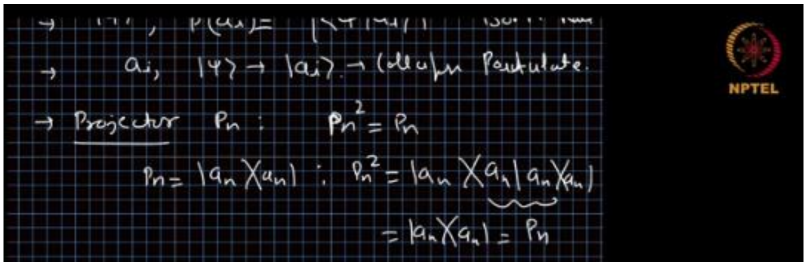

We can see that when we substitute Pn, we get an outer product an, Psi, which is same as Psi an inner product mod squared. So, we can write the Born rule in terms of the projector Pn. And the state collapse, Psi going to the projected state becomes Pn acting on Psi divided by P of an. So, this becomes the collapse postulate. So, in this way by refining the projector Pn we can write both the statements of the projective measurement.

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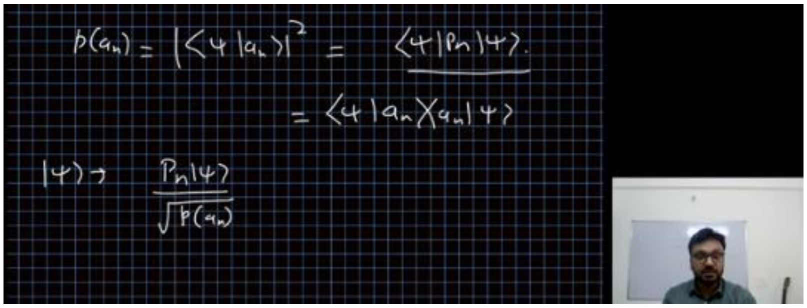

If we have not a pure state but a mixed state rho which is given by sum over n lambda n phi n outer product phi n. If you have a density matrix rho, then the probability of getting mth outcome will be sum over n lambda n am phi n phi n am sum over n lambda n the Born rule for each of the phi n am mod square and the mixture, the statistical average over the mixture. This is how the probability of m-th outcome, which is what we are just writing, short form as p m, we call it p of am, is given by this. Now, in terms of projectors, it becomes sum over n, lambda n, phi n, Pm, phi n. And if you remember the cyclic property of the trace, this whole thing can be written as trace of Pm rho. This also can be written as expectation value of Pm.

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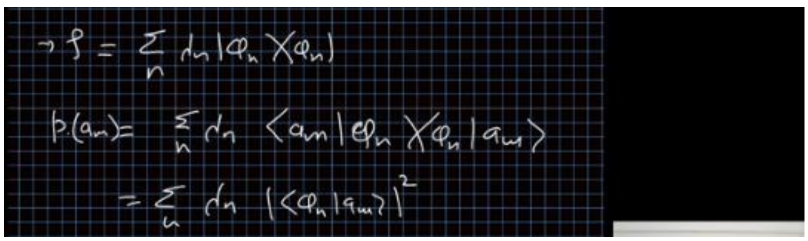

Similarly, here also the probability this can be written as expectation value of Pm. Pm here. So, the probability of the outcome is the expectation value of the projector Pm. And the state collapse rho going to rho tilde after measurement becomes Pm rho Pm divided by p of, small p of am and that's nothing but am am. So, in that way by defining the projectors, we can reformulate the projective measurements in a much simpler, much concise way. But our definition of projector was P square equals P. This is the definition of projector. Then if we take, let us say, a1 outer product a1 plus a2 outer product a2, and let us call it P, then P square, since a1 and a2 are orthogonal, will also be a1 outer product a1 plus a2 outer product a2.

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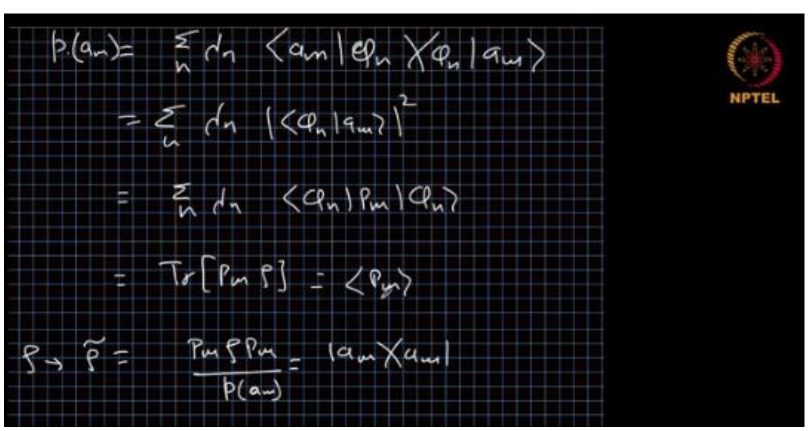

So, this is also a valid projector. So, it means we can have a measurement setup where we have state coming in, rho, and outputs are there. And there can be scenarios where we cannot distinguish between, let us say, a1 and a2. So, both of those outcomes give you the same, both of those cases will give you the same outcome. Similarly, you can have a bunch of other measurements which will give you one click.

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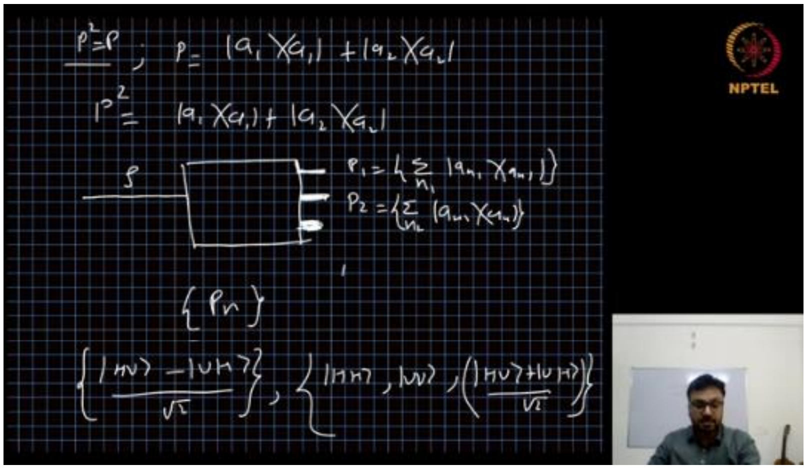

So, it means you will have p1, which is a sum of many many sum over n1 an1 an1 and then you have other projector p2, which is sum over n2 an2 an2 and so on. You can have a set of many pn's which are not rank 1 projectors but more than rank 1 projector. And so, you can have a set of many pn's which are not rank 1 projectors but more than rank 1 projector. They can be rank 1 or more projectors. These kinds of measurements are called coarse grain measurements. In some scenarios where we cannot distinguish between many outcomes, then we see this kind of coarse grain measurement.

For example, when we have two photons coming in with different polarizations, when we merge them on a beam splitter, then the output can be either a symmetric state or an antisymmetric state of the two photons in the polarization ratio. So, they are called singlet and triplet. In a linear optical experiment, we cannot distinguish between the singlet and triplet generally, unless we put some extra effort. So, in those cases, the singlet state, which is given by the horizontal, vertical, minus vertical, horizontal, the two photons being in different polarization, divided by root 2. This is antisymmetric state in the sample, if we interchange the two photons, we get the negative sign out.

And then we have triplet, where we have both photon in H or both photon in V or one in H other in V plus one in V other in H. This is the symmetric combination of the different polarization. So, this is a singlet and this is a triplet and this is a singlet state. Singlet state is the anti-symmetric state and triplet state is the symmetric state. Since the phonons are bosonic in nature, so, in a linear optical experiment we cannot distinguish between these three states. So whenever we have any of this state we get the same click out and when we have this, we get another click so in that way we have one projector let's call this state as zero and this is one two and three state.

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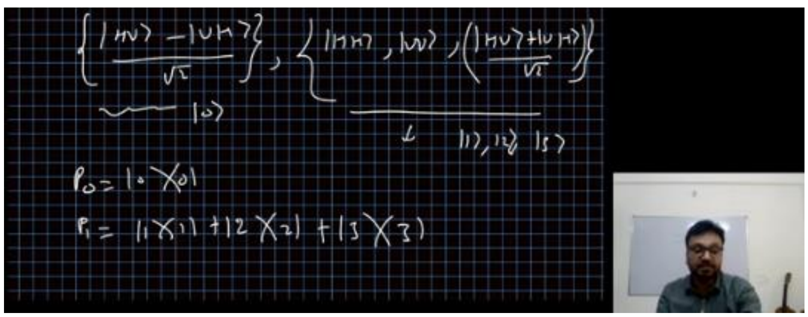

So, we have one projector P0, which is 0 outer product 0 and other projector P1, which will be 1 projector 1 plus 2 projector 2 plus 3 projector 3. So, in that way, we have some physical examples of when more than one orthogonal state being a part of a projector. So, in these cases, what happens to the Born probability? So, the Born probability P of n is still given by the expectation value of the nth projector. We do not care whether this projector Pn is rank 1 projector or rank 2 projector or rank 3 projector.

What we mean by the rank is the number of non-zero eigenvalues. So, since Pn square is Pn, then the only eigenvalues it has are either 1 or 0 because the eigenvalues should satisfy the equation satisfied by the operator. So, Pn square minus Pn equals 0. So, only 0 and 1 can satisfy this equation. Now, how many ones we have and how many zeros we have will tell us what is the rank of the projector. So, when we have p to be an outer product an then it's rank one. When we have more than this, then we have rank two, three and four depending on how many non-zero ones we have. So, while defining this Born rule, the general Born rule, we do not care what is the rank of the projector under consideration but the probability is given by the expectation value of pn, so if it is a pure state it becomes psi pn psi and if it is a mixed state, then it becomes trace of rho pn, this is for pure states and this is for mixed states.

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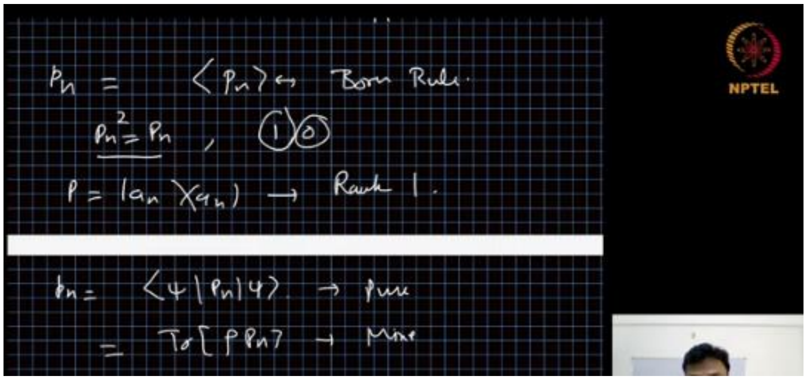

Now the collapse postulate can be written as psi collapses to pn psi divided by root pn if it is a pure state and if it is a mixed state, rho goes to pn rho pn over root over pn. Till now in terms of the representation in terms of the expressions for the Born rule and the collapse state, there is no difference between rank 1 projector and rank 2 projector but or rank higher than projector, but one thing we have to be carefully is when pn is a rank 1 projector then if pn is a rank 1 projector, then rho collapses to pn rho pn over small pn, which will be an outer product an. It means it's a pure state. It means rank one projectors project any density matrix rho to a pure state. But if it is rank more than one, let us say two, then we can write Pn to be an1 an1 plus an2.

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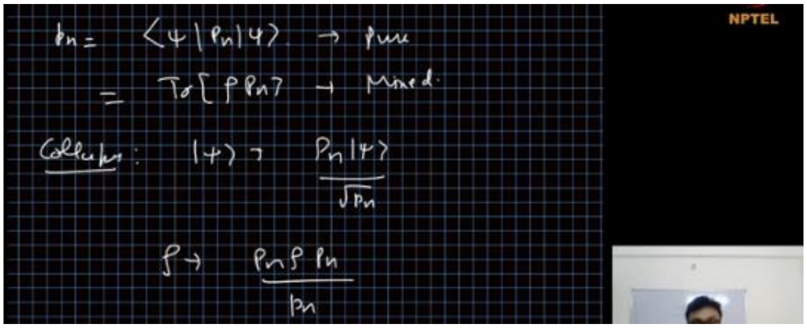

Then rho collapses to Pn, rho Pn over Pn, which is not a pure state. It is actually a mixed state, generally. Of course, if there is no component of an2 or an1 in rho, then we can get pure state also or nothing. So, in that case, things will be different. But in general, for the projectors more than rank 1, the projected state will not be a pure state, it will be a mixed state.

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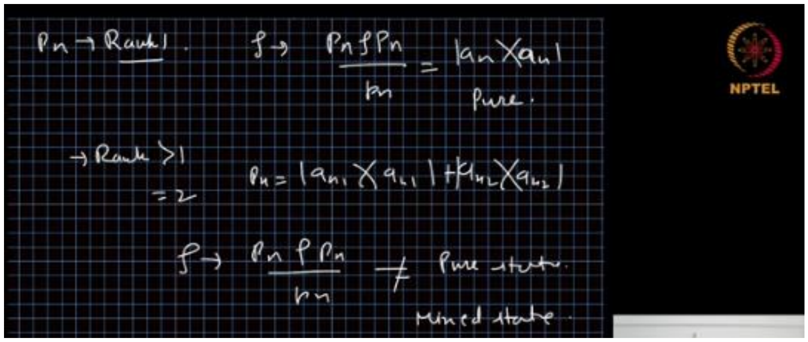

Let us consider a set of projectors Pi, such that Pi Pj is pi delta ij. It means if i is not equal to j, then it is 0, the product is 0. If i is equal to j, then it is a projector. So, it is Pi square is Pi. And if sum over i Pi is identity, just let me remind you that identity is also a projector. It's a projector because identity squared is identity. So, identity is also a projector. And if some over i Pi is identity, then it's called informationally complete set.

Set of projectors. So, this is a valid measurement basis. So, if you want to perform a complete measurement on the quantum system, we need to find one such set. One previous example of one regular example of such set is the set of the projector a i ai ,that will amount to the regular von-Neumann measurement. This is the regular projective measurement we have been talking about from the beginning of this course. Then other can be coarse-grain measurement where more than one projector combined, more than one projector combined to make rank more than one projector and then we get coarse-grain measurement. One very trivial example is a set of just one element, which is identity, which is also a projector, which satisfy all the conditions given above.

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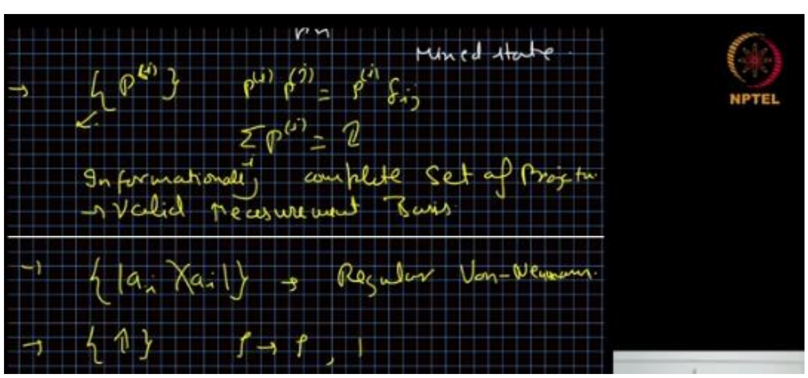

So, this measurement will yield nothing. So, you have a state going in the setup and the state comes out. The probability of that happening is one and no information is captured about the system in this way. But this is a mathematically valid measurement. Next is measurement on composite system. Let us consider a bipartite system HA tensor HB.

For the sake of ease, we assume both of them are the same dimensional system and same basis they have. So, an arbitrary state psi AB can be written as sum over i j alpha i j i tensor j. An arbitrary state can be written in this form where i and j are the computational basis. It's a two, it's a bipartite system, it's a composite system, so we can think of it like there are two quantum systems A and B and they are in some joint state. This is just artistic representation of two entangled or joint quantum system. But they can exist physically apart from each other. So, this can be in one lab and this can be in another lab. So, there is a possibility that we perform measurement on subsystem B, but not on subsystem A. So, how do we represent such scenarios where we have a composite system and we perform measurement only on a subsystem of it.

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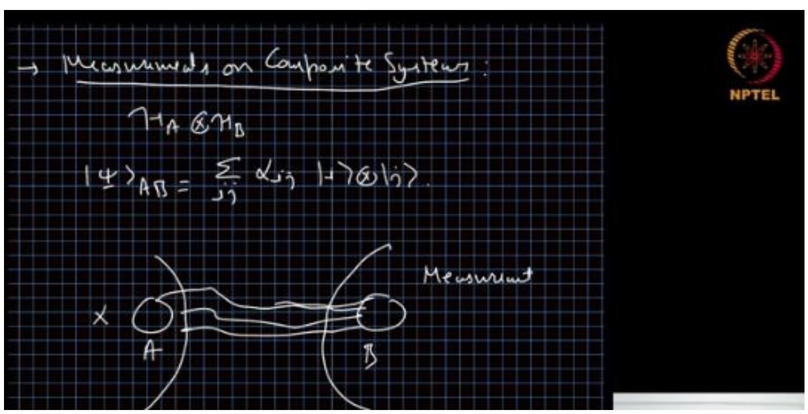

So, partial measurements, let us call it so let us say the measurement basis we are choosing on subsystem B is phi j basis. It's the orthonormal basis, it's a rank one projector. So, it will belong to rank one projector. So when we perform measurement on subsystem B, so the projectors we can think of a, we can say that there are Pi projectors or Pj projectors which are given by phi j outer product phi j. So, in general the joint measurement of A and B will be the measurement on subsystem A and measurement on subsystem B. Since we are not doing anything on A, the projector on A can be thought of as just identity projector on A and Pj's on B. So, this is the basis in which we will be performing measurements. The total basis will be identity tensor pj. We can see that sum over j identity tensor pj equals identity tensor sum over j Pj.

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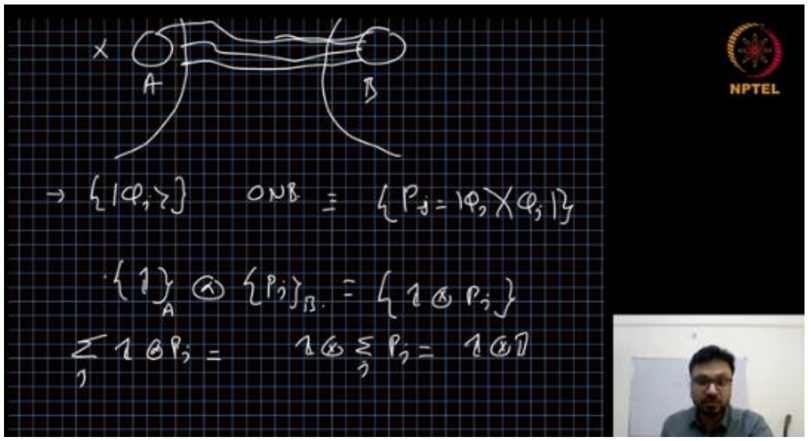

This is equal to identity tensor identity. So, this satisfies the criteria of the informationally complete basis in terms of projector. Next, we can see that identity tensor Pi times identity tensor Pj can be will be identity tensor Pi Pj which is identity tensor phi i phi i phi j phi j and these are orthonormal basis. So, if this is delta i j, it becomes identity tensor Pi delta j, so these set of projectors satisfy all the conditions of an informationally complete basis, so this is a valid measurement basis to perform on the subsystem, on the composite system AB. So, the measurement will yield psi AB going to after measurement identity tensor Pn on psi divided by root Pn where root Pn is the probability of the outcome that is given by psi identity tensor Pn. Now we can see that, let us call it Psi tilde which is the projected state will be 1 over root Pn sum over ij alpha ij and there is a let me write identity tensor Pn i tensor j.

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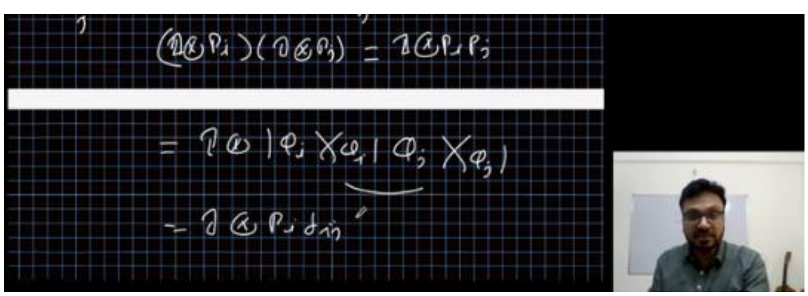

This will be 1 over root Pn, sum over ij, alpha ij, i tensor, Pn is phi n, outer product phi n, j. It can be simplified, root Pn, sum over ij, alpha ij, we have phi n, j, and we can take this also and it is i tensor phi n. Now if we look at it carefully, this can be written as chi n chi n tensor phi n where chi n is 1 over root of p n sum over i j alpha i j phi n j and i this chi n is the state of the subsystem A And phi n is the state of the subsystem B after the measurement. So, from the subsystem B point of view, after the measurement, the state collapse to the eigenstate of the outcome, phi n, because phi n was the outcome we got. And in response, the state of the subsystem A also collapse to some state and that state is given by chi n which depends on the outcome of the subsystem A or subsystem B that is phi n and the coefficient, joint coefficient of A and B so measurement on B affects the outcome of A also. That is one thing that the measurement on AB affects the state of A and this is what is used in many of the quantum computation and information protocols.

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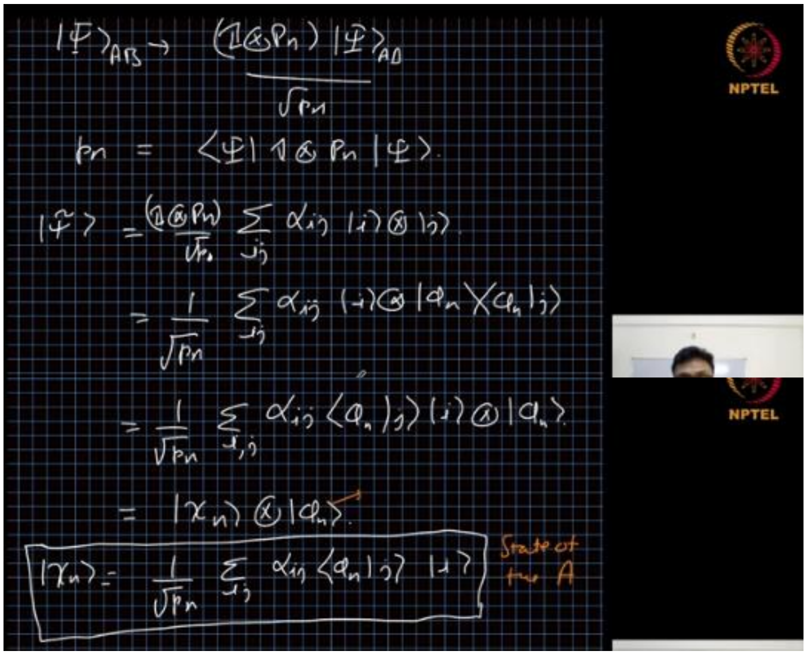

The quantum teleportation is performed using this principle. Another interesting thing to notice here is after we perform measurement on subsystem A, the state type AB which was the initial state, does not matter what was initially the state, the outcome after measurement is the product state. This is a product state. Or in other words, it is a separable state, non-entangled. So, a measurement on subsystem A, a projective measurement rank one projective measurement on subsystem A yields a separable state between A and B. it destroys all the entanglement if there was any. We can generalize this result to the mixed state also but those are trivial exercises, I will recommend everyone to try those things at home.

What will happen if what would have happened if our projector was a rank two projectors. It means Pn let us say was phi n1, phi n1 plus phi n2, phi n2. So, we want to explore the case when Pn is rank two projector instead of rank one projector. In that case, psi AB will go to psi tilde which is again given by one over root Pn, sum over ij alpha ij i tensor pnj. Now pnj is is phi n1 phi n1 j plus phi n2 phi n2 j. Let us call it p n1 j pi n1 just p r let us say r n2 j pi n2 and we can substitute it there, so our side delta will be 1 over root p n sum over i j alpha i j I tensor r n1j phi n1 plus r n2j phi n2. Now, the issue here is this state depends on both n which is n1 and n2 and j. We cannot separate this in terms of like in n and j index, we cannot write this state which is independent of j. So, in that way, we cannot write the whole state here as some chi tensor some phi n. Hence, this is an entangled state.

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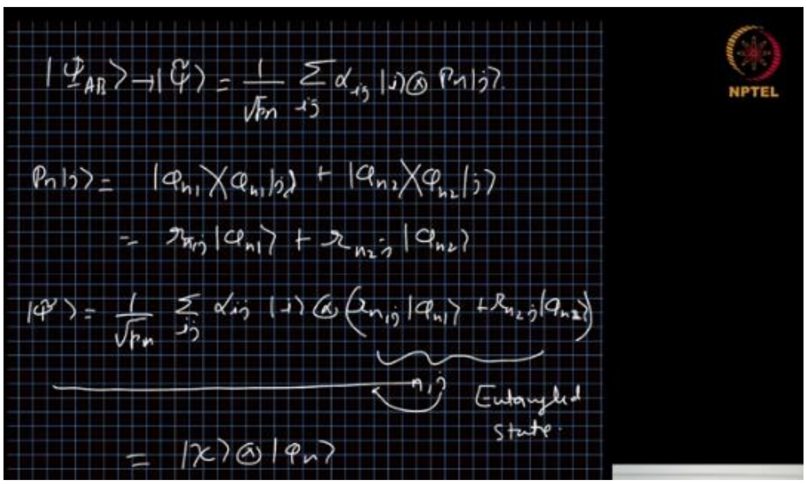

Unlike the rank 1 projector where we got a separable state, if the measurement is done in a coarse grain manner, then the resultant state need not be a separable state. In fact, in general, it will be an entangled state. So, this is the crucial difference between the rank 1 and rank 2 projectors when they apply on a subsystem. Next, we will be discussing the indirect measurement. An indirect measurement consists of a system on which we want to perform the measurement.

Let us represent the system with S. It's a Hilbert space HS and the initial state is rho S. And we want to perform measurement on rho S. To assist in the indirect measurement, we need a probe. Often, we call it ancilla A. The Hilbert space of the ancilla is HA. And the initial state is rho A. We will choose rho A according to our convenience so that we can make the measurement easily and in a more effective way. Next is the coupling or interaction between A and S. The interaction between any quantum system given by a unitary transformation, let us represent it by U. It is a unitary acting on HA and HS together in a joint manner. So, the total Hilbert space H is HA tensor HS.

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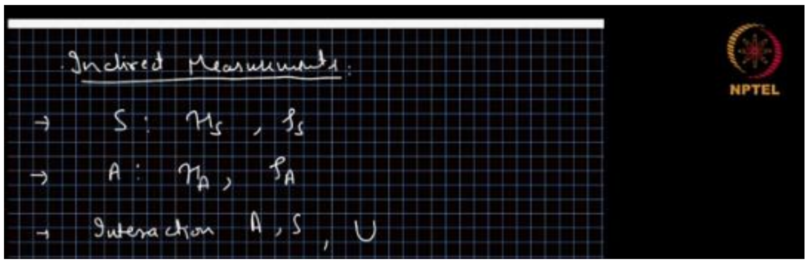

So, the unitary U belongs to the set of operators setting on H. So, initially our total state rho is rho A tensor rho S. After the interaction rho tilde becomes U rho A tensor rho S U dagger. And after that, we perform measurement on the Ancilla. So, to perform measurement, let us say we perform projective measurements without complicating things. So, we choose the basis of the measurement to be n basis. So, this is the orthonormal basis in the Hilbert space of the ancilla.

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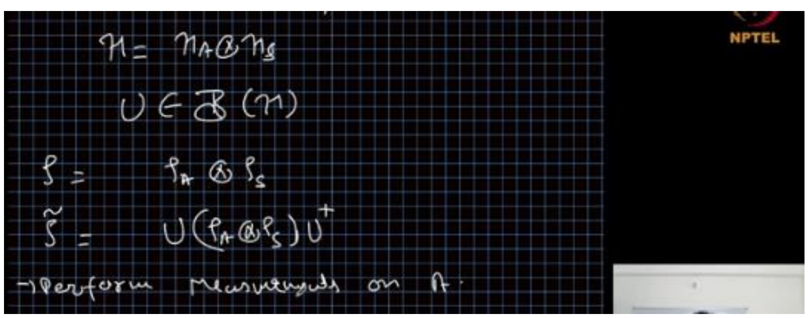

When, in the case of nth outcome, we get the outcomes, the state of the system and ancilla collapses to the following state, pn tensor identity, U rho A tensor rho V, U dagger, pn tensor identity over the probability. We will talk about the probability pn in some time, but the projector pn is nothing but the projector over the state n. Let us say rho A, the ancilla state is a pure state given by psi, psi. In that case rho tilde n becomes pn tensor identity u psi psi tensor rho s u dagger p n tensor identity. If we replace p n with n n with n n and then we defined Kn to be n tensor identity, U n tensor identity. Then this becomes the operator acting on the Hilbert space of the system. This is the Kraus operator.

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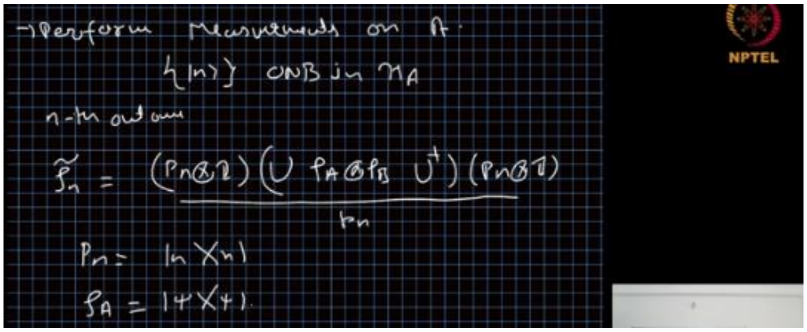

We have discussed this thing in the completely positive maps. Then rho n tilde can be written as n out of product n tensor kn rho kn dagger. There was a probability Pn down there and there is a probability Pn. Then Pn becomes the trace of rho tilde n which is same as trace of kn rho kn dagger. So, this is the probability of the outcome or the measurement outcome and this is the state after collapse. What we have done is we have extended our system of interest the Hilbert space of the system HS to HS tensor HA or HA tensor HS.

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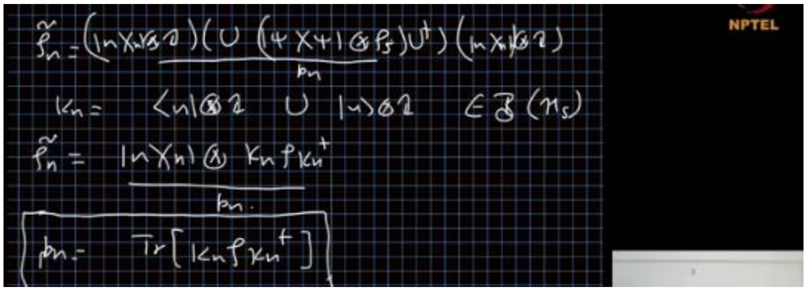

So, we have converted a simple system into a composite system and after that we have utilized a unitary transformation U to create interaction between system and the ancilla. Following that, we have performed a von Neumann measurement on the ancilla and the effect of that can be seen on the system S in the following form that the state of the system rho s goes to Pn rho s Pn dagger over Pn upon measurement. This is a normalized state, so this is the the result of an indirect measurement on the quantum system on the quantum system S. This outcome happens with the probability Pn. So, the total average outcome state of the system can be written as sum over n Pn times Kn rho S Kn dagger over Pn will be n Kn rho S Kn dagger.

This is the average outcome state after the complete measurement. K n's are called measurement operator. The trace of the outcome state should be always one because it's a valid state. This implies that, which remains exercise, which we have already done also that sum over n K n dagger K n should be identity. This is an additional condition over the measurement operators. Now, what we have done, what is the difference between the projective measurement and the indirect measurement? In the projective measurement, we had a set of projectors P n, such that sum over n, P n equals identity and Pi Pj equals Pi delta ij.

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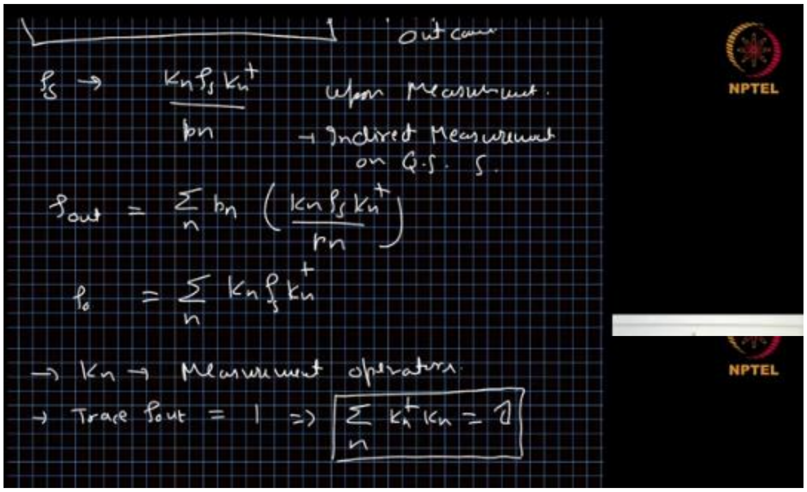

They are orthonormal. So, this is the projective measurement. In the indirect measurement, we have set of K n such that sum over n K n dagger K n equals identity and that is all. In the projective measurements, the probability Pn was given by expectation value of Pn and the state rho n was after the measurement was Pn rho Pn over Pn. In the indirect measurement, the probability Pn is given by K n rho K n dagger trace which also can be written as trace of K n dagger K n rho or expectation value of K n dagger K n and the state after collapse is given by K n rho K n dagger over K n.

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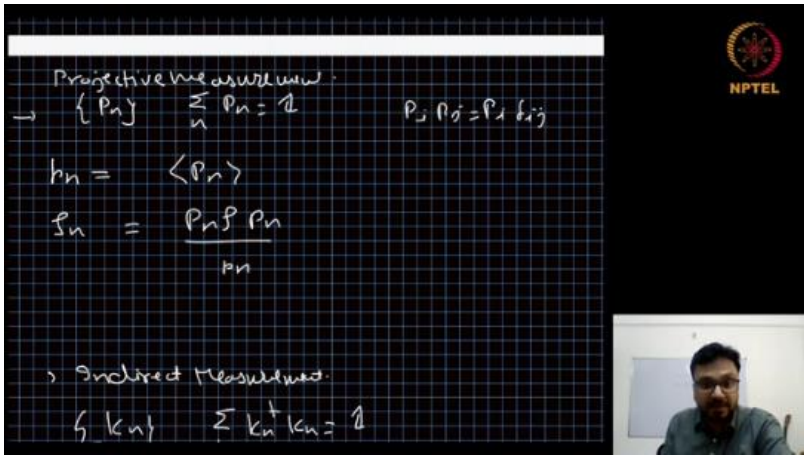

In that way, there are certain differences in projective measurement and indirect measurement, but mathematically they are very much alike in spirit and we can see that the projective measurement is a is a subclass of the indirect measurement. We can choose K n's in such a way that they become projective measurement. Till now we have seen that given a unitary, which is the interaction between the ancilla and the system and the initial state of the ancilla and the measurement basis will yield a set of Kraus operators K n or measurement operators K n, which will give us the indirect measurements on a system, or generalized measurement on a system. The question can be asked, for a given set of measurement operator K n, can we find a unitary operator U, a state of the ancilla, initial state of the ancilla psi and the measurement basis n corresponding to K n. So, to answer this thing, we see that KN was defined as N tensor identity U psi tensor identity.

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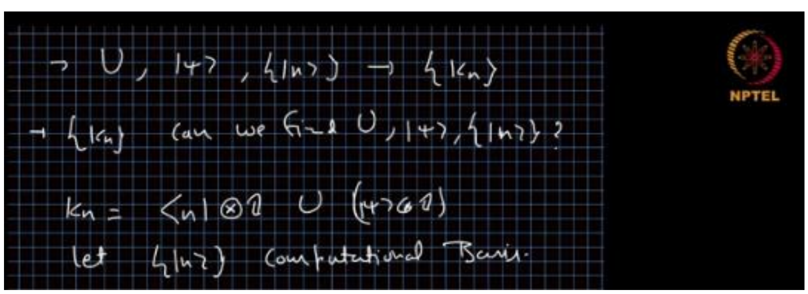

Let us say n is the computational value. It means vector n is defined as a vector of zeros with only one 1 at nth location. And let us say psi is the zero state. And we know that any arbitrary U which is acting on a composite system can be written as ij, i outer product j, tensor U ij where U ij are the operators acting on the system. Or in other way, we can write it as U00, U01, U02. U10, U11, U12, these are all operators.

So, we have written U, capital U as the matrix of matrices. Then we can see that n U, n tensor identity, U 0 tensor identity is U n 0, which is by definition Kn. So, if we had a U, then the first column of vector, first column of operator is equal to the K n we have as measurement operators. So, for a given set of measurement operators, we can choose a U such that the first column is the column of the measurement operators, K 0, K 1 up to K n and rest of the matrix is, can be chosen arbitrarily in such a way, let us say the rest of the matrix is R, which you can choose R in such a way that U is a unitary. So, we say R, we choose in such a way U is unitary.

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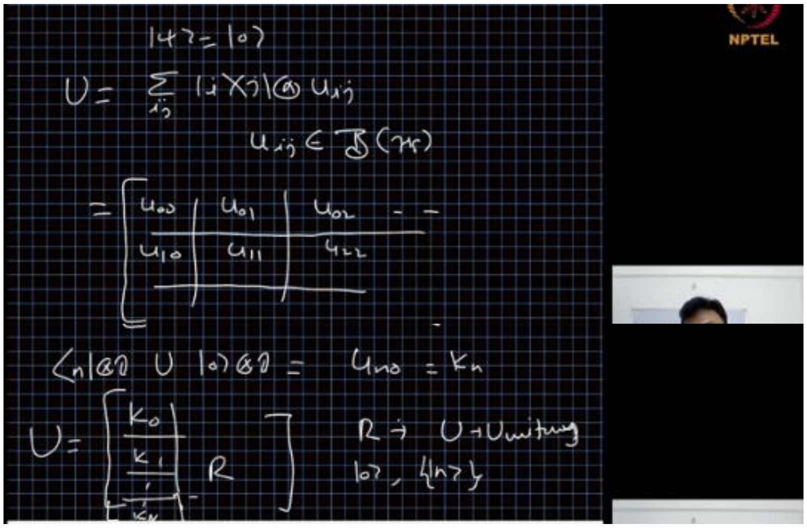

And this can always be done. So, there is a huge choice in terms of R for a given set of measurement operators K n, that U is a unitary and here, what we have assumed is the initial state of the system is, ancilla is zero and the measurement basis is the computational basis. Of course, if we change these things then U will also change but since our task is to find the simplest U possible, we will choose the simplest initial state and simplest basis, but if the given problem demands something else, then we can choose those also and our unitary matrix will also change. So, in that way for a given set of measurement operators K 0, k=K 1 up to K n, we can always construct a unitary U for the initial state 0 and the measurement basis n such that it gives you the indirect measurement.