FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-06
Lecture-17

Quantum Maps: Completely Positive Maps - Part 02

So, what we have proved in the second choi theorem is that if you have a completely positive map omega, the only condition we have is omega is a completely positive map. Then we can find at least one set of operators K n such that the map, the action of the map omega on operator rho can be written as sum over n K n rho K n dagger. And this we achieved by assuming or by finding a very special operator E such that the ij-th block in that E is the ij-th matrix. So, because ij matrix is the basis for an arbitrary operator, if we can find the action of the map omega on ij, we can find the action of omega on any arbitrary operator rho. This is what we did and we found the operator sum representation or Kraus operator representation of the omega.

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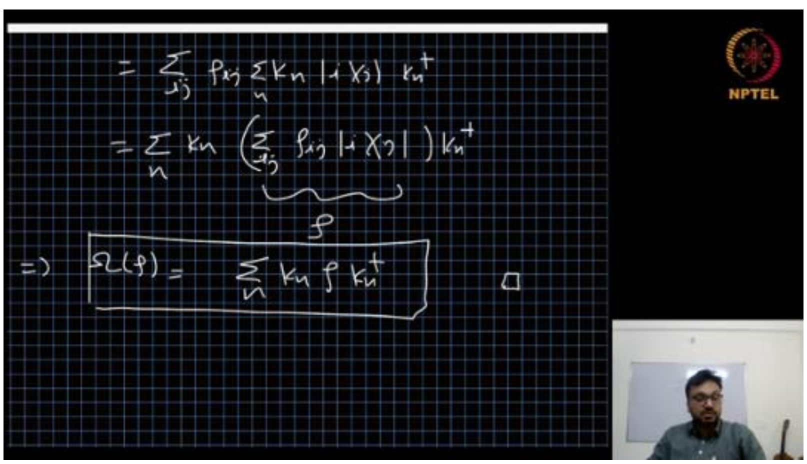

So, what the first theorem of choi is that operator sum representation implies completely positive map. And second one says that completely positive map implies operator sum representation. So, it means all the positive map can be written in the operator sum representation and operator sum representation can be written in terms of the positive map, is a completely positive map. So, with this, we are done with the proof of the choi's theorem. Just to connect it with what we have already done, for positive operator, positive maps also, we found that H can be related to the sum over n lambda n, K n outer product k n.

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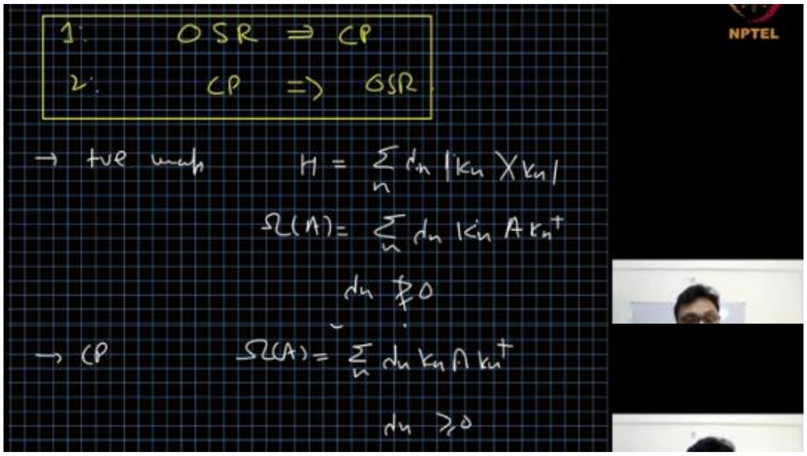

Okay, or the action of omega or a matrix A could be written as sum over n, lambda n, K n, A, K n dagger, but lambda n need not be all positive. Need not be all positive semi definite. Okay, but for completely positive map, omega of A can be written as sum over n lambda n K n a K n dagger where lambda n are all positive this is the difference between positive map and completely positive map but this condition is not enough for a positivity of the map, because this condition only assumes, this condition can be only derived from the hermiticity condition not the positive positivity condition of the map but all the positive maps are Hermitian also so this condition is valid for, this condition is necessary but not sufficient for all the positive maps but this is necessary and sufficient condition for the completely positive map, so in that way there is a relation between positive maps and completely positive maps. If we want to see then there is a set of all the maps.

It's a graphical representation, so all the maps we have, then there is a subset of these maps which are Hermitian maps then there is a smaller subset of it which are positive maps and there is a smaller subset for the smaller subset inside it which are CP maps. So, all the map, the Hermitian map, then positive map and all the CP maps. This small subset of maps is what represents the physical states on a quantum system. Only these maps, not all the other maps. So far, we have been only saying that completely positive maps represent physical processes and all the physical processes can be represented by completely positive maps.

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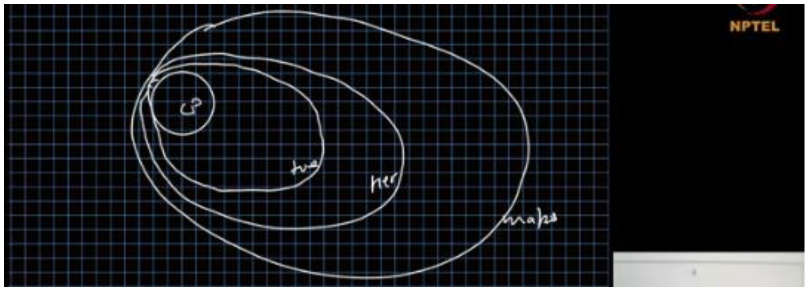

But in this lecture, we will establish that connection and give a formal proof of this statement. But before that, we need to understand what we call a physical process. In quantum mechanics, only unitary transformations are valid transformation. It means only unitary transformations can transform one state to another state. So, as far as quantum mechanics of an isolated quantum system is concerned, only unitary processes are physical processes.

If we assume that there is nothing beyond our existing universe, we see if there is nothing outside that universe that you know then that entire universe forms an isolated quantum system it means the dynamics of the whole universe will be a unitary transformation. So, if we are interested in the dynamics of a system as then we can think of it S plus the entire universe or we call it ancilla, some assisting universe or assisting quantum system. So, the total Hilbert space is HA tensor HS. So, any process which can be defined any process on Hilbert space HS if we can define it as a unitary process on HA tensor HS and the reduced dynamics of that only on the system, then we will call it a physical process. What I mean is if we have a dynamic or a process which takes rho to rho prime where rho belongs to the set of operators acting on HS then rho is a state of system S and the process takes rho to rho prime.

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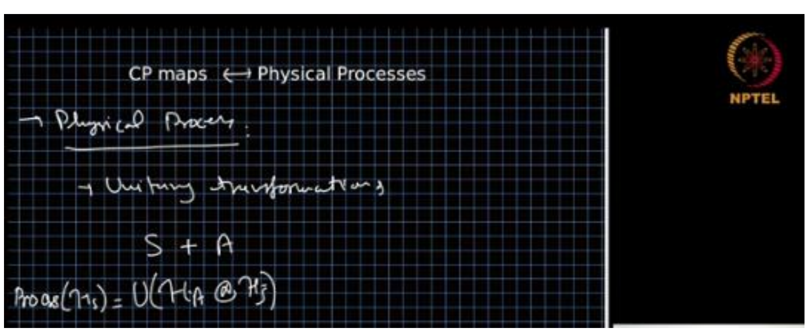

If we can write it as a unitary process on a state rho A of ancilla tensor rho S or rho of the system U dagger and then traced over ancilla, if the dynamics or the process we are interested in the physical process, if that can be represented by the unitary action of on the system and ancilla and tracing out the ancilla, then we will call it a physical process. So, in other words, if the desired process can be achieved by a unit transformation on an enlarged system, then this process must be physical. So, whenever we say physical process now onward, this is what we will mean. Now the point is from the Choi's theorem, we know if omega is a completely positive map then action of omega on rho can always be written as sum over i K i rho K i dagger, so this is the crux of the two Choi's theorem theorem and if we want the trace of the achieved state to be preserved so omega of the trace of omega of rho to be one for all the rho's then it means sum over i trace of Ki rho Ki dagger must be equal to one. One equality I would like you to verify that if we have trace of product of three matrices A, B and C, then it is same as trace of C, A, B. This is the cyclic property of the trace.

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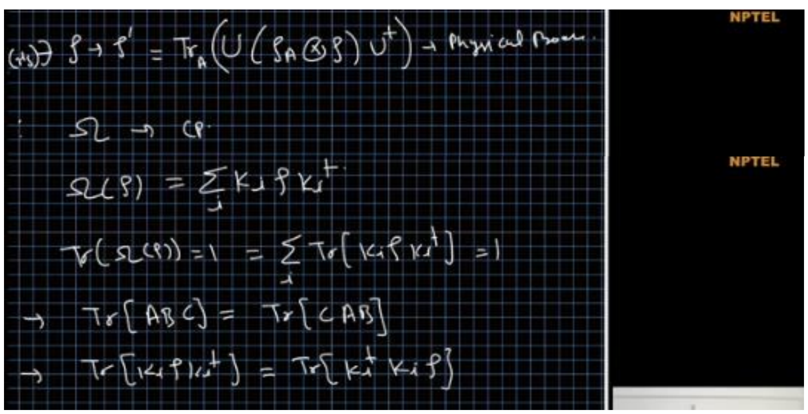

We can use this property. Then we get trace of Ki rho Ki dagger to be equal to trace of Ki dagger Ki rho. Hence, trace of omega rho becomes sum over i, trace of Ki dagger Ki rho to be equal to 1 for all the states rho such that trace of rho is 1. This is true for all the states rho. It means that sum over i, Ki dagger, Ki must be identity.

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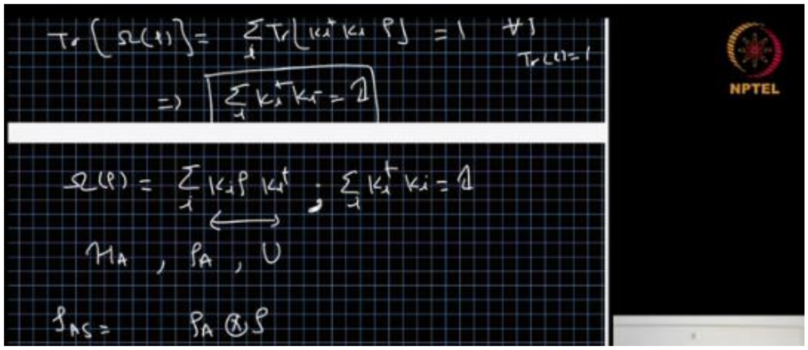

So, this condition over the operators Ki is the condition for trace preserving nature. So, omega of rho for a completely positive and trace preserving map can be written as sum over i Ki rho Ki dagger and Ki dagger Ki sum over i is identity. Now, we have to find a unitary process. We have to, in order to show that this completely positive map can be achieved, it represents a physical process, then we need to find an ancilla, HA, the Hilbert space of the ancilla and the initial state rho A of the ancilla, a unitary operator U such that the action of U on rho A tensor rho and then tracing out the ancilla gives us a completely positive map. The initial state let us say rho AS is rho A tensor rho where rho is the state we are interested in, after that we do the unitary transformation then we get rho tilde AS to be U rho A tensor rho U dagger.

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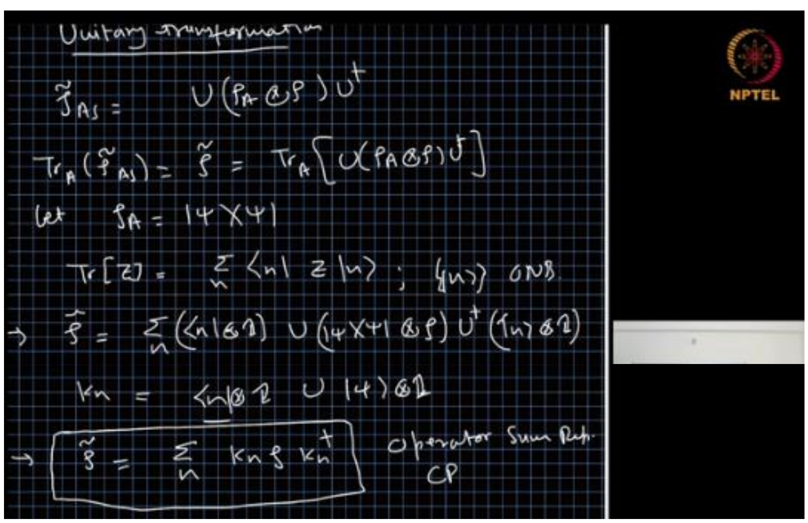

And then we trace over ancilla, we get rho tilde, which is trace over ancilla U rho A ancilla rho U dagger. Let us assume that rho A is a pure state. And let us remember the trace over operators Z is sum over n n Z n where n is the orthonormal basis. So we use these two and we get trace over, we get rho tilde to be sum over n, we are getting trace over only over one subsystem, so n tensor identity on the other, U rho A is psi tensor rho U dagger n tensor identity. So, this is our rho tilde. Now, let us define, now let us define Kn, which is n tensor identity, U psi tensor identity. So, we are projecting the ancilla space on the state n on the left side and psi on the right side.

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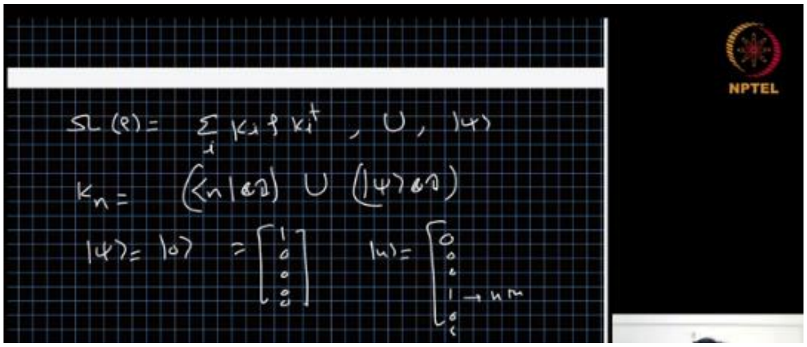

The U is an operator acting on ancilla and system. And we are projecting it in the n size space so that the leftover operator is the operator acting on only the system so K n are the operators acting on the system. So, with this definition we can see that rho tilde is sum over n k n rho k n dagger So in that way, every physical process represented by the unitary operator U and the ancilla states psi, we can always get operator sum representation. Hence, it is a completely positive map. But next question we want to answer is given a completely positive map omega, whose action is represented by sum over i, Ki rho Ki dagger, can we find a unitary operator U and the state psi corresponding to this set of operators Ki's, okay. Now, you see Ki is or Kn is n, which let us take to be the computational basis, U tensor identity U and psi tensor identity. For simplicity, let us say psi is the zeroth computational state, so it means it is one zero zero zero zero and so on and n is also a computational basis so it will be zero zero zero one zero zero zero where this is the nth position And U can be written as sum over ij, i outer product j tensor small u ij, okay. So, it means we can write it as u11 u12 u13 u21 u22 and so on, where u ij are the operators acting on the Hilbert space of the system.

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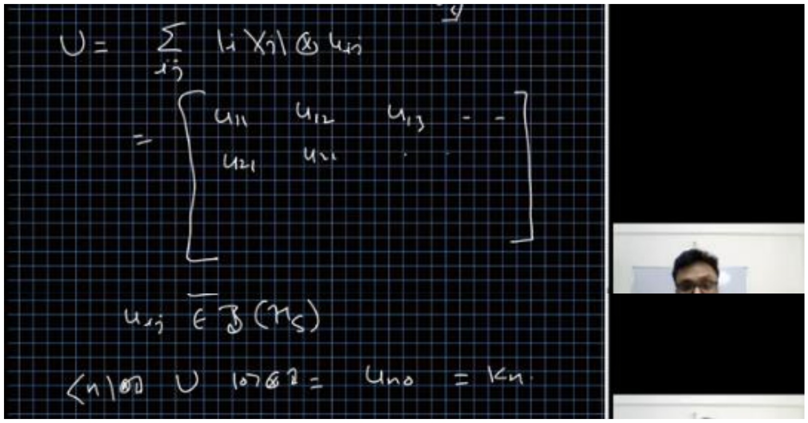

So, it means n tensor identity U 0 tensor identity is U n0 and which is K n. So, it means u 11, sorry it is 0. This is one, the counting is from zero not from one, so u 1 0 u 2 0 u 3 0 and so on are k 1 k 2 k 3 and so on. So, it means the unitary, if we choose unitary to be of the form k 0 k 1 counting is from 0, so let me make this also 0, k 2 and k n where n is there where n projectors or n k operators. From here we can see that the first column of operators is occupied by the Kraus operators k i's which represents the completely positive map omega and you know and U must be a unity operator so the rest of the matrix R is chosen in such a way that the U is unitary and a square matrix. And this R is completely arbitrary. We can choose anything as long as U is unitary. This with the initial state of the ancilla to be computational state 0, we have found a physical process corresponding to the completely positive omega given by the Kraus operators, represented by the Kraus operators k's. So, in that way, we have shown that completely physical process given by unitary operator U and initial state of the ancilla psi. this corresponds to a set of Kraus operators k n. In a set of Kraus operators k n, we can always find a unitary U acting on ancilla and system together jointly and the state of the ancilla to be psi.

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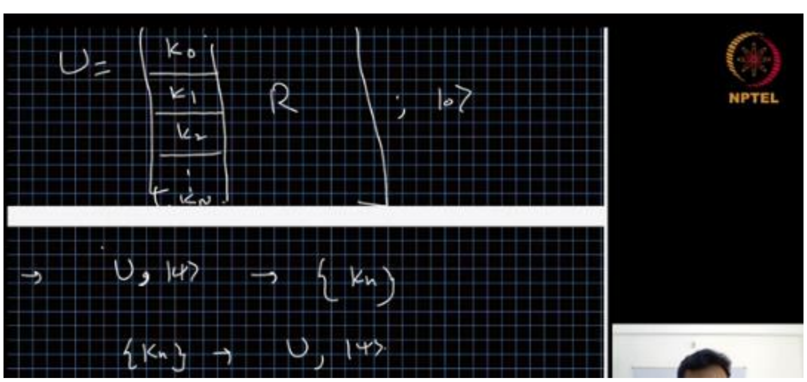

Let me repeat. What we have shown is for a given unitary acting on ancilla and system and the initial state of ancilla, this correspond to a completely positive map CP given by the Kraus operator kn. Alternatively, if we are given a set of Kraus operator, we can always find a unitary U acting on system and ancilla and an initial state of the system. So, in that way, completely positive maps represent physical processes and physical processes represent completely positive maps. And hence, we have established the relation between the completely positive maps and physical processes.

There are some properties of Kraus operators and completely positive maps, which we need to keep in mind. First of all, for a map omega sum over n, K n rho K n dagger, the choice of Kn is not unique. So, there exist R m such that R m rho R m dagger, sum over m is also omega of rho. To prove this thing, let us say R m is chosen in such a way it is a W nm K n. Okay, so and sum over n then R m rho R m dagger, sum over m can be written as sum over n, n prime and m W nm K n rho W star n prime m K n prime.

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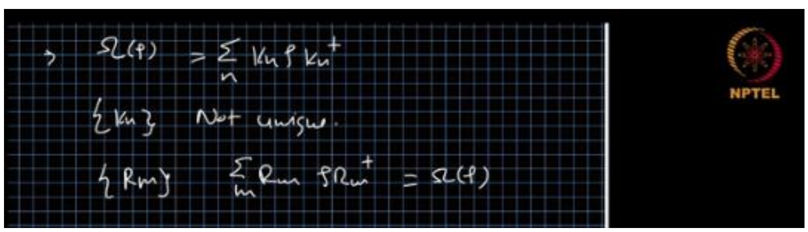

We can write it as sum over n, n prime, W n m, sum over m, W n m, W star, n prime m, K n rho, K n prime dagger. Sum over n or m, W n m W star n prime m is sum over m W n m W dagger matrix and m n prime element. Now you can see this is the product of W and W dagger, this is the product of W and W dagger and they're jointly n n prime element Now if we choose, it means W is isometric. Then W dagger, WW dagger nn prime is delta nn prime. It means sum over m, R m rho R m dagger becomes sum over n n prime delta nn prime K n rho K n prime dagger, which is same as sum over n K n rho K n dagger.

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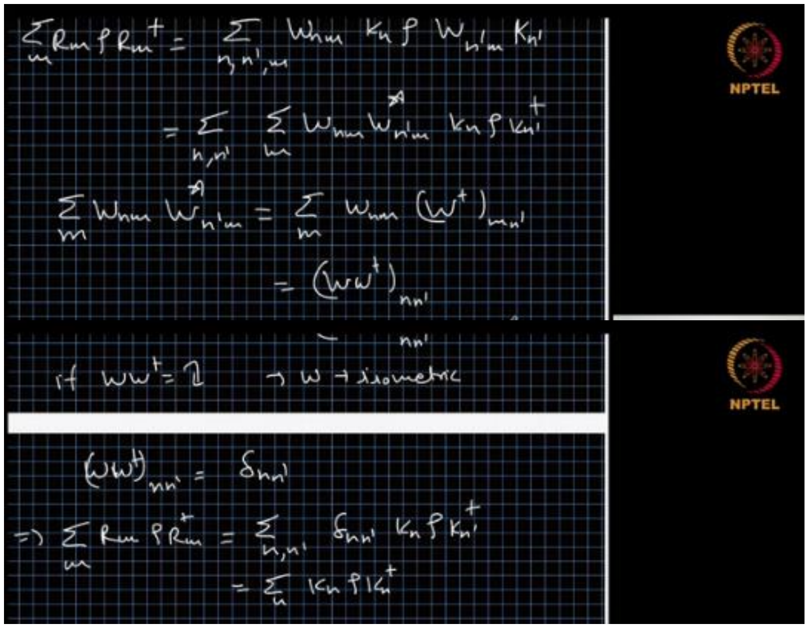

So, in that way, if the two set of Kraus operators are related by an isometric matrix, then they will represent the same completely positive map and the choice of the Kraus operator is not unique. To find the set of Kraus operator for a given completely positive map omega, what we need to do is, we apply omega and find the kl element, sum over ij M ij kl rho kl. So, from here we can find the matrix representation of Omega that is M, from M, we can find the H representation and these two are related as follows M ij kl equals H ik jl We have done it earlier, when we are talking about the positive maps and for since H is positive, we can show that for completely positive maps, H is positive and H is Hermitian of course, then H can be written as sum over n lambda n k n k n where lambda n are the eigenvalues and k n are the orthonormal eigenvectors of H.

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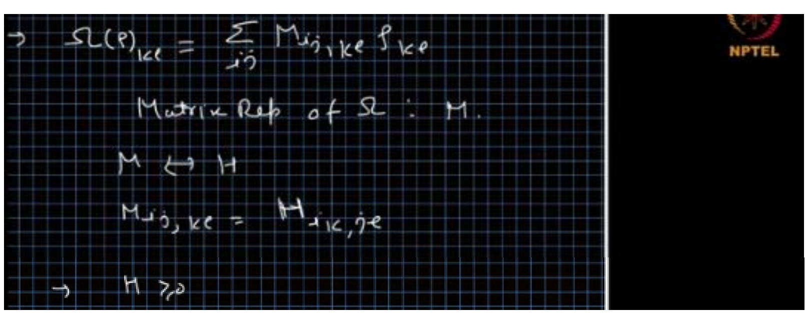

And from here let's put tilde here for normalized and sum over n k n k n where these are unnormalized, k n is defined as sum over lambda square root of lambda n k n tilde. k ns are the unnormalized vectors. From here, H sum over n k n lambda n, we can show that omega of rho is sum over n k n rho k n dagger where k n are the folded version of the k ns. It means we convert a vector into a matrix and then that is how we get the Kraus operator k n. This is one choice of Kraus operator and this is in some sense, it is a canonical choice of Kraus operator because they are achieved from the spectral decomposition of the H matrix. Now, we can use the isometric transformation on k ns and we can generate large number of set of operators which represent the same CP map and we can choose the most convenient Kraus operator for us.

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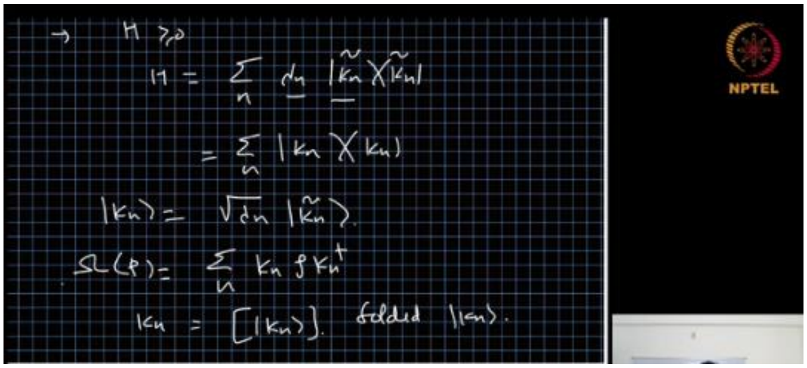

In that way, we can find the set of Kraus operator for any given completely positive map.