FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

Dr. Sandeep K. Goyal
Department of Physical Sciences
IISER Mohali
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Quantum Maps: Completely Positive Maps - Part 01

Today we will be discussing completely positive maps. This is in sequence, in continuation with the positive maps and stuff we have been studying so just to recapitulate a positive map, omega is something which takes positive operators to positive operators. So, rho is a positive operator it means its eigenvalues are positive semi-definite. We mean when we say positive, we can mean positive definite or positive semi-definite, it's a very small difference for us but in mathematics it can be a big difference, but we are not worried about that for the moment. So, rho is a positive operator so that eigenvalues are positive and rho is Hermitian and it will be mapped to another operator rho prime which has a similar property that its eigenvalues are positive and its Hermitian. So, a positive map is Hermitian as well as positivity preserving. Now what will happen if we apply this map omega which is a positive map on one subsystem, on of a larger system. So, if we are applying omega on subsystem B and nothing on subsystem A, so we can write it as i tensor omega i means applying nothing identity operation and if we apply it on a bipartite state rho AB, so rho AB is the state of a mixed state or pure state doesn't matter, it's the state of a of a bipartite system of a composite system. So, what will happen if we do that will it still be a positive map. So will omega still map positive operators to positive operators. So, the question is if identity tensor omega maps rho AB to rho AB prime, some other state of the same subsystem.

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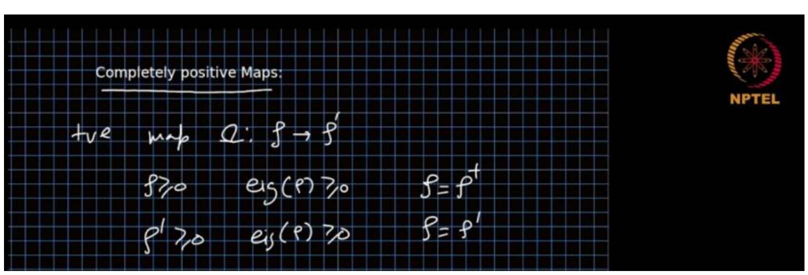

To see that, let us take example, let us say rho AB is in the separable space. It means rho AB can be written as sum over n, p n rho A n tensor rho B n, where p n are positive numbers such that they add up to 1. It means sum over n p n equals 1. and rho An is a positive operator, which represent a density matrix in a subsystem A and rho B n is a positive operator representing a density operator in subsystem B. So, when we apply the positive map omega on one side of rho AB on one subsystem of rho AB, then it will be it can be written as p n rho A n, means nothing on An tensor omega acting on rho B n. So, if omega is positive and it is positive then rho B n will be rho prime B n which are also positive operators. So, it means the transformed state rho tilde AB which is this here, is nothing but sum over n p n rho A n tensor rho B n prime.

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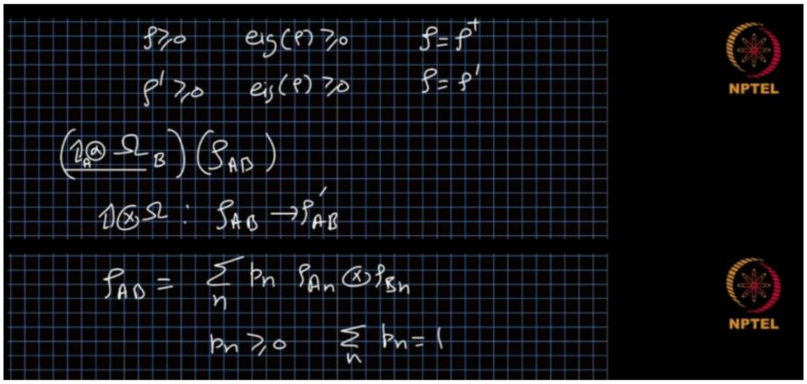

It's a state rho A n rho prime B n, pn, all of those are valid things, whatever we needed. And needed instance like this is a valid state of a system B, this is a valid state of a subsystem A. So, the total thing is a valid state, then we are taking the mixture of these valid states. So, this is also a valid state. it is possible that omega is just positivity preserving positive map but not just preserving in that case this will have different trace so it will be a unnormalized valid state but still a positive operator and we can normalize it and we can get a state we want so in that way identity tensor omega acting on rho AB results in a positive operator rho prime AB. But this is the case when rho AB is a separable state because we were writing rho AB as sum over n pn rho A n tensor rho B n and we have already discussed that these kinds of states are called separable states.

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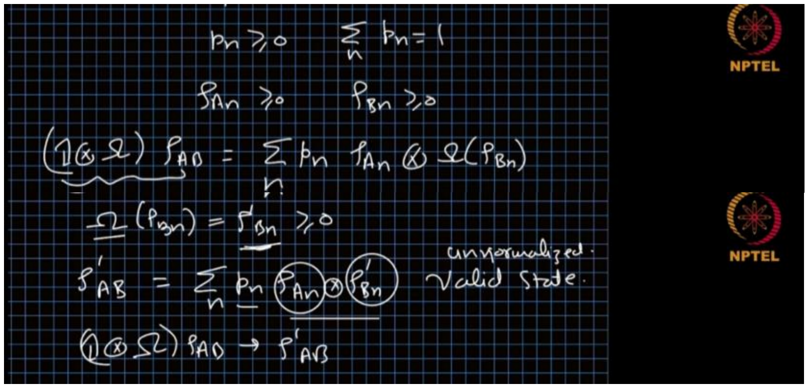

So, there's no wonder that a positive map goes to positive map because the matrix representation or like it's not because but we can get a hint that this will be true from the matrix representation of omega, the matrix representation of omega was M and from here we went to H and we know about one, what we know about H is it's positive, if we have psi tensor phi H phi tensor psi, this quantity is always positive. So, the H matrix corresponding to a positive map is always positive for all the product states and separable states is just a mixture of product states. So, in that way, we can get a hint that probably this statement is always true. What we want is, is it always true, is it always that, I tensor omega acting on rho goes to rho prime always positive for all rho positive. If rho omega bar goes to rho bar prime positive for all rho bar positive, it means is it always true that if omega is a positive map ,then it acting on a subsystem of a composite system will also be a positive map. Now, a general proof might be very difficult, but one counter example should be enough to prove that this is not the case. So, the example we have, our counter example is a transfer map.

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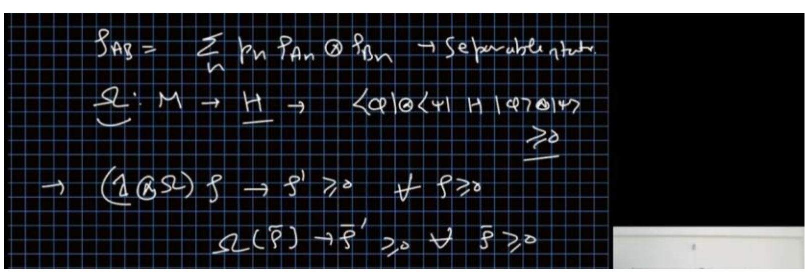

So, transpose map takes rho to rho transpose. If rho is positive, then rho transpose is positive. If we apply identity tensor transpose map on rho AB, which is sum over n p n, which is a separable state, if we apply it on a separable state, rho B n transpose, then this also is positive. So, it satisfies all the properties so far of a positive map. So, transposition is a positive map but what will happen if we apply it on a state which is not a separable state, so let us take a state psi which is zero one minus one zero over two. Okay, so in a vector form it will be 0 1 minus 1 0 over root 2. So, the density matrix corresponding to this will be 1 over 2 0 1 0 1 plus 1 0 1 1 0 minus 0 1 1 0 minus 1 0 0 1.

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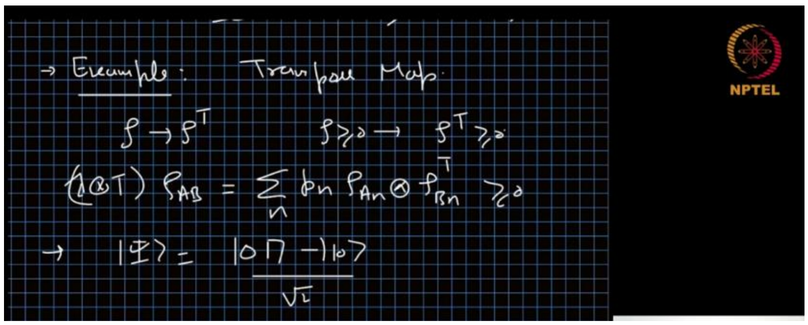

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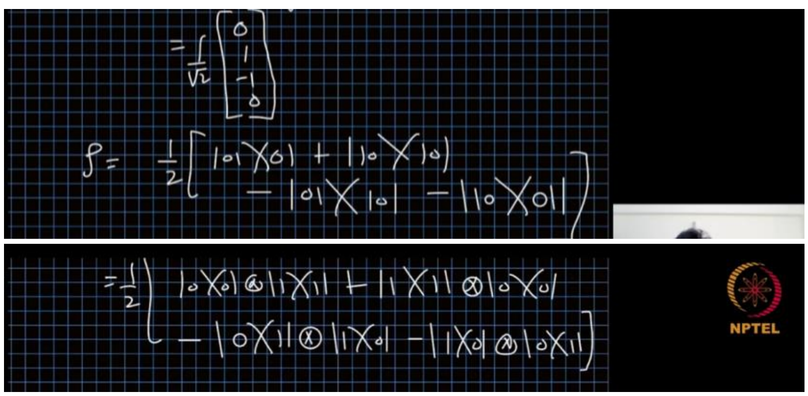

We can write it in a simpler form which will be you see here we have 0 tensor 1 and 1 0 tensor 1 and 0 tensor 1 here, so it can be written as 0 outer product 0 tensor 1 outer product 1 plus similarly here, one outer product one tensor zero outer product zero and this term here minus zero outer product one tensor one outer product zero minus one outer product zero tensor zero outer product one. So, when we take partial transpose over this state rho A B, this is the rho A B, we have then it will be the transposition on the second subspace so transposition on one one it will remain one one zero and one real state so there is no complex conjugate coming here one one tensor zero zero will remain zero zero minus one zero or zero one tensor one zero will become zero one after taking transposition minus one zero tensor one zero. So, this is our state or this is our matrix after transposition now we can write it and this can be a decent exercise problem that this matrix will look like the following, just I'm putting the grids to this is, minus one minus one and this is one one all other are zero. So, this is our matrix after applying partial transposition and we can see that the eigenvalues of this are plus half and minus half, plus half appears three times and minus half appears once. So, these are the eigenvalues of this matrix.

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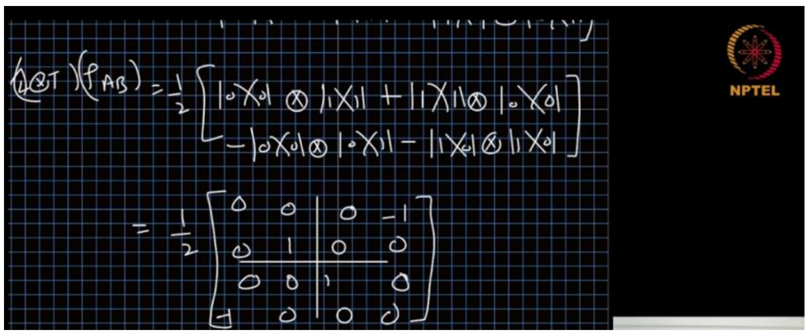

So, in that way identity tensor transpose acting on a subsystem of a larger system need not be a positive operator even if even if rho AB is positive. So, this is one counter example where we see that not all the positive maps which map positive operator to positive operator are positive maps, when we apply them on a subsystem of a larger system. So, it means even if rho is positive does not imply that identity tensor omega is positive. It means if omega is positive does not mean that identity tensor omega is a positive map. So, here we define a new operator.

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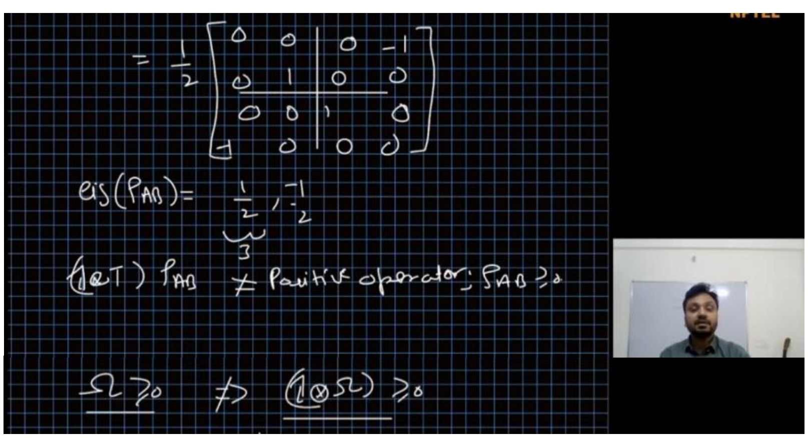

We call it m positive map. M positive maps are those maps, where, we have identity, we have a subsystem which is m dimensional and our map omega, this is positive for omega positive. If you have a map which is positive map omega, it is also positive when we apply it on a bipartite system, one side of the bipartite system where the other party is m dimensional then it is called a m positive map. But we are not interested in that. We are interested in the completely positive maps. Completely positive map is a m positive map for all m. If we have a map omega, which is positive and it is m positive for all the values of m, then it is a completely positive map.

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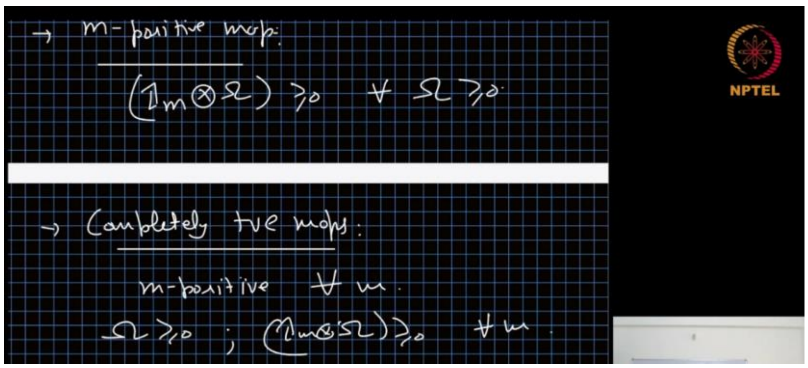

So, let me repeat if we have a map omega which is positive and I m tensor omega is also positive for all m then it is called a completely positive. So, the benefit or advantage of a completely positive map is that a completely positive map omega or completely positive or CP we will often say a completely positive or CP map omega is, will map density matrices to density matrices no matter on what subsystem on what system on what dimension we are applying them. So, a completely positive map will always map positive operators to positive operators. Hence, all the physical processes can be represented by a CP map. Because a physical process will always take a state of a quantum system and it will map to another state of a quantum system.

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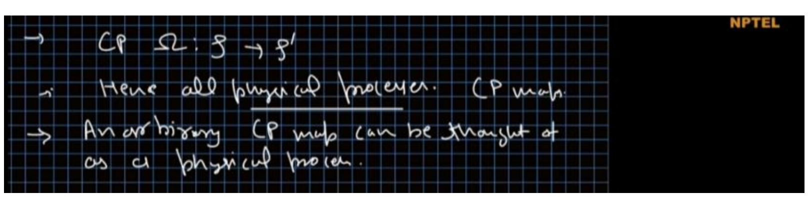

So, in that way, it's a process, it's a map which maps positive operator to positive operator and hence it can be represented by a completely positive map. An arbitrary completely positive map can be thought of as a physical process or in simpler words, if we have an arbitrary completely positive map then we can always find a physical process representing that map. So, in that way, completely positive maps represent physical processes and physical processes can be represented by a completely positive map. In that way, the set of completely positive maps is equivalent to the set of all the physical processes we can perform on a quantum system. There are some very interesting theorems, demonstrating these concepts, proving these concepts so and making them more formal. So, they are called choi's theorems.

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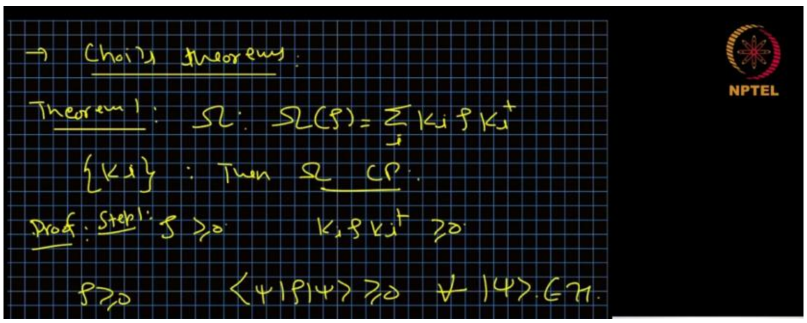

There are two theorems and the theorem one, the first theorem states that that if we can write the action of this map on an operator rho as some more i ki rho ki dagger, for an arbitrary set of operators ki, then this map omega must be completely positive. Let me repeat it. If we are given a map such that the action of this map on an operator rho can be written as sum over i Ki rho Ki dagger. For an arbitrary set of operators K i, then omega must be a completely positive map. Proof is reasonably straightforward.

First, we see that if rho is positive, then K i rho K i dagger is also positive. Let us call it step one. Okay, so how do we prove that the definition of positivity rho being positive is that its expectation value is positive for all the states psi in the Hilbert space. Now, if we have rho prime, which is K i rho K i dagger, then psi rho prime psi which is equal to psi K i rho K i dagger psi, let us call them phi rho phi and K i need not be unitary, they can be any arbitrary operator. So, phi need not be normalized, so we can put a normalization constant n outside so that phi is normalized and n is a normalization constant, so it's the positive number. And we know that rho is a positive operator, so phi rho phi is also a positive operator, so this is always positive for all the phi's. Hence for all the psi's and hence for all the K i's so this implies that K i rho K i dagger is a positive operator good.

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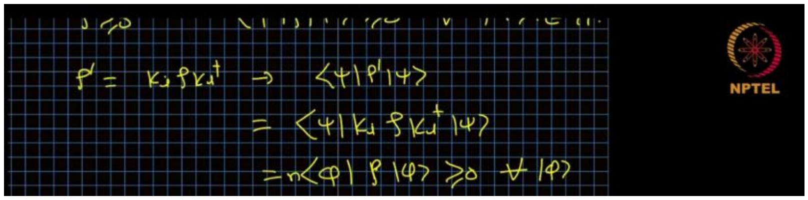

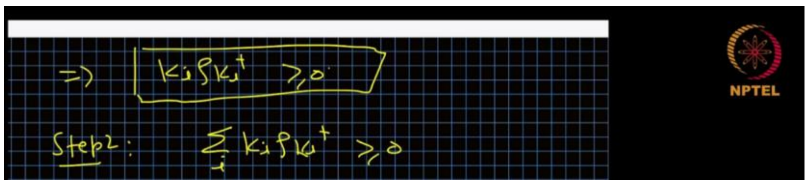

Step two, sum over i K i rho K i dagger is positive, of course some of the positive operator is also a positive operator, so we don't know, there's nothing to prove here. Step three, Identity tensor omega rho acting on rho is rho AB let us say, bipartite system is sum over i identity tensor K i rho identity tensor K i dagger there is nothing to prove here also this is the definition of the map omega and the partial implementation of it so if rho AB is a positive operator. So, is identity tensor K i rho AB identity tensor K i dagger and hence sum over i, identity tensor K i rho A B identity tensor K i dagger positive and here we have not assumed anything about rho A B other than the fact that it is a positive operator. This implies that if omega of rho is sum over i, K i rho K i dagger, then identity tensor omega acting on rho AB is positive for all rho AB positive. Hence, omega is a completely positive number.

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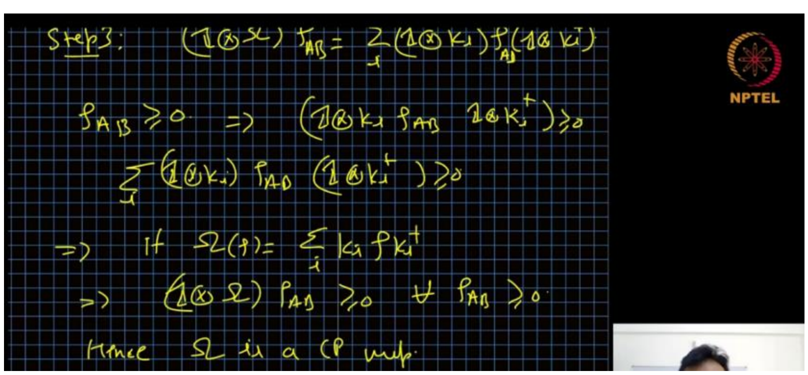

So, that's the proof of the first theorem. Now the second theorem. If we have a CP map, omega, then there exists a set of operators K i such that omega of rho can be written as sum over i K i rho K i dagger. So, the second theorem says that for every completely positive map, the representation sum over i K i rho K i dagger exists. It means we can always find a set of operators K i such that omega over rho can be written as sum over i k i rho K i dagger.

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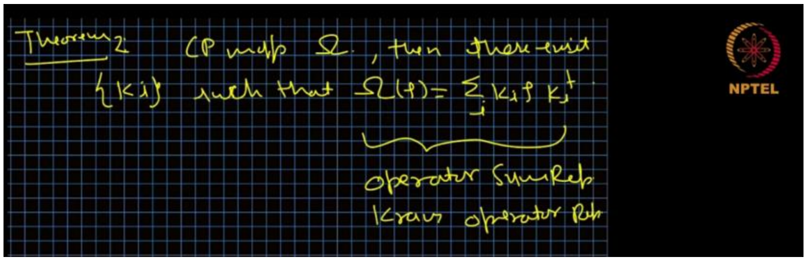

This is called operator sum representation or Kraus operator representation and the proof of this goes as follows. Since omega is a CP map, then identity tenser omega is also completely positive. Now, first let me say, let's start with the computational basis i, and what do we mean by computational basis that i vector is zero, vector of zeros and exactly one one at i th position and rest zeros. Okay, so we have a computational basis and we can define computational basis of the bipartite system it will be i tensor j where i and j runs from 1 to n or d. What is the dimension, whatever is the dimension and we are assuming the bipartite system has two parties of equal dimensions here. Now, we can define a state or a vector. So, normalization for our proof, the normalization does not matter here, but we just say we have a vector here, which is sum over i from 1 to d, i times i, so it is ii.

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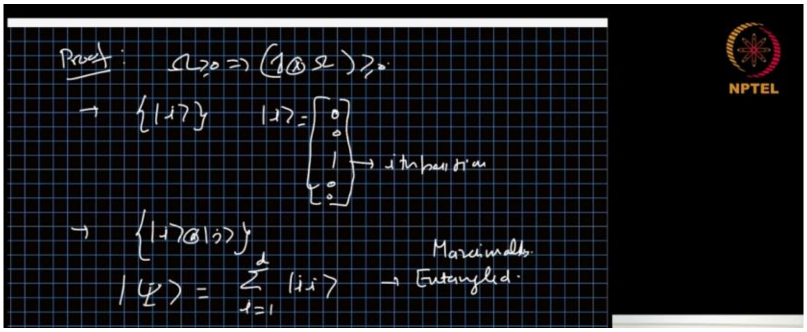

Later on, we will realize that this is an entangled state, unnormalized or maximally entangled state actually. But for the time being, that is irrelevant, so from here we can define operator E which is psi outer product psi. Since it is operator E which is psi outer product psi, so E is a positive operator. We can see that it is a positive operator because again like for any phi E phi it will be phi psi psi phi which is phi psi mod square which is a positive number or 0. So, in that way E is a positive operator. Now, we come back to our completely positive map. So, if we apply omega on one side of E, if it is a positive operator, then from there, we can say something about the representation of the omega that is the operator sum representation sum over i K i log K i. So, we have to prove that omega acting on E is positive, but how that will result in K i's? will that also we have to answer eventually. So, first before going there, we need to see over the structure of E. E can be written as sum over i j i i outer product j j. This is what it means by taking the outer product of psi with itself. And if we simplify it, it will be ij there will be i outer product j tensor i outer product j.

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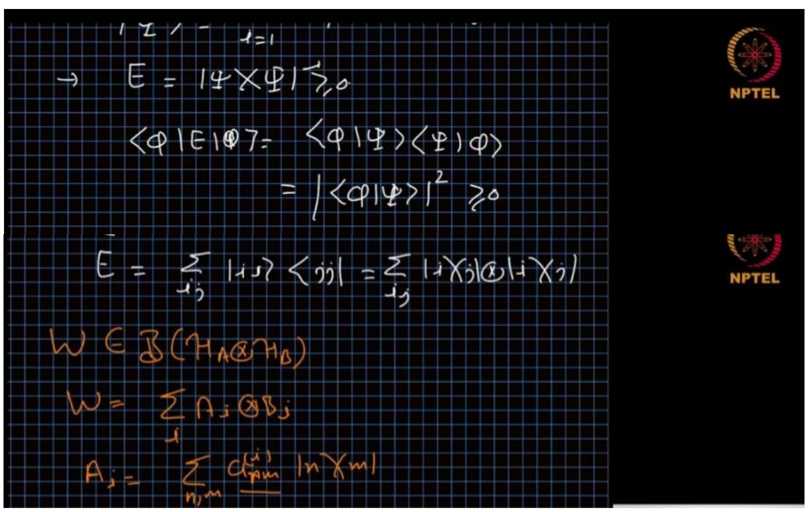

So, to understand, to appreciate the structure of E, first we notice that if we have any arbitrary operator W, which acts on the set of operators, which belongs to the set of operators acting on HA tensor HB, then this W can be written as some over i, Ai tensor Bi. So, Ais are the operator acting on HA and Bis are the operator acting on HB. Now, any matrix A, the Ai here, for example, can be written as sum over n, m, small a, nm, n outer product m. So, in this, if n is a computational basis, then a nm, small a nm, is the coefficient of matrix Ai. Let me put i as somewhere here. Then Ai n m is the coefficient of matrix Ai at a location n m.

So, we have matrices, we have locations at 1, 1, 1, 2, 1, 3 and so on. So, this will be the element at n m location. So, if similarly, we can write for B but we don't need to here, so W becomes sum over n m i, if we substitute for a i here then it will be it will be n outer product m tensor a i n m b i, the summation i can go inside this so we can write it as sum over n m, n outer product m tensor B nm. The B nm is an operator acting on Hilbert space B. B nm is nothing but sum over i a i nm B i. So, what is the advantage of this?

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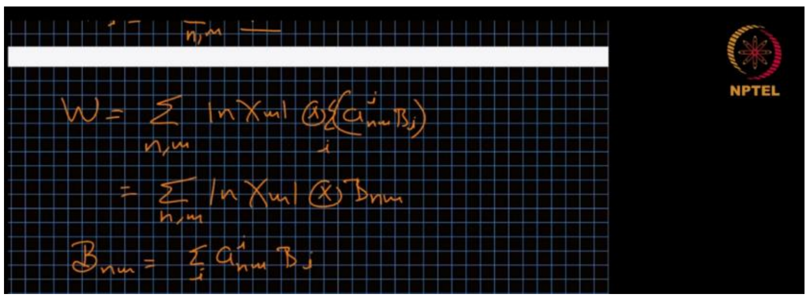

It means we can write W as a matrix of matrices. I'm just putting this grid for the reference purpose. So, this W can be written as a matrix of matrices where each element is a matrix here, which is B11, B12, B13, B14, B21, B22 and so on. So, this block, if we write just along this block, it will be one outer product one tensor B11. Similarly, ijth block will be i outer product j tensor b ij.

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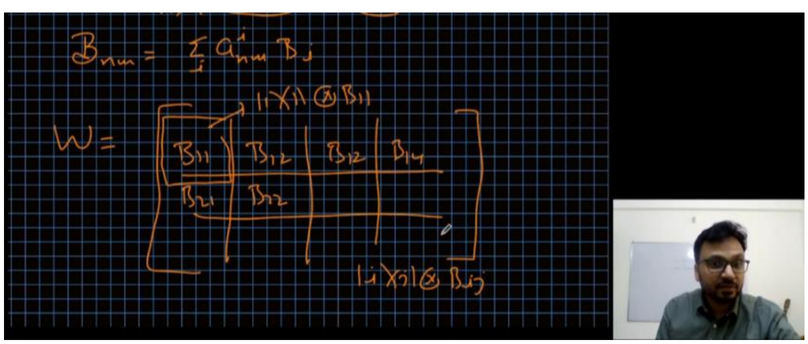

So, it means an arbitrary operator W can be written as the location of the block tensor the block matrix. So, going back to our E. E, which was i outer product j tensor i outer product j, sum over ij says that in the ijth block, the matrix is ij and i j is a matrix with one at ijth location and zeros everywhere else. So, this is the special matrix in that sense. Okay, and it will be useful for our proof. Now, if we apply the completely positive map on E, it will be sum over ij, i outer product j, tensor omega acting on ij.

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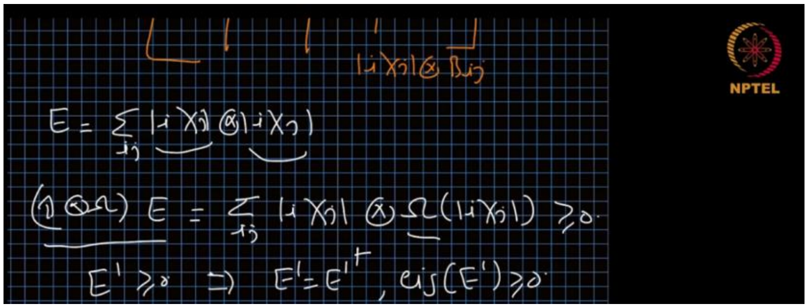

So, we do not know what is the exact action of omega on ij element. But we know for sure, the only thing we know is since omega is a completely positive map and since E is a positive operator, then i tensor omega acting on E is also a positive operator. Let us call it E prime. So, E prime is a positive operator. The property of a positive operator is, E prime is Hermitian and the eigenvalues of E prime are positive semidefinite.

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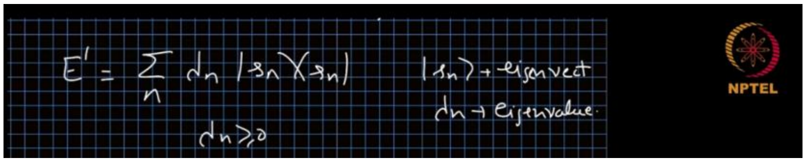

What does it mean? It means that. It means that E prime possess a spectral decomposition sum over n, lambda n, sn, sn. Where sn are the eigenvectors and lambda n are the eigenvalues. Since E prime is a positive operator, lambda n's are positive semi-definite and sn are the orthonormal basis.

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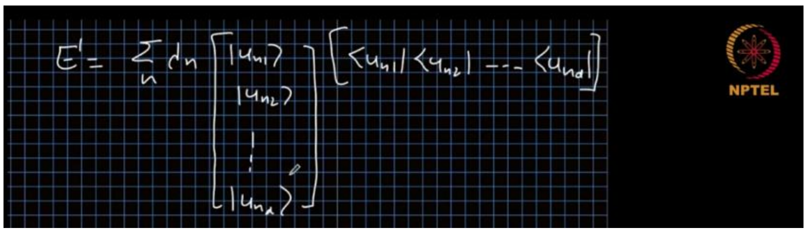

Now let us take Sn and since we know E prime is a d square dimensional matrix because we have chosen each subsystem of d dimensions, then sn is a d square dimensional vector. So, it is a vector d square dimensional. Let us say we have first d element, we call it a vector u1 or un1, the next d element we call u n2 vector and so on. They are all normalized, the u nd, the d element vectors of d dimension each, we have and we are calling them u n1 u n2 u n3 and so on. Okay, we are just writing them for the time being, for the ease of calculation, so E prime becomes sum over n lambda n u n1, u n2, u nd and its dagger, which is u n1 ,u n2, u nd. And if we expand it, we can keep the sum over n outside and lambda n, it will be u n1 u n1, u n1 u n2, un1 und. un2 un1, un2 un2, un2 u nd and so on. So, if we look at the ijth block of it, ijth block, we can write it as ij tensor the block and that block will be sum over n lambda n u ni u nj and sum over ij. So, this is the whole E prime matrix and E prime, if you remember its identity tensor omega acting on. Now, compare it with E and we will see that omega acting on i j is nothing but sum over n lambda n u ni u nj. And this thing can be written as sum over n since lambda n are positive numbers so we can define u ni tilde u nj tilde, where u ni tilde is nothing but square root of lambda n u ni and if we define a matrix Kn such that the first column is u n1, second is u n2 and we have u nd. Then we can see that Kn acting on computational basis i will give us u ni, so which means omega acting on i j can be written as Kn tilde i j K n dagger. Now, if we are given any operator rho, we can always write it as rho i j, i outer product j sum over i j.

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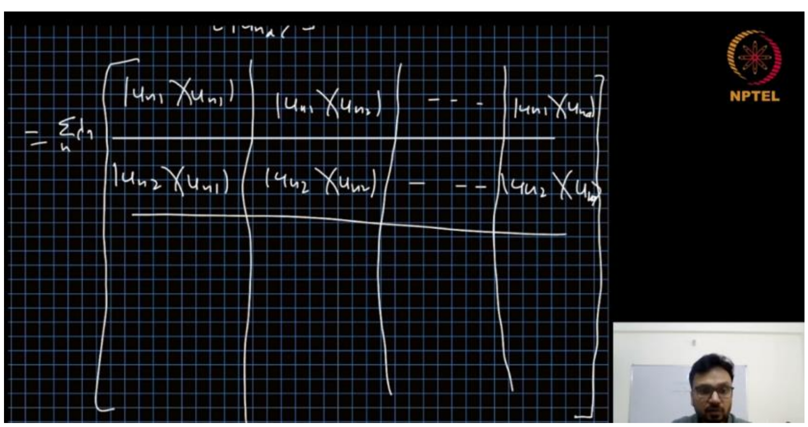

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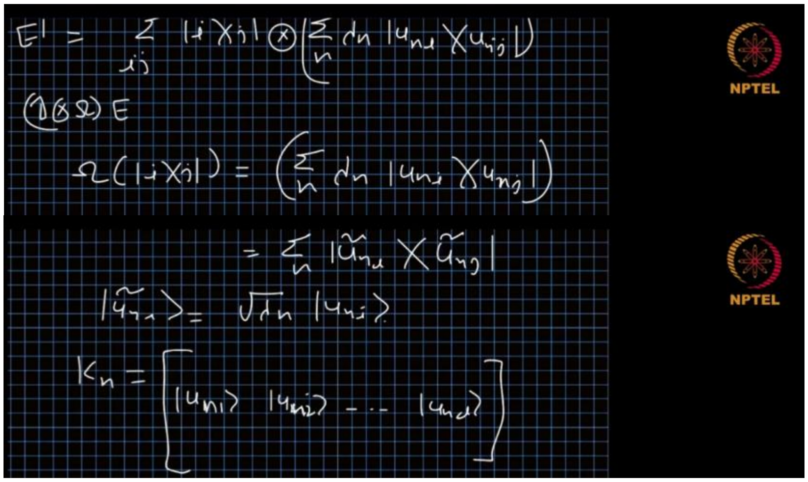

There was sum over n also. Then omega acting on rho can be written as sum over ij rho ij omega ij and we have shown that omega ij can be written as Kn, sum over n, ij Kn dagger. We can take rho ij and summation over ij inside the summation over n, we get k n sum over ij rho ij i outer product j Kn dagger and this is rho, so omega acting on rho can be written as sum over n Kn rho Kn dagger. Hence, proved.

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