FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Lecture-15

Quantum Maps: Positive Maps

So, in this and the next lecture, we will be talking about quantum maps and they will play a very important role in entanglement analysis and in answering the question whether a given state is entangled or not. They are also the basis of physical processes and how we represent them mathematically and many other things, so, this is a very important unit in this course, let us say we have a map omega, it's a mapping between the operators acting on Hilbert space to operators acting on Hilbert space. For simplicity, we will consider the same Hilbert space, so operators acting from one Hilbert space to itself. This can be a map from a set of operators acting on one Hilbert space to set of operators acting on another Hilbert space. So, it is just a mapping between operators to operators, the dimensions can be decided later on, but for simplicity, unless we say it explicitly, we would be assuming the same Hilbert space.

So, a linear map, a map will be linear if you have operators A and B with coefficients, complex coefficient alpha and beta and it gives us alpha omega A plus beta omega B. Where for A, every A and B that belongs to the set of operators and for alpha beta complex or any element of the field. So, if a map satisfies this property, then we call it a linear map. And we know very well that whenever there is a linear map, there will be a matrix representation for it. So, what do we mean by matrix representation? So, we have, for simplicity, we are assuming that A and omega of A belongs to the same set of operators.

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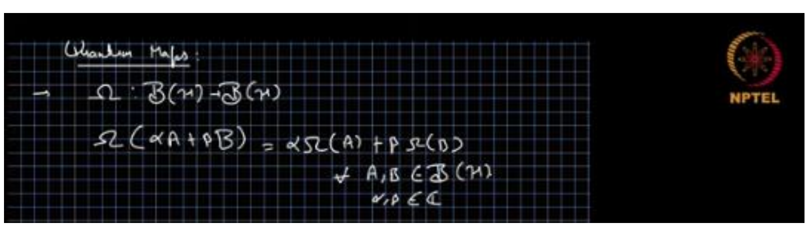

Of course, it can be generalized to different set of operators. So, but for the time being, we are just assuming that. So, we can say the map A, the map omega acting on A is a matrix, of course, and we consider the ij-th element of this new matrix. This will depend on the elements of the matrix, original matrix A. That will be kl and a matrix relating the elements of A with elements of omega A. So, it will be i j and k l, sum over k l. So, this is one way of representing the action of omega on a matrix A. So, M will be the matrix representation of omega.

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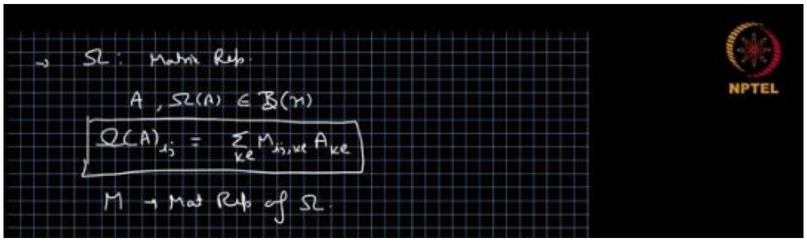

So, we are using double index for, so if we identify ij with small n and kl with small m, then we can say that omega of A n is M nm A of m, sum over m. So, in that way, you can see that we have a vector, which is omega of A and that is equal to some matrix acting on vector A. So, how is this vector A is related to the matrix A? So, here is the recipe. So, if A is, let us say a simple example, if A is a 2 by 2 matrix, it will be a11, a12, a21, a22. These are the four elements of it. Then the vector A will be the first element, a11, the second element a12, the third element a13 and the fourth element a21 and a22. This is how we map the matrix A and the vector A and this we will call unfolding of the matrix. In the literature might be known with many other names.

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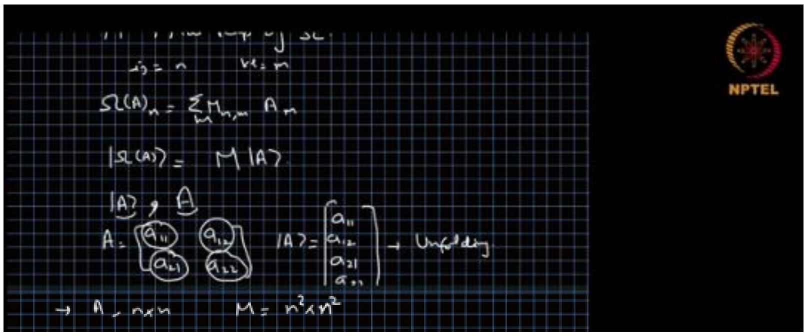

So, in the similar fashion, we can unfold the omega A matrix and we get a vector omega A and then the transformation connecting with these two vectors will be represented by a matrix M. So, if A is an n by n matrix, then M will be n square by n square matrix. Okay, because it's mapping a set of operator to itself. So it will be n square by n square. But if it were mapping from different sets, then this matrix would have been, it can be different. Now, this is a general map.

We are interested in more specific type of maps. So, the map we are interested in is Hermiticity preserving map. Or we will call it Hermitian maps also. So, the definition of Hermiticity preserving map is a map, we have omega such that omega of A dagger is omega of A dagger for all A in the set. So, how is it Hermiticity preserving if A is Hermitian?

Then omega of A is same as omega of A dagger and omega of A dagger by definition is omega of A dagger. So in that way, omega of A and omega of A dagger are equal, so, in that way so if we say omega of A is B then B equals B dagger So, in that way, this map will map Hermitian operators to Hermitian operators. So, in that way, it's a Hermiticity preserving map. Now, for a Hermiticity preserving map, how does the matrix representation M of this map look like? See, we have a map omega, which is Hermiticity preserving.

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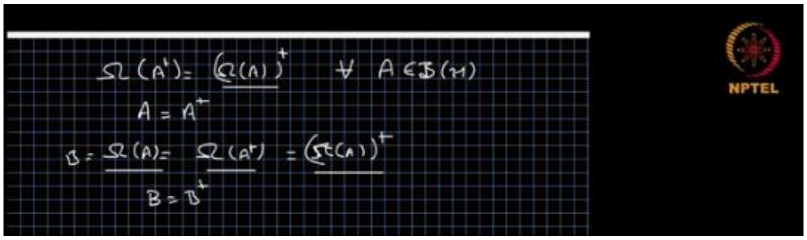

So, omega of A dagger is omega of A dagger. And it means if we have omega of A, its ij-th element is same as omega of A star ji element, because, if this is true if aij is equal to a star ji. This is the condition that A equals A dagger and we are saying this implies that omega of A equals omega of A dagger. Now let us see omega of A i j, like we said earlier will be M i j k l A k l, sum over k l and omega of a star j i is equal to sum over k l, we just take, we perform this transformation here. We get M star ji.

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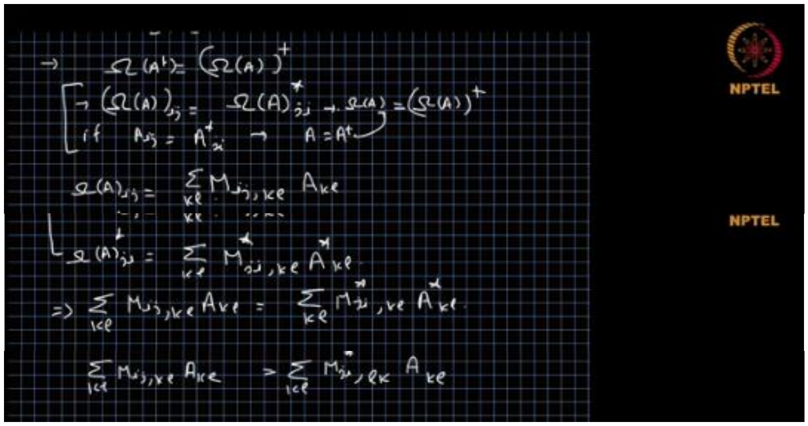

We have to interchange i and j. So, we interchange i and j. We take the complex conjugate. It is A star kl. And if omega is A Hermitian map, then these two conditions are same. So, we get sum over kl M ij kl, A kl equals sum over kl, M ji star kl, A star kl. But these two conditions are same only when A kl equals, if A is a Hermitian operator. So, A kl is equal to A lk star.

So, we can replace this, M j i star kl, and A star kl is equal to A l k, we have to take transpose and we have to take complex conjugate, so this is the case. Since k and l are dummy indices, we can interchange them here, l k and it we have to interchange here also, kl now we have i j k l A k l. Now, this is the final equation we have, not final, but almost final equation. Now, we write it as matrix representation M, the matrix M acting on unfolded vector A equals M bar unfolded vector A, where M bar is defined in such a way, it is ij kl element is M star ji lk. So, if we remember our linear operator lecture, this is true for all vectors A, all the Hermitian vectors A. So, if we remember our lecture from the linear operators, if such conditions are met, then M must be equal to M bar. This implies that M ij kl must be equal to M star ji lk.

If I give you a matrix representation of a map omega, and M is the matrix, then we interchange ij element and we interchange kl element and take the complex conjugate. And if these two numbers are same, then it must be a Hermitian map. So, if this condition is met, then we have a Hermiticity preserving map. This looks a complicated condition. So, what we do is we define a matrix H such that H ij kl is M ik jl.

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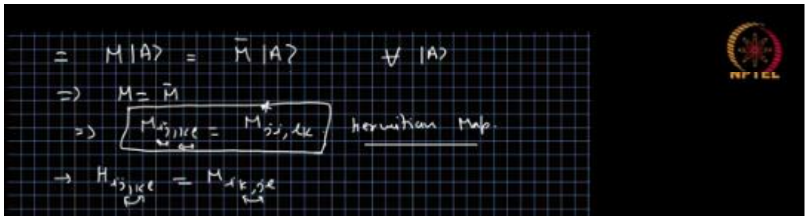

So, what we have done is we have just interchange j and k elements. And H matrix contains the same element as M but reshuffled. So, they are kept at different places but it contains all the elements. We will show what this transformation look like. So, M ij kl is H ik jl and M star ji lk will be H star jl ik. So, it means the condition, this Hermiticity condition on the map will look like H Iik Jjl equals H star Jjl ik. Now, how is it a simplification?

If we identify ik with n and jl with m, then the condition reads H nm equals H star mn. So, in other words H is equal to H dagger, so, H itself is Hermitian. Let us go over it again, we have a map omega which is Hermitian map. We have matrix representation of it M which has a very complicated condition for it to be Hermitian map. From here we go to H and we get a very simple condition that is H should be Hermitian for a Hermitian map. When we go from M to H then it is very straightforward to check whether a given map is Hermitian or not. Next, we will see the relation between the matrix M and H in terms of their position of the location of the elements.

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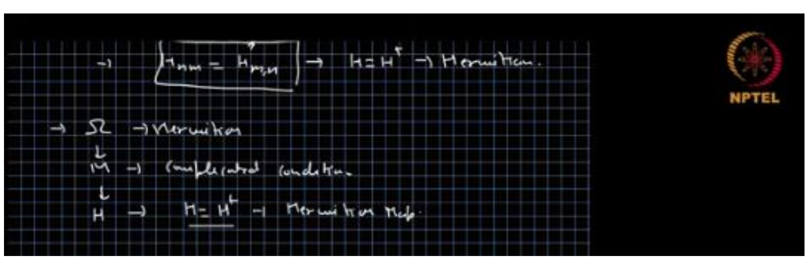

There are two different matrices with the same elements but at different locations. So, let's start with a simple example of M being 4 by 4 matrix, 4 dimensional by 4 dimensional. And just for the ease of reading the elements, I'm putting this grid here. So, the element will be m 1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121, 2122, 2212, 2221, 2222. These are the 16 elements.

And we have two interchanges. You see the relation that M ij kl is same as H ik jl. So, we have to interchange the middle two indices. If we do that, then we get a matrix. we just keep the local position as such, so it's one one one one, one one one two, line here this will become one two one one and one two one two, now one one two one, one one two two, one two two one and one two two two.

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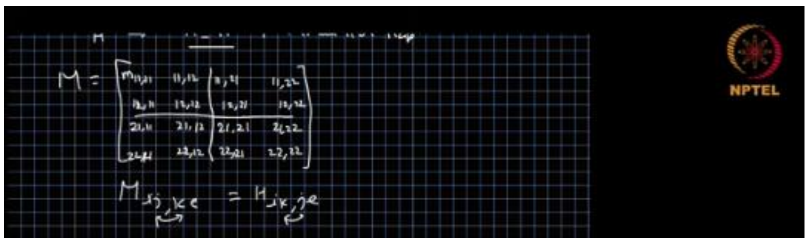

Now 2 1 1 1, 2 1 1 2, 2 1 2 1, 2 1 2 2 and 2 2 1 1, 2 2 1 2, 2 2 2 1, 2 2 2 2. So, if we just interchange the elements then this is the matrix we get right so but these are still. So, H matrix will be h 1 1 1 1, h 1 1 1 2, h 1 2 2 1, h 1 2 2 2 and so on, okay. So, these are h elements here h h h h So, if we see that if we convert H into M. then this will be the conversion but if we want to write the H matrix, then these elements here will come at this location, okay. These elements you can see let me put double bar here will come here. Okay, so it means these blocks let me use a different pen these blocks have been interchanged literally you will see when we do more calculation these elements and these elements have been interchanged or if we want to, so M is 4 by 4 matrix with A, B, C, D being 4 2 by 2 matrices. That will go to H will be, so A vector, B vector, C vector, D vector and the transpose.

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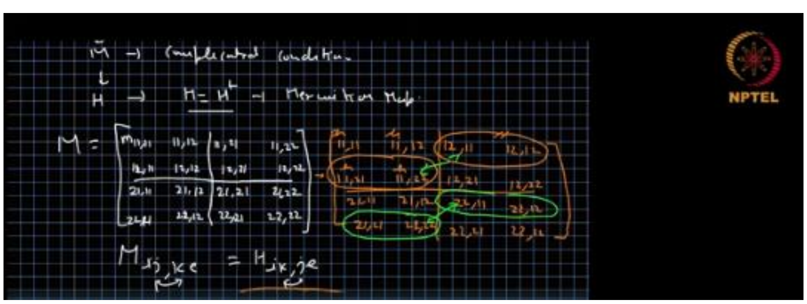

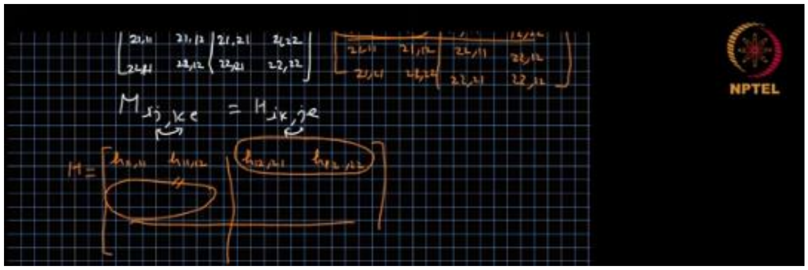

If we do this, then, we can go from M to H, where A vector is the unfolded or unfolding of the matrix A, B vector is the unfolding of matrix B, C is the unfolding of C and D, and we put them as columns and then we take the transpose, so in that way we can go from M to H. If M is a bigger matrix, if M is some matrix, okay, let us say 9 by 9, so we divide it into 9 blocks, so every block is a 3 by 3. And let us say we have A11 block, A12 block, A13 block, A21, A22, A23, A31, A32, A33. Then if we go to H, which will be A11, A12 and A33 block and then transpose it. This is how we can go from M to H matrices. So, we are given a map omega, we find the matrices representation, we go to H and then we check whether it's Hermitian or not.

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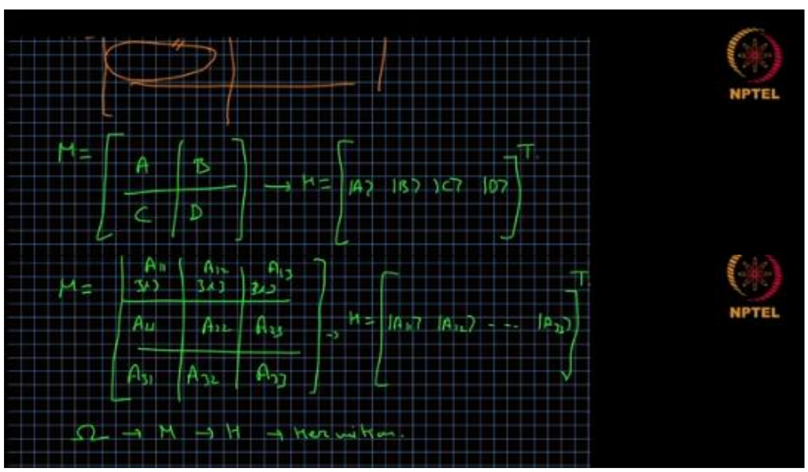

This is the chain of sequence we need to see whether a given map omega is Hermitian or not. next interesting map we have is a positive map or positivity preserving map, so, it's a map Omega such that if omega of A, is map omega such that omega of A is positive, is a positive operator for every A positive. The definition of positivity is psi A psi is positive for all state psi in the Hilbert space H, then phi omega A phi should also be positive for all phi in the corresponding Hilbert space. Okay, so if a map is satisfy this condition, then it is called positivity preserving map or positive map for sure. Now, the matrix representation of this positivity preserving map will be, next we will derive that matrix for the condition over the matrix M for a positive map. So, omega of A i j like this have M i j k l A k l. So, this is the linear, this is the definition of the matrix representation.

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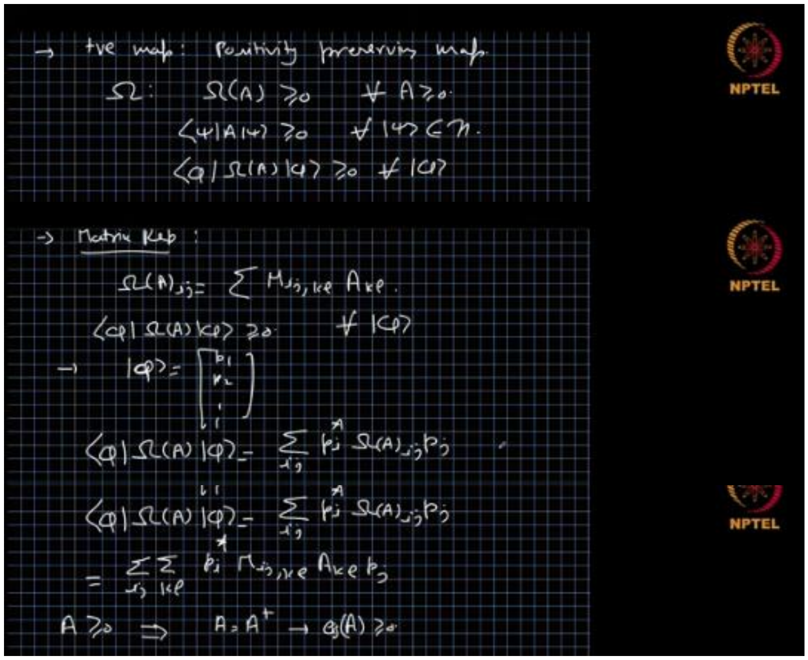

Now for every positive operator a phi of a is also positive so positivity condition on phi of a is this for every every phi it means if phi is a vector p1 p2 and so on then we can say phi omega A of phi is in a sum over ij p i star omega A ij pj. And we can substitute it in this equation and we get sum over i j, sum over k l, p i star, M i j k l A k l p j. Now, we see that A is a positive operator and we have already seen in the linear operator lecture that this implies that A is Hermitian and the eigenvalues of A are positive semi definite. So, it means we can write the spectral decomposition of A which is sum over n, lambda n psi n psi n. From here, we can calculate the k l element, which will be sum over n lambda n psi n k psi n star l. That will be the kl element of this matrix, we can substitute it in this expression, we get phi omega A phi equals i j kl and n now, lambda n, p i star, p j, psi n k psi n l star M ij here. We have all the elements here. What we do next here to simplify it, we use the H matrix and we replace M with the H matrix. So, the M, the relation, let me recall the relation of M and H matrix, the M ij kl is actually H i k j l.

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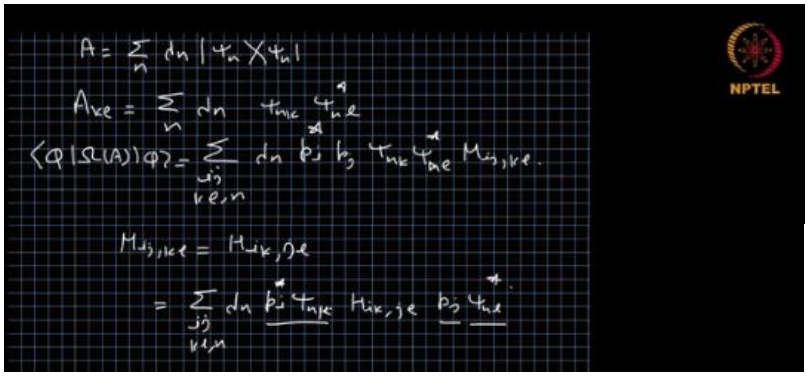

We just interchange the elements, so we get i j k l, n, lambda n p i star psi n k, I'm just reshuffling these also to get a form I require, and p j psi n l star. So, we have kept the p i star and psi nk on one side and pj and psi nl star on the other side. This is because we have ik and jl segregation in the indices in the H matrix. So, if we define a vector eta ik vector, which is phi tensor psi n star. Then we can write the previous equation phi of A omega of A phi as eta ik, H matrix, eta ik.

Sorry its a star here and star here and lambda n sum over n. This equation is same as this equation with this, okay, so, and just a reminder that lambda n are positive because A is a positive operator, positive over zero. So, for this to be positive for all, then we get, eta ik H eta ik should be positive. Because if any of them is negative, then we can choose a set of lambdas where that element is one and all other are zeros. So, we get a negative number. We don't want that.

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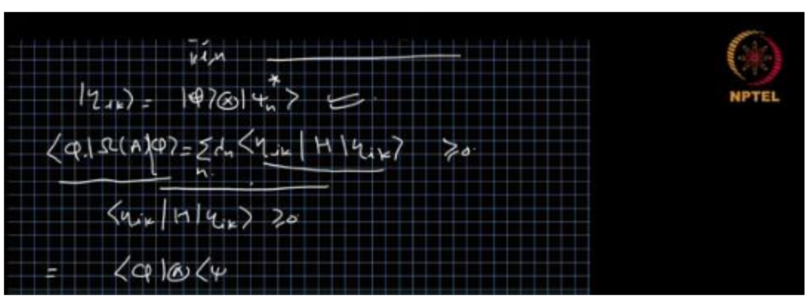

That's why all of them individually need to be positive. Now, this is same as phi tensor, it's actually phi and psi star. Psi star H phi tensor psi star should be positive for all phi and psi. If it is for all phi and psi, then we can remove star also. So, it is phi tensor psi H phi tensor psi should be positive.

So, we can summarize it that a map omega, which is a positive map. this implies that the H is positive operator, for all the product states. So, what we are saying is we have a map omega, from here we go to the matrix representation M, and then we go to the matrix representation H from here. Then H is a conditional positive operator or it's not a completely positive operator. It's a conditional positive operator. It is positive only on the product states, psi and phi for all psi and phi.

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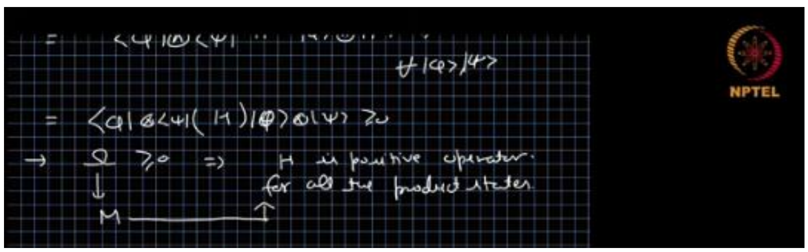

So, if that is the case with H, then we will know that the map omega is a positive map. So, it will map positive states, positive operators to positive operators. Why this set of maps are important? Because these are the maps which will map states to states. So, in a way, any physical operation which we require, which we can consider in quantum mechanics, which is allowed in quantum mechanics, they take a state of the quantum system and they take it to another state of the quantum system.

So the mathematical representation of that comes from positive map. I'm not saying positive maps represent the physical transformations or physical operations but this is at the beginning and then we will see how the positive, how the physical operations are represented by such maps. You know there is an exercise here. Consider the transposition map. Find the M matrix, H matrix, show that it's it's Hermitian map. And try to see if it is a positive map. So, in summary, we have a map omega, there is a matrix representation M and there is a matrix representation H for it, so, H equals H dagger means it's a Hermitian map. H being positive for all product states implies it is a positive map.

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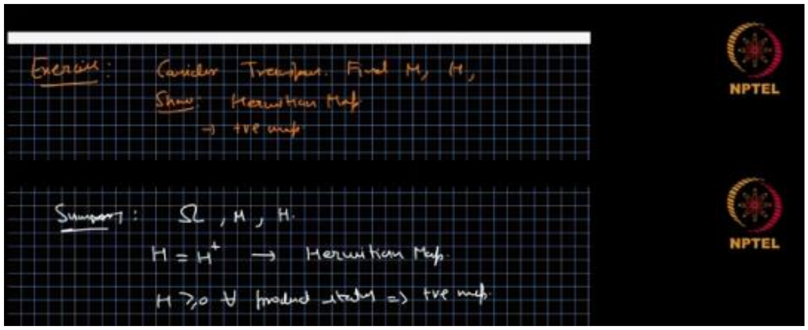

There is another condition it's called trace preservation, map. So, that just means trace of omega of A equals trace of A, so the operation does not change the trace. For example a unitary matrix acting on a matrix A like U A U dagger has the same trace as A, so this is a trace rendering map the exercise can be find the condition on H for it to represent a trace preserving map. An interesting property for representation, for a Hermitian map, H is Hermitian. It means we can write a spectral decomposition for it, sum over n, lambda n, s n, where s n are the eigenvectors and lambda n are the eigenvalues. It means H i k j l is the element, will be lambda M s n, i k, s n star j l. And this is equal to M i j kl. So, a map omega acting on A ij element is this, kl ij kl A kl.

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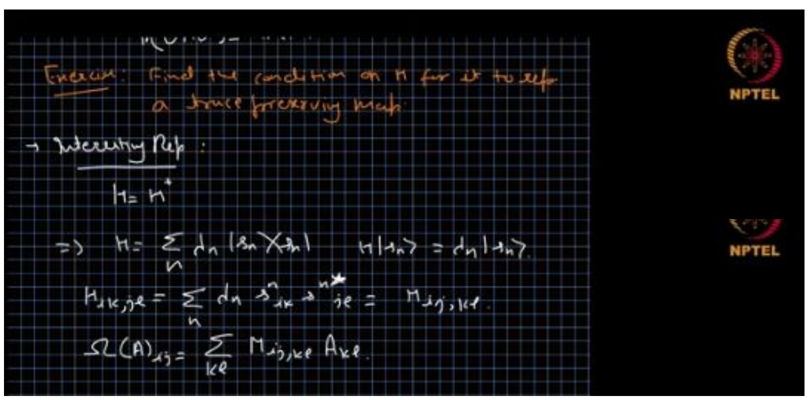

Now, we can substitute this expression for the element M ij kl that will be sum over M, sum over kl, lambda n, s n i k, s n star j l, A k l. This can be simplified further by sum over k l, lambda n, we have taken out, it is s n i k, A k l, s n dagger l j. So, I have written this element, the star, the complex conjugate is there, s n dagger, and we have to change the indices j and l to l and j. So, this will be equivalent to s n, A s n dagger and i j. This implies that omega of A is nothing but sum over n, lambda n, s n A s n dagger or M matrix will be sum over n, lambda n, s n tensor s n star. So, these are the representation we can get very easily. And H actually is lambda n. This is the s n, outer product s n.

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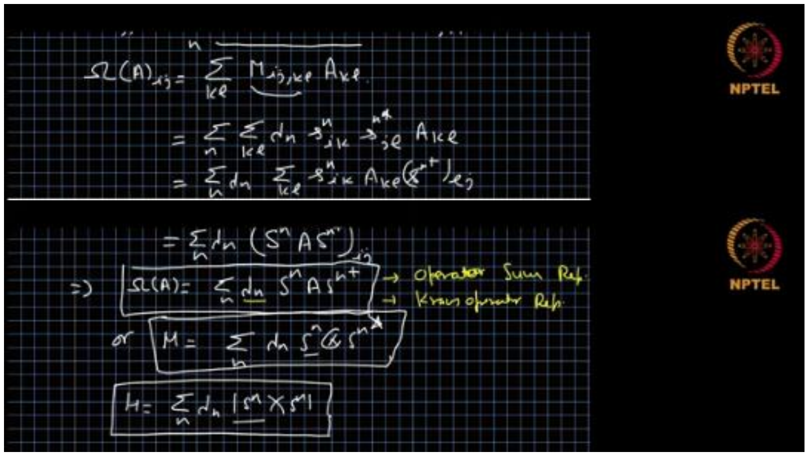

So, s n are the unfolded vectors of matrix s n. This representation of the operator or of the map here is called operator sum representation or Kraus operator representation. So, here, there is no condition over lambda n because H is only Hermitian, so lambda n can be positive or negative the only condition is they are here real. Second thing, when H represents a positive map, it is naturally a Hermitian map first. So, positive maps also have this representation. And another thing is that lambda, again there is no restriction over the elements lambda n. They can be positive or negative as long as they are real.

Okay, so this representation operator sum representation or Kraus operator representation holds for positive maps and Hermitian maps in the same way. One application of the positive maps is the following if we have a state, bipartite state rho AB, And it is written in the i p i rho A i tensor rho B i. That is, if we have a bipartite density matrix, which is written in the separable decomposition. Then a positive map acting on a subsystem, let us say B subsystem here, will give us the following. Sum our i p i, rho A i tensor omega of rho B i. Since we know that omega is a positive map, then omega of rho B i is always a positive, because rho B i are valid density matrices.

So, omega of rho B i is positive. So, we can write it as some over i, B i, rho A i, tensor rho B i, tilde, some other density matrix, but density matrix of a B subsystem. So, this is a positive operator, the whole thing. But, if rho AB cannot be written in the separable form. Then sometime, this need not be a positive operator.

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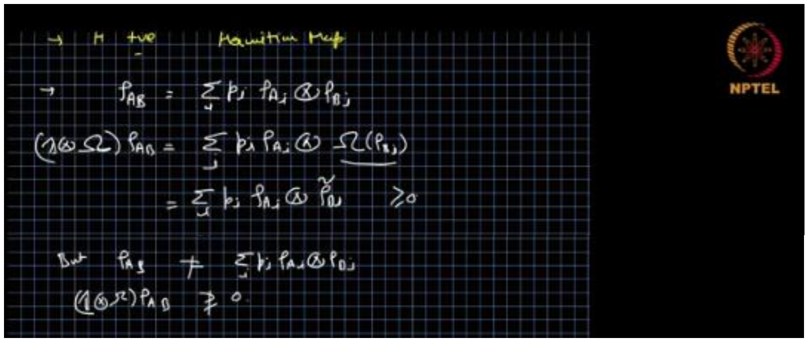

Although the map omega was a positive map, when we apply it on a subsystem, sometimes for entangled states, for non-separable states, we don't get the positive operator back. So, as an example, we can consider the transposition map. So, rho goes to rho transpose is the map we have. So, we can see that the eigenvalues of rho is same as eigenvalues of rho transpose. So, in that way it maps all the positive operators to positive operators. So, in that way it's a positive map. So, it means if we if there is a separable decomposition, we apply a transpose on one side, we will get another positive operator and in that way it satisfies what is required but now consider a state psi which is 1 over root 2, 0, 1, minus 1, 0.

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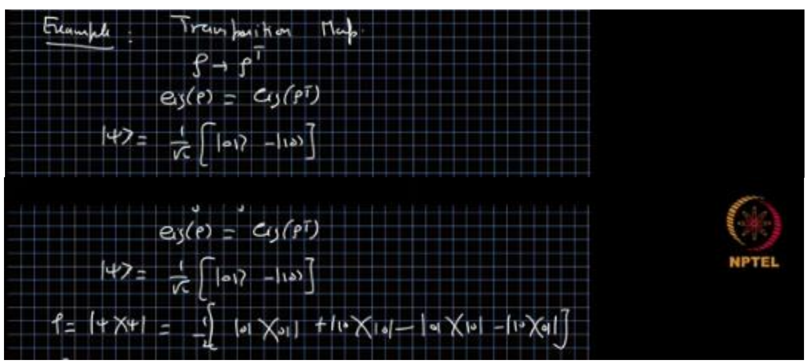

This is the singlet state. And psi outer product psi, which is the rho, will be 1 over 2, 0, 1, 0, 1, plus 1, 0, 1, 0, minus 0, 1, 1, 0, minus 1, 0, 0, 1. So, now we can apply transposition on the second part of it. So, we have identity on the subsystem A and transposition on the subsystem B. It means first, okay, first let us write rho as half, so, this is 0 1, 0 1 can be written as 0 0 tensor 1 1, plus one one tensor zero zero, minus zero one tensor one zero, minus one zero tensor one one. We have written it as a product of A and B subsystems. Now we apply the transposition on the B subsystem, doesn't matter, we have applied on the subsystem A also, but here we have to choose one, we are choosing B. one one goes to one one transposition on zero zero gives you zero zero.

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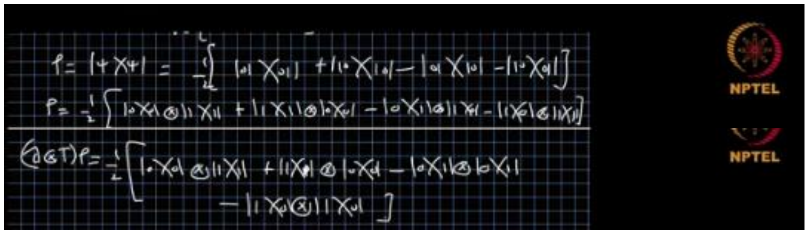

We are writing this the operator of A as such and we are performing the transposition on the other one, we get one zero goes to zero one minus one zero tensor zero one zero one zero one goes to one zero. if we write the matrix form of it, we get one one minus one minus one zero zero zero zero zero zero zero zero zero zero and zero zero zero. These are the, this will be the matrix of this partial transpose of rho and if we find the eigenvalues of this it will be half half half and minus half. It means this is no longer a positive map, positive operator, hence the state, underlying state must be entangled. Because a positive map applied on one side of it gives us non-physical answers. So, in that way, the positive maps can be useful in many things, especially in the estimation or detection of entanglement.

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