FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-05
Lecture-13

Composite Systems: Pure States, Schmidt Decomposition, Operators Acting on Composite Systems-Part 02

So, we can have a formal treatment of finding out whether a state is an entangled state or not and what is the canonical form of a motion rule state, okay. We call it Schmidt decomposition. So, in this Schmidt decomposition let us take state psi, which is sum over ij, alpha ij, psi i tensor phi j. We have this. Now we have matrix A such that alpha ij is the ijth element of matrix A. Now we will invoke something called singular value decomposition. Singular value decomposition, it states that an arbitrary X matrix of dimension N by M can be written as some U, which is a N by N unitary, some D, which is an N by M generalized diagonal matrix, and V transpose, which is M by M, another unitary. So, U is unitary.

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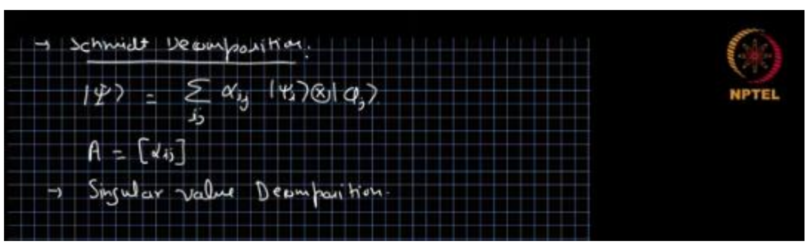

V is unitary. If V is unitary, V transpose is also unitary. And D is a diagonal matrix. But D is a rectangular matrix. What do we mean by diagonal?

So, of course, there can be two type of diagonals. One is this type of diagonal. Then the diagonal matrix will look like D1, D2, D3, so on. And rest everything is zero. If there is a other type of the long diagonal matrix, the matrix, then we have d1 d2 d3 and everything else is zero. So, this is what we mean by generalized diagonal matrix.

Another interesting point about the singular value decomposition is the elements D1, the diagonal elements D2, D3, they are all real and they are all positive. They can be zero, but they cannot be negative. They are non-negative numbers. So, all the elements in D are non-negative numbers. So, this is very important decomposition and we will be using it in our present treatment.

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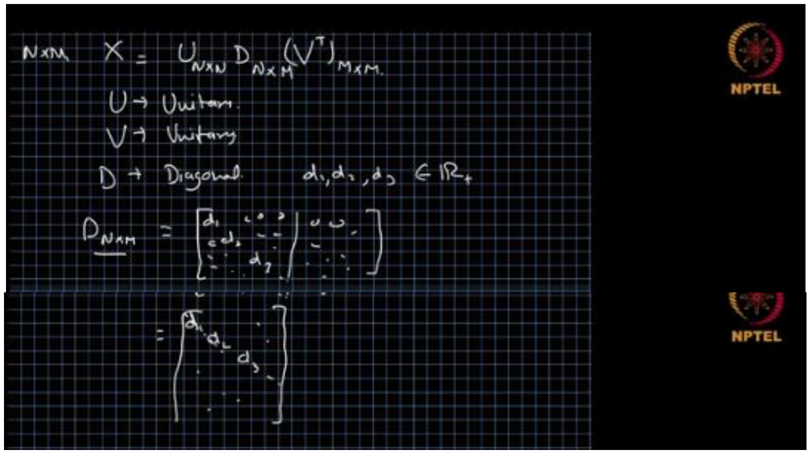

So, we have a matrix A, which is d1 by d2 matrix. So, we decompose it, we find a singular value decomposition of it where U D V transpose, where U is a d1 dimensional unitary, V is d2 dimensional unitary and D is the diagonal, generalized diagonal matrix. So, Aij which is alpha ij, can be written as sum over k l U i k D k l V transpose l j. This is how we can decompose element wise matrix multiplication, so the element of the product can be written as the product of elements of the matrices U D V here. D kl we know, it is a dk and delta kl because D is a diagonal matrix. So, we can write alpha ij as sum over kl U ik dk delta kl V transpose l j. It is V transpose l j. So, in terms of V, we can write it as j l. So, this is the decomposition we get and we can simplify it by using the delta function. So, sum over k U i k v j k and d k. This is how we can write alpha ij in terms of elements unitary U and V and the k is the singular values of matrix A. Now we have expression for psi in terms of alpha ij and it is sum over ij alpha ij psi i tensor phi j, and we can substitute it here sum over i j sum a U i k V j k d k psi i tensor phi j. Now let us say we have a vector e k defined as sum over i U i k psi i. Psi i is the orthonormal basis in first Hilbert space in Hd1. Now what is e l e k, inner product, that will be sum over i, sum over j, U i k, i l star U j k psi i psi j.

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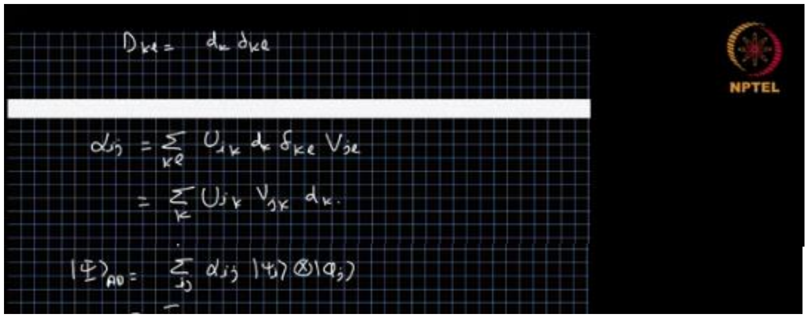

I'm just using this expression for e l and e k and I get this. That will be, since psi i is the orthonormal basis, so, psi i psi j is the delta function, delta i j. So, we can sum over j and replace wherever we have j by i, we get sum over i U i l star U i k. And this we can write as sum over i U dagger l i U i k, so, we are using that U star i l is the element U dagger element of U dagger l i. Okay, this is the identity we have used.

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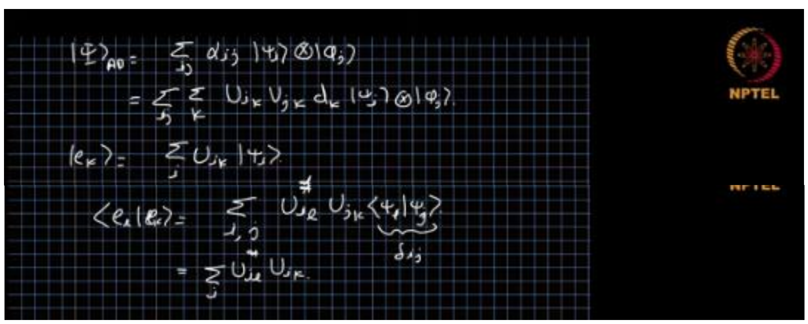

Now we have U dagger li times U ik. So, the right index and left index matches and the summation is over that matched index. So, it will be a product of the two matrices U dagger U and the lk element of that. U is a unitary matrix. This is the condition of the singular value decomposition.

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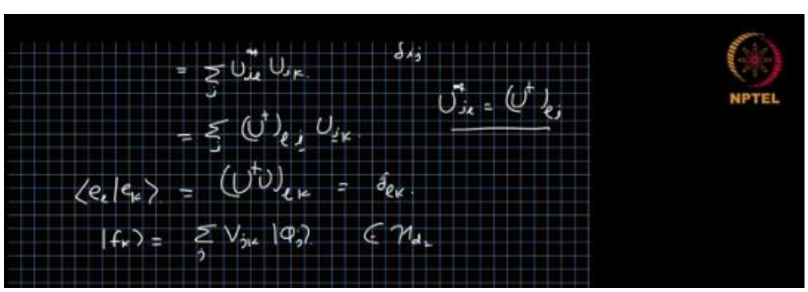

So, U dagger U is identity. So, U dagger U lk element is delta lk. So, from here, we can see that el, ek are orthonormal states. Similarly, we can have fk, which is sum over j, U, jk, V, jk and phi j, for Hd2. And we can verify again that fl fk is delta lk that means it's orthonormal basis. Why did we go through all this trouble to arrive at some expression, because this expression is, means psi AB can be written as sum over k dk ek tensor fk, where, ek is an orthonormal basis and fk is an orthonormal basis in Hd2. And dks are real and positive non-negative numbers. So, this decomposition is the canonical representation of the state psi. This is the minimum you can achieve in terms of number of terms. And this is called Schmidt decomposition.

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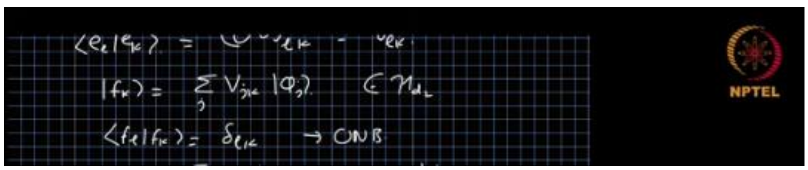

Now, you see we have a set of these. They are the singular values of the coefficient matrix A, then, the number of non-zero dks is same as the rank of matrix A. It means if only the first one or only one dk is non-zero and all other dks are zero, Then we have a product state, then psi can be written as, if we keep these in the descending order, so, the largest one is non-zero and that is equal to 1, so, it becomes e1 tensor f1. So, from this decomposition, we will immediately get the product form of the state psi. But, if we have more than one dk non-zero then uh the state psi cannot be written in this form, then the state psi will be this is when rank is one rank of A is one, when the rank of A is not one then psi will be at least e1 tensor f1 with coefficient d1 plus d2 e2 f2 and other terms.

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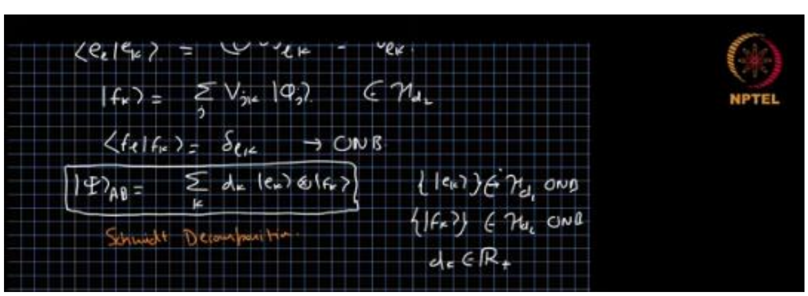

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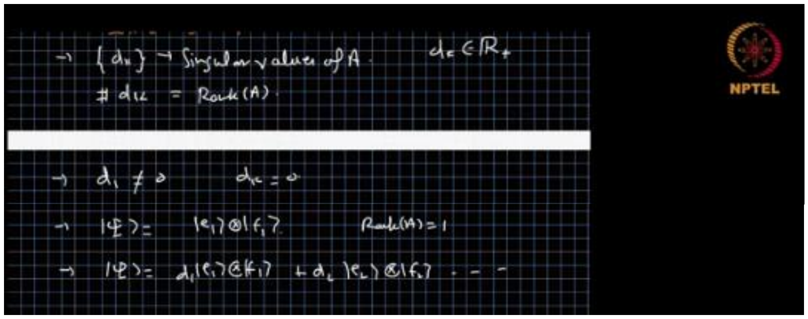

So, in that case, it is guaranteed that it is not a separable state. In that way, Schmidt decomposition will give us very quickly, will tell us whether a given state is a separable or an entangled state. The number of non-zero dk's are also called the Schmidt rank. The number of non-zero dk's, they are called, they are also called Schmidt rank. Now we will discuss operators acting on Hd1 tensor Hd2, so they belong to the set of operators acting on Hd1 Hd2. So, if T is the operator acting on Hd1 and S is the operator acting on Hd2, then we represent the operator Z, which is T tensor S, we will define what we mean by tensor here, that will act, that will belong to Hd1 tensor Hd2.

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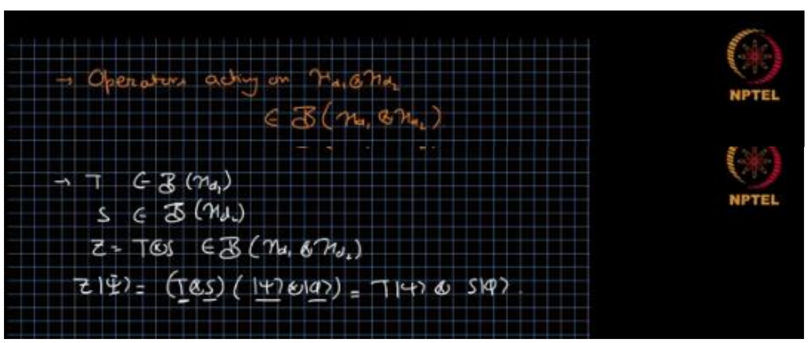

So, if we apply Z on the state psi and let us say psi is the product state, then T tensor S psi tensor, phi, so till now there is no correlation, there is nothing here to worry or to consider. So the operator of subsystem A will act on the state of subsystem A and operator on subsystem B will act on state of subsystem B. So, it will be T psi tensor S phi. So, this is how the tensor product operators acting on the tensor product state is defined. So, now if we have the more general state, which is sum over ij, alpha ij, psi i, tensor phi j. Since the matrix product is linear, then T tensor S, acting on psi will be sum over ij, alpha ij, T, psi i tensor S phi j. it will individually on each of the tensor product it will act in this way. T is the operator on subsystem A and if we don't apply any operator on subsystem B that is represented by T tensor identity.

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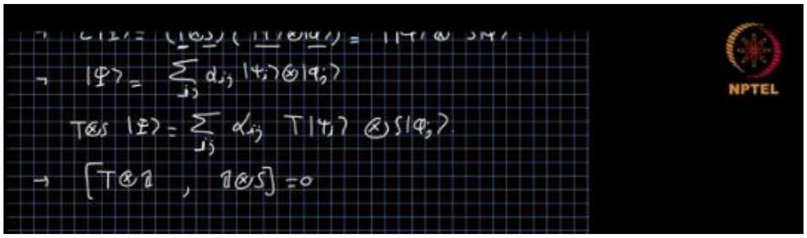

Identity means doing nothing on a subsystem. Similarly, identity tensor S will represent the operator acting only on subsystem B, nothing on subsystem A. And these two operators always come. The operator acting only on subsystem A and operator acting only on subsystem B, they commute. If we have A, operator A and the elements are a11, a12, a13, a21, a22 and so on, we have an operator with these elements and if we have another operator with elements b11, b12, b21, b22 and all these elements then A tensor B, is defined as a bigger matrix so till now we did not mention the dimension of A and B, it can be any dimension, they can be square they can be rectangular they can be even vectors, so we have already seen the tensor product with vectors a11 times B, a12 times B, a13 times B and so on. a21 times B, a22 times B a23 times B and so on and so on.

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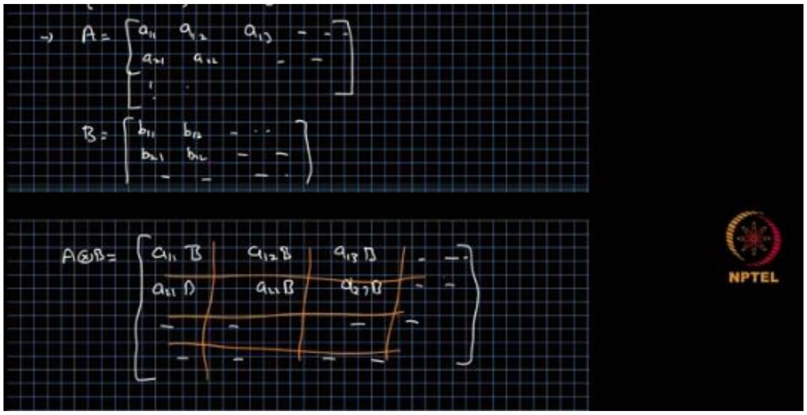

This is how we define the tensor product we can make some grid here to understand it slightly better. The dimension of this grid is same as this block is same as the dimension of B, so if you write just this block, it will be a11 b11 a12 a11 b12 a11 b13 and so on a11, b21, a11, b22, a11, b23 and so on and so on. a11, b21, a11, b22, a11, b23 and so on.

This is just this block. So, here we have taken the B matrix and multiplied every element with number a11.

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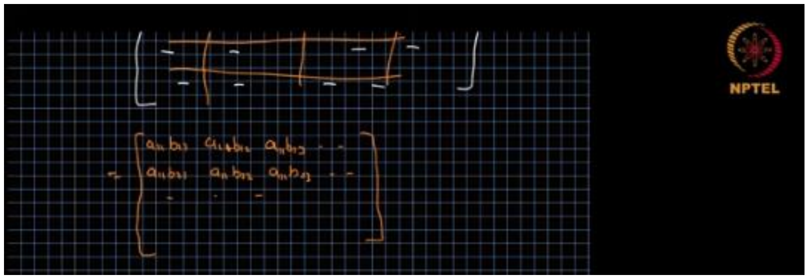

Similarly, in this block, we have the whole B matrix and multiply those elements with a12 and so on. So, these lines, these blocks are just to understand. But ultimately we have a much bigger matrix. And the dimension of matrix is defined like this. If A is N by M matrix and B is P by Q matrix, then A tenser B will be NP by MQ matrix.

This is how the dimensions multiply. For example, if n is the one-dimensional vector, so, it means A is the one-dimensional vector. So, if M equals N is N and M is 1, so it is N by 1 matrix and P is also N by 1, P by 1 matrix. Then A tensor B will also be a vector and the dimension of it will be NP times 1. NP by 1.

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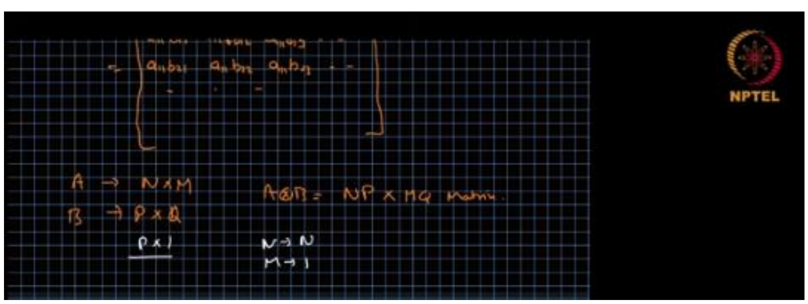

And this example we have seen already when we define the tensor product of states. So, in that way the definition of tensor product is same for operators and states if we consider states as one-dimensional matrices. Now if we have A tensor B and we want to find the element MN here. So, of course, A tensor B is a matrix and finding MN element is a very well-defined question here. But it will be much more useful notation if instead of mn element, if we say ij kl element.

So, what we have done is m we are writing as ij and n we are writing as kl. So, for example if our A is a 2 by 2 matrix and B is also 2 by 2 matrix, then A tensor B is a 4 by 4 matrix so instead of m going from 1 to 4, m and n both going from 1 to 4, we are saying m is 1 1, 1 2, 21 and 22, similarly, n is 1 1, 1 2, 2 1 and 2 2 . We just change the notation from the decimal we have gone to whatever is the convenient basis here, the i j. So, i and j can be from 1 to d 1 and 1 to d 2, respectively. So, you will see in time how this will be a more useful representation for an element. Now we are seeing A tensor B, ij, kl element. So, A, we can write as, we can open it in some basis, A ij, let me say ik, i k.

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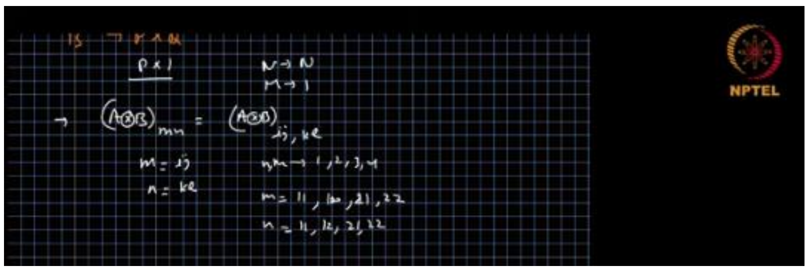

So, we can write matrix A in terms of the computational basis i, which is 0, 0 and 1 at the highest place and 0, 0 everywhere. Then k is also like that. So, A matrix can be expanded in terms of the coefficient A ik and the basis i, the computational basis. Similarly, B, we can define as B, jl, j outer product l. Let me put a bar here to say that they do not need to be from the same Hilbert space.

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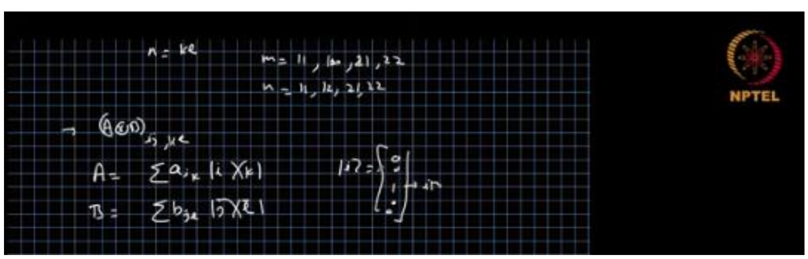

But we can always find a computational basis in both the vector spaces. Then A tensor B will be sum over, this was summation is over i k, this summation is over jl. So, ik, jl, all four summations we have here, that will be a i k, i k, tensor B j l, j bar l bar. We can reorganize the symbols i j, k l, a i k, b j l, they are scalar, they come out on one side. We have i k tensor j l. So, we have the operator of subsystem A and the operator of the subsystem B. I should not say A and B to avoid the confusion. But this we have operator of A and this is operator of B. This can be written as ij kl a ik b jl, i tensor j and k tensor l, j bar and l bar. So it becomes ij kl.

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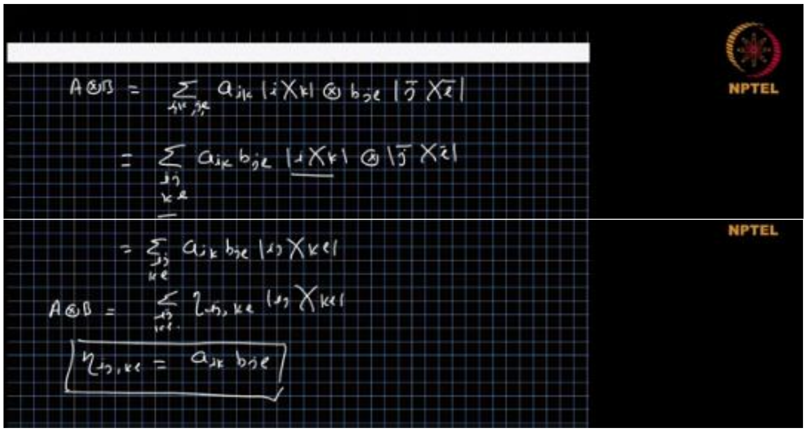

So, a i k, b j l, i j, k l and let us call it eta i j, k l, i j, k l, sum over i j, k l, where we have to redefine eta i j, k l to be a i k b j l. And this is A tensor B, the whole matrix A tensor B. Now, if we compare this with this definition so we had a computational basis i j i k and computational basis of B was j l and A can be written as i k and a i k and this becomes the i kth element from here. A i k is small a i k. The i kth element of the matrix A is just a i k, the coefficient of i outer product k. Similarly, B j l element is just b j l. So if we compare this representation with this, so here the computational basis is i j, so this becomes the orthonormal basis in Hd1 tensor Hd2 and this is the orthonormal basis. So, the element what we were looking for that is A tensor B ij, kl element is actually eta ij and kl, because this is the coefficient in the computational basis. And this is nothing but a ik and times b jl.

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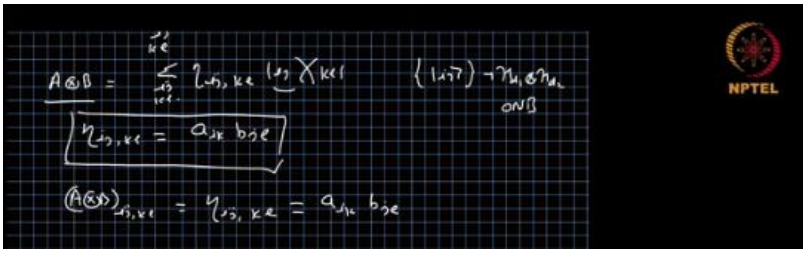

Just look at the indices. Indices are very important here. The ij kl element is not a ij and b kl. It's a ik and b jl times b jl. So, there is a small shuffling of the indices here to get the ij kl element of the tensor product.

This will become very important with time and when we talk about the positive and non-positive, completely positive maps, when we talk about those things or the Kraus operators and super operators, these indices will become very, very important. One can figure out after doing this thing, it's not very difficult to check that A tensor B times C tensor D is actually A times C tensor B times D. I call it A B C D rule. I don't know if there is a rule, a name for this rule but it is very very useful. This is, this we have already used multiple times without without us realizing for example when we when we said T tensor S acting on psi tensor phi is T psi tensor S phi. We have actually used this rule precisely. Further, when we said ij tensor ik tensor j bar l bar is equal to i tensor j bar times k tensor l bar, we have used this rule but in reverse, we were given this and we took it to this.

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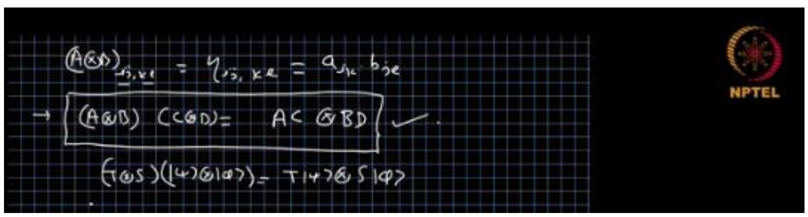

So, in that way, without us realizing we have used this rule multiple times and when we work with tensor products space, this rule will appear multiple times and we will use it without registering it in our mind. So, these are the properties of the tensor product of operators and there is the exercise, this can be fun exercise and slightly involved. If I define the matrix H, H tensor identity plus identity tensor H. And H is Hermitian. So, calculate the unitary, which is exponential of minus i H T over h bar. You can use small u, which is exponential of minus i small h T over h bar. So, we have to calculate this in terms of u.

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