FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-05
Lecture-12

Composite Systems: Pure States, Schmidt Decomposition, Operators Acting on Composite Systems-Part 01

Hello everyone. Today's topic of discussion is composite systems. Here we will talk about tensor product space, how the entanglement comes into the picture, how we find the reduced density matrices, how do we perform measurements on a composite systems and things like that. This is one of the most important topics in foundation of quantum mechanics because this gives us the foundations of entanglement and how to deal with it. And entanglement is the resource which makes the quantum advantage possible in quantum computation and quantum key distribution and quantum communication in general.

A quantum system with more than one particle is called composite system. But there is a small confusion in this statement. That is, whenever we talk about a quantum system, we say an ensemble of particles. So, there are many, many particles. Like if you remember, when we were talking about the measurement, we had n identical particles going in the measurement setups.

And then from measuring upon those, we are finding the probabilities of different measurement outcomes. So, in that way, whenever we deal with a quantum system, we always talk about a large number of systems. Then what is so special about a composite system? So, here, when we say more than one particle, it means the single unit of quantum system, when we say, like we were saying earlier, qubit. So, qubit is one single unit, which has one particle in it.

But if we have pairs of qubits coming into the picture always, this is a single particle we are seeing single particle even if we have an ensemble of particles, they are coming one by one to us, so, it's always a single particle but repeated copies multiple times. Now if we have a pair of qubits and we consider that as one quantum system, so, we have qubit A and we have qubit B and these two together we call it a quantum system. They need not be together in the sense that they need not be present spatially, physically at the same place they can be at different places but this qubit A and qubit B belong together okay, so, in that way we will talk about the state of this two qubit system together and then this becomes our composite system. Of course, one pair of qubit is of no use in quantum mechanics, so we will have an ensemble of a pair of qubits, of course, again we are just representing them for the sake of representation but these pair of qubit need not be at the same location and they're like, n tending to infinity or very large number of those pairs.

For example, in a typical quantum communication protocol we have a source which is producing pair of qubits and it is sending those pairs at different locations. If they are in at the same location then they are not useful for communication, so we are sending them at a different location and we are tagging them with time, so, the pair produced at time t and t plus delta and t plus two delta and so on and so forth. So, we tag each pair of qubits with time. So, that becomes the identification. So, the two photos or two qubits we are getting in this way, they belong together.

So, all the processing is done on them together. Together in the sense like their measurements will be correlated or their operations will be correlated in some sense. So, this becomes a composite system. This is two qubits, we can have three qubits also, we can have four qubits, we can have not just qubits, we can have larger systems, n-dimensional systems, d-dimensional systems. So, all of them are composite systems.

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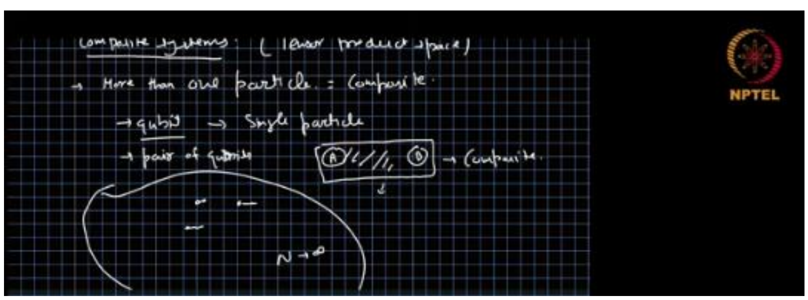

So, here we are making distinction between single particle and quantum system and more than one particle quantum system. So, more than one particle quantum system is the composite system. So, now let us take a very simple case to understand the whole scenario. Let us talk about two particle quantum systems. So, there is a party one or A and party B they can be of like five qubits and it can be of 11 qubits or some such number like we don't care just that we have divided the whole system into two parts, part A and part B. Dimension of part A is d1 and dimension of part B is d2, they can be same they can be different they can be as low as two. The part one or particle one or subsystem one sub subsystem A, as the state of that, let us call psi and they are from the Hilbert space Hd1 where d1 stands for the dimension of the Hilbert space and we are assuming the dimensions. So, the subsystem B, the states, let us call they have phi states and they belong to the Hilbert space Hd2.

We can have operators, let us say T1, T2 are the operators acting on Hilbert space Hd1. So, they belong to the set of operators acting on Hd1. Okay, let us represent the set of operators acting on Hd1 by this B of Hd1 and S1 S2 are the operators acting on subsystem B, so, they belong to set of operators acting on Hd2. Now our aim is to write the joint state of A and B because we want to represent these two subsystems together, to make a bigger quantum system which is the bipartite quantum system, so, there are two parties, so, we can, we generally use the name bipartite quantum system. So, we want to write the joint state of the two, so we have psi for state A, system A and phi for system B. We can use some symbol here and the most favorite system symbol is this, let me zoom in, this is a circle with a cross and this is called tensor product, okay, symbol for tensor product, we will see what are the properties of this.

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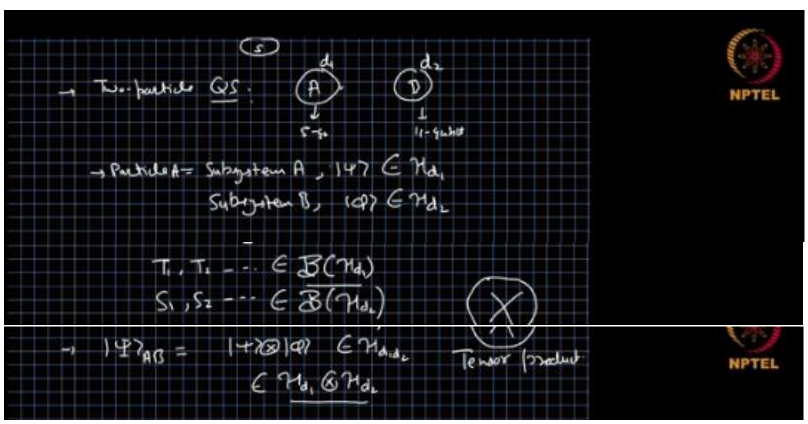

So, whenever we write the state of two systems, where the state of subsystem A is psi and subsystem B is phi, then the joint state is psi tensor phi. And they belong to Hilbert space Hd1. This belongs to Hilbert space Hd1 tensor Hd2. Okay this is a symbolic way but actually it belongs to Hilbert space H of d1 d2, okay, so we'll see that, let us say there's a orthonormal basis in Hd1 and let us represent it with psi i where i is from 1 to d1. This is an orthonormal basis in Hd1.

Similarly, we have orthonormal basis in Hd2 and we represent it by phi j. j is from 1 to d2. Okay, so now the basis for the Hd1 tensor Hd2 will be psi i tensor phi j where i is from 1 to d1 and j is from 1 to d2. So, in that way there are d1 times d2 vectors in the basis. Hence the dimension of this space is d1 times d2 so that's why we are writing space d1 d2. Now let us discuss about the properties of the tensor products or some properties in general.

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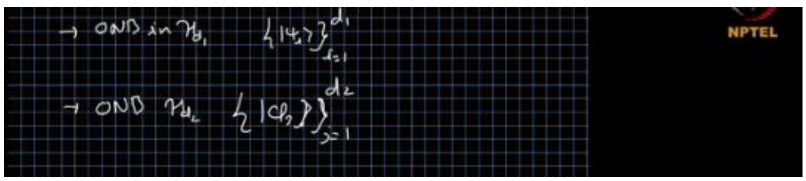

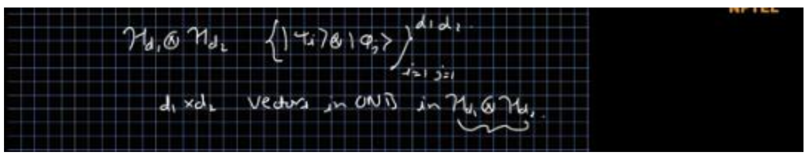

So, if we multiply the product this tensor of psi tensor phi with a scalar alpha then it is same as alpha multiplied with psi tensor phi and this is same as psi tensor alpha phi so it's linear in both the arguments psi and phi if we have psi 1 plus psi 2 tensor phi 1 plus phi 2, then it will be it can be opened up like a normal product this is just keeping the order, psi 1 tensor phi 2 plus psi 2 tensor phi 1 plus psi 2 tensor phi 2. This is another property the tensor product. We can have inner product. Let us say we have two vectors, psi 1 tensor phi 1 and psi 2 tensor phi 2. And we want to calculate the inner product.

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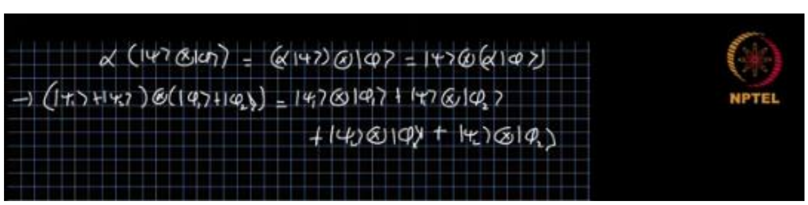

That will be psi 1 tensor phi 1, psi 2 tensor phi 2. And that will be, so, the subsystem A states will have inner product separately and subsystem B states will have inner product separately. So, it will be psi 1, psi 2, tensor phi 1, phi 2. And since this is scalar and this is scalar, the tensor product does not act on scalar, so it will be just product, psi 1, psi 2, phi 1, phi 2.

This is the definition of inner product and this can be understood in the following sense that, subsystem A, if we have two states in subsystem A, psi 1 and psi 2, and we have states in subsystem B, phi 1 and phi 2, and they are not correlated in any sense, till now we have not talked about correlations, then they don't feel the existence of the other system. So, A does not feel the existence of subsystem B, and B does not feel the existence of A. So in their labs, the inner product will be just as if there was nothing else. So, for A it will be just psi 1 psi 2 and for B it will be psi 1 psi 2. When we have correlation which we will discuss shortly, when we have the correlations then these inner products might be slightly more complicated, but it will satisfy this definition and then little bit of generalization of that.

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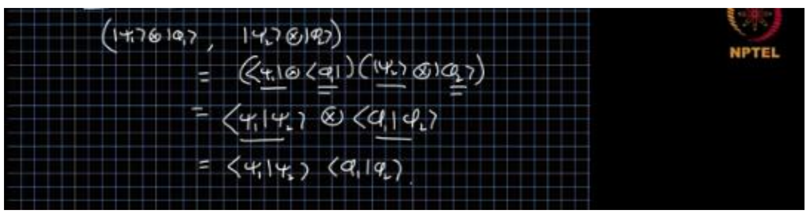

Now, what we mean by the tensor product is, let us say psi is the vector, s1 s2 up to sd1, some vector we are not talking about the normalization or anything is just a vector, a matrix vector, so, and phi is another vector, let us say with elements p1 p2 up to pd2 where d1 and d2 are the dimensions of the respective vectors then psi tensor phi will be s1 times phi s2 times phi s3 times phi and so on, so it will be a vector of d1 d2 dimension d1 times d2 dimension and the elements will be written like this, so, it's just a shorthand we are writing if we expand it fully then it will be psi tensor phi will be s1 p1 s1 p2 up to s1 p d2 there's no line here s2 p1 s2 p2 up to s2 p d2 then it will be s d1 p1 s d2 p2 up to s d1 sorry d1 p d2. These will be the elements of the vector the tensor product vector psi A B, it's such a long vector that it is not coming in one screen, so that the size of this vector will be d1 times d2 and it belongs to a Hilbert space dimension d1 d2. So, this is the like working definition of defining definition of tensor product.

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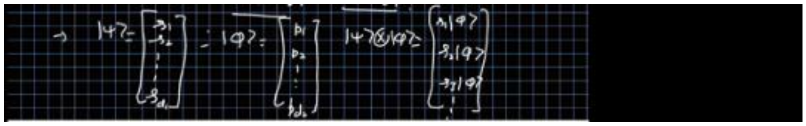

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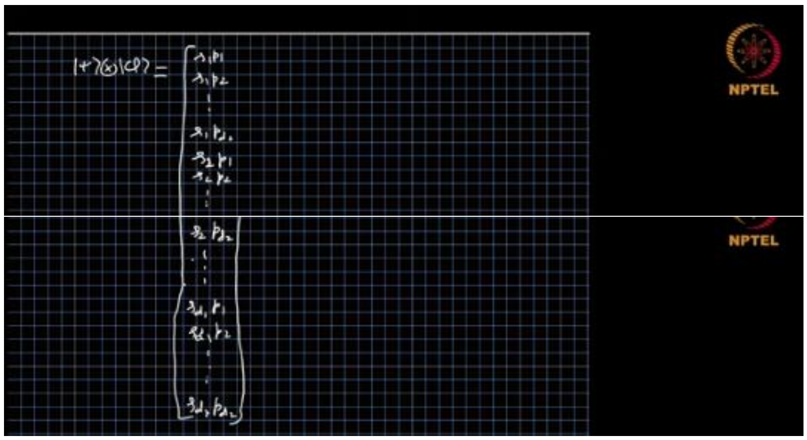

Now, if we have psi 1 and psi 2 states. So, these states belong to Hd1 tensor Hd2 and it's a valid Hilbert space it's a valid quantum system and stuff like that if it is a it if this bipartite quantum system can be treated as a single quantum system then by the exams of quantum mechanics the first example of quantum mechanics the system can be in a state which is superposition of any of its states, so, a system can be alpha ij psi i tensor phi j sum over ij. This is the most general state of the composite system, where psi i and phi j we have taken as the orthonormal basis or any state which exists in the Hilbert space. Now, can we always write these kinds of states as some state psi tensor some state phi? This is the question we should ask now before we start talking about the quantum correlations.

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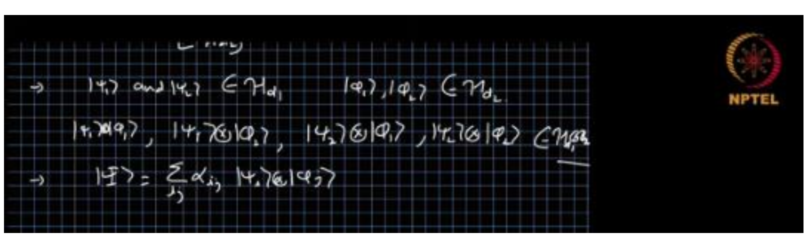

So, whenever we can write a state as a tensor product, psi tensor phi, then it means we have individual identity of subsystem A and subsystem B. So, it's as if we have two qubits, for example, and one qubit is in lab A and other qubit is in lab B. We prepare the qubit A in state psi and prepare the qubit B in state phi. But the state of the composite system, more general state is of this form. Now question is, can we always write it as psi tensor phi? Now let us see that we are saying that psi i is the orthonormal basis,

In Hd1, this is the orthonormal basis and phi j is the orthonominal basis in Hd2. It means we can write psi as sum over i s i psi i and we can write phi as sum over i p i phi j. What we are saying is if the composite state capital psi A B can be written as a product psi tensor phi tensor product psi tensor phi in for some choice of psi, some particular specific choice of i and phi then psi is some state from Hd1, so we can write it as a superposition of its orthonormal basis. We don't know yet what is s i, but it can be written in this form for some suitable choice of s i, the coefficients. Similarly, the state phi can also be written as a superposition of the orthonormal basis with phi j with some unknown coefficient p j. Now, this product, psi tensor phi, will be sum over ij, s i, p j, psi i tensor phi j. Right? But we are given that this state psi is sum over ij, alpha ij, psi i tensor phi j.

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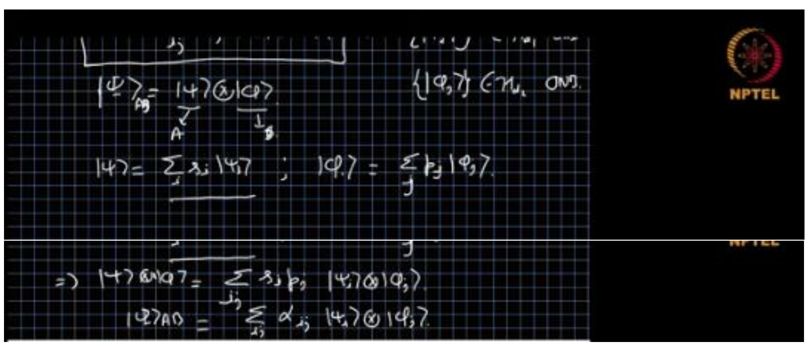

Comparing these two, it turns out that alpha ij must be a i, s i and p j. Now let us say we have matrix A such that its ij th element is alpha ij. We have a vector S, such that its is vector is s i and we have a vector p such that its jth element is p j. Then what we are saying is from this equation, this equation is equivalent to A matrix being S vector times P vector transpose, okay, so, these two equations are identical, that alpha ij being s i p j and A matrix being S times P transpose, S times P transpose is the matrix.

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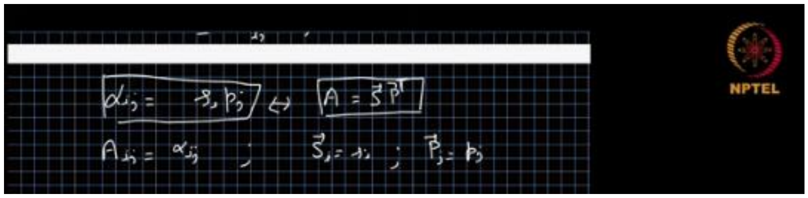

So, these two statements are identical. Now, what is the condition over alpha ij for it to represent a state psi? The condition on alpha ij is alpha ij mod squared sum over ij should be equal to 1. That is the normalization. Other than that, there is no restriction over the choice of alpha ij.

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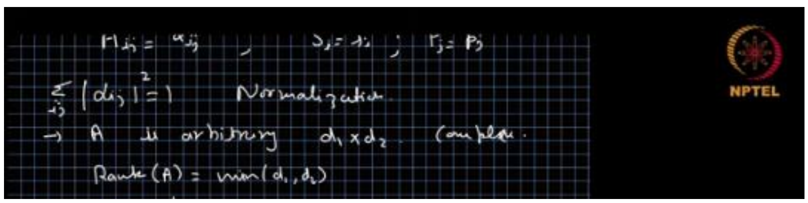

So, in that way, A is arbitrary. And the dimension of A is d1 by d2. And it's complex. So, A is an arbitrary d1 by d2 complex matrix. So, the rank A will be minimum of d1 and d2 in general. There can be specific cases where it will be less than this but in general or there exist lot of matrices A for which rank is minimum of d1 d2, so, the rank of A is minimum of d1 d2 d1 and d2 but what is the rank of matrix S times P transpose.

The rank of matrix S times P transpose is one, okay, it's a one one rank one matrix, so, it means a is not generally equal to S P transpose or what we can say is there exist enough states which cannot be written as psi tensor phi so, psi need not always be psi tensor phi state. What I am saying is the state psi can be written as a superposition of tensor product states but it need not be a tensor product state itself. So, the states, which can be written as tensor product psi tensor phi, they are called product states. They are called separable states and they are called unentangled non-entangled states.

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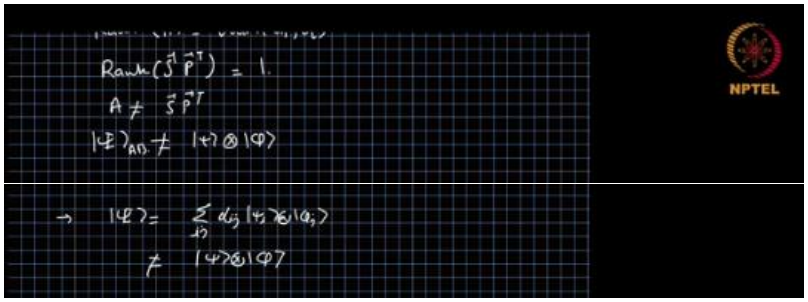

States which cannot be written as side tensor phi, they are called entangled state. Here entanglement can be thought of as the correlation between subsystem A and B, which cannot be achieved by preparing the states locally in A and B. If we prepare the states of subsystem A and B locally, then it will be psi tensor phi. So, entanglement is the correlation which cannot be achieved by local operations, and classical communication, because when we have classical communication, that point will be clear later on, because till now we are only considering the pure states when we talk about the mixed state and classical communication also will come into picture. So, these states cannot be prepared locally.

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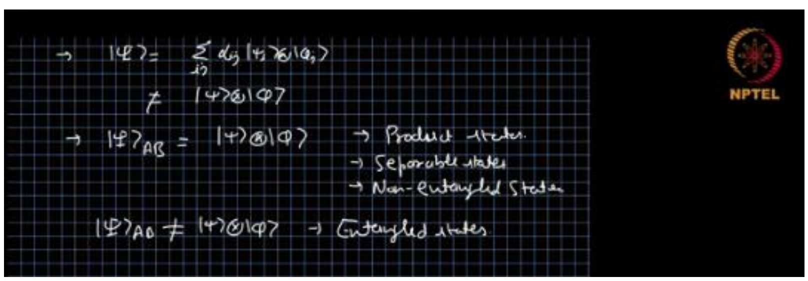

We cannot have two qubits where we prepare their states independently and then put them together. They will not be entangled. So, two qubits have to be simultaneously prepared in some state to get entanglement. These things will be more clear when we talk about mixed states. Consider the state psi to be 0 tensor 0 plus 1 tensor 1 over root 2.

So, here, alpha ij is, there are four alphas, alpha zero zero, alpha zero one, alpha one zero and alpha one one, there are four alphas alpha zero zero is one over root two, alpha zero one is zero, alpha one zero is zero and alpha one one is one over root two. So, we have two qubit system, d1 and d2 are two and we have this four coefficients. So, the matrix A becomes one over root two one zero zero one okay, so, this is the matrix A and the rank of this matrix is two, hence, psi is entangled is an entangled state. This was a trivial example, let us take slightly more complicated example, let us say psi A B is zero, so, for shorthand known word we will be most often we will be representing psi tensor phi as psi phi. So, it will be 0, 0, 0, 1, 1, 0 and 1.

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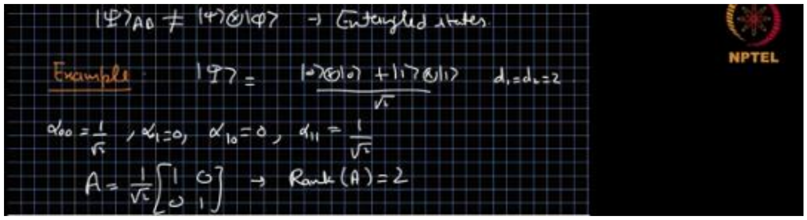

So, let us take the coefficient of this plus, plus, plus and 1 over 2 for the normalization. So, in this case we have alpha 0 0 to be 1 over 2, alpha 0 1, alpha 1 0, alpha 1 1. So, here the matrix A is 1 over 2 1 1 1 1 now the rank of the matrix A is what, is the question. To find the rank of the matrix we see how many independent rows and columns we have. We have one column here and we have, we have the same repeating, so, we have only one independent column and similarly one independent row. So, the rank of A is one. In fact, if you see, this A can be written as a vector one one times, this is the ket vector this is the bra vector and there is a vector one over two outside. So, this can be calculated to be one one one one, okay. So, the rank of matrix A is one, this implies that the state psi is a product state. So, we can always find psi, a small psi and small phi such that psi can be written in this form.

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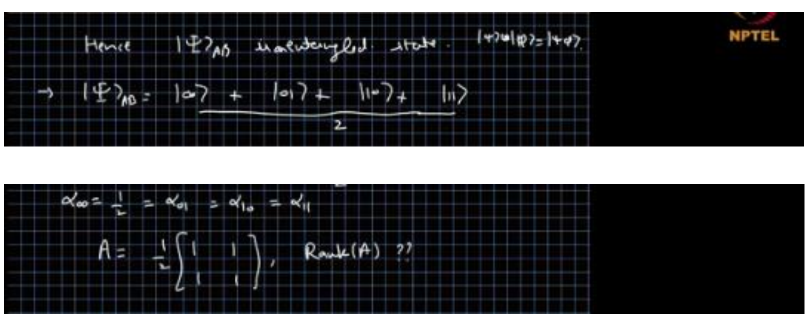

Now, in our earlier treatment, we saw that if A is S times P transpose, there are vectors, then it's, it's a, it's equivalent to capital psi to be psi tensor phi. Then there we, here we see that A can be written as S vector and P vector where S vector turns out to be 1 over root 2, 1, 1, and this is same as the P vector. Hence, the state psi can be written as s i psi i, psi i are zeros and ones, so sum over i ,so, this will be one over root two, zero plus one. Similarly, phi, everything is coming out of the same so it will also be one over root two zero plus one. So, you can see that if you take these psi's and phi's then we get the same the state capital psi which was given here. So, this was the example of separable states.

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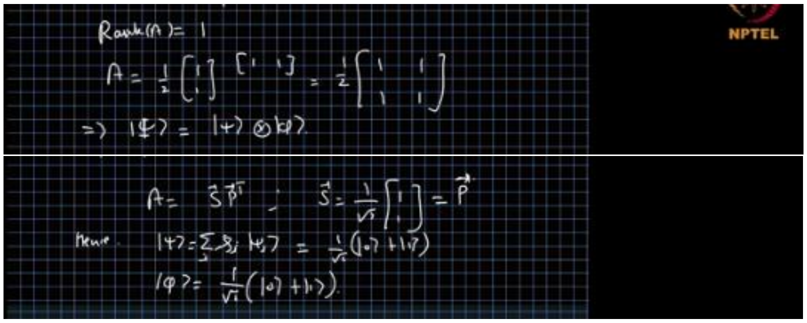

Now, let us take another example just to finish things off. Let us say again we have 0 0 plus 0 1 plus 1 0 minus 1 over 2. This looks very much similar to the state considered in the previous example. But in this case, let us see alpha 00 is 1 over 2, that is equal to alpha 01, that is equal to alpha 10, that is equal to minus of alpha 1. So, the matrix A becomes 1 over 2, 1, 1, 1, minus 1.

What is the rank of this matrix? Here again, we see that the one of the column here is 1 1 and other column is 1 minus 1. These are actually orthogonal to each other. So, they are two independent columns, similar two independent rows. So, the rank of the matrix A is 2, which is not equal to 1.

Hence, psi is an entangled state. I hope with these examples the entanglement, like it's clear what what we mean by entanglement and how to find out whether it's entangled or not.

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$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\rightarrow |\Phi\rangle = \dfrac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2}$

$\alpha_{00} = \frac{1}{2} = \alpha_{01} = \alpha_{10} = -\alpha_{11}$

$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ Rank(A) $\overset{?}{=}$??

Rank(A) = 2 $\neq$ 1.

Hence $|\Phi\rangle \rightarrow$ Entangled state.