**FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH**

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**Lecture-11**

# Qubits: Bloch Sphere

In this lecture, we will discuss the geometric representation of qubits. What we mean by this is we will talk about the something called Bloch sphere. So, to understand Bloch sphere, let us start with the representation of the pure state, which is cos of theta over 2 sin of theta over 2 exponential of i phi. And this is the mathematical representation of a state of a quantum system and the experimental representation of that will be the expectation of the sigma x, sigma y and sigma z. And they are given by sine of theta cos of phi, sine of theta sine of phi, cos of theta. So, this vector psi, which belongs to the two-dimensional Hilbert space, complex Hilbert space, and this vector of expectation values, let us call it r vector that belongs to r3, they represent the same information about a quantum system.

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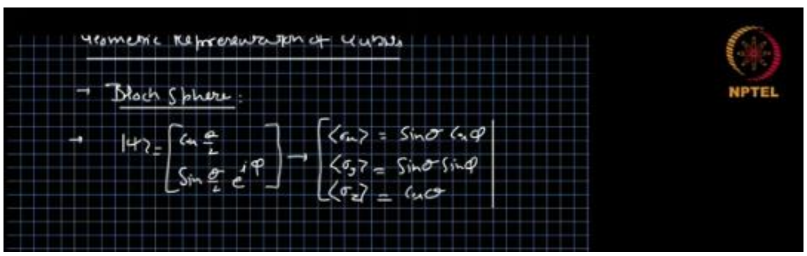


And if we look at it carefully, this vector r, which is from the three-dimensional real space, the r vector magnitude is 1. We can calculate it. That is sigma x square plus sigma y expectation square plus sigma z expectation square square root. Sigma x square plus sigma y square, we can get and it will be sine square theta plus sigma z square is cos square theta which is this is one and square root of that is one so the vector r the norm is one. So, if we take the set of all the vectors r of this form they form a sphere in R3, in three-dimensional real space.

In fact, these sigma x expectation values, sigma y expectation values and sigma z expectation values are the polar coordinates of a sphere. So, if we have three axes, then, we can have a we can have a vector, let's call it r vector, the angle of this, this is x axis, this is y axis, this is z axis so angle of the r vector with the z axis is theta and its projection in the xy plane the angle of that with the x axis is phi. So, the r z component here can be written as cos of theta and r xy component that is a projection in the xy plane that can be written as sine of theta and r x that's along the x axis will be sine theta cos phi and r y will be sine theta sine phi. So, this vector in the polar coordinates is represented by these three quantities, sine theta cos phi, sine theta sine phi and cos theta.

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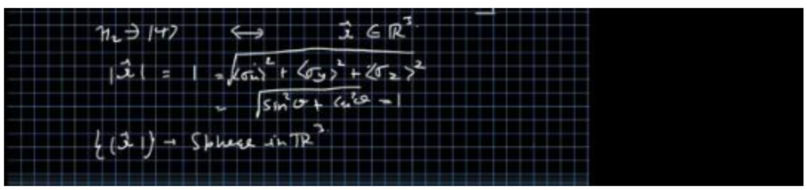

So, collection of all such vectors in a three-dimensional real space will result in a sphere of unit radius. So that will be a sphere of unit radius. Here you can see that theta goes from 0 to pi. The r vector can be along the z-axis or it can be opposite to the z-axis. So minus z axis, so theta is allowed between 0 and pi and but phi can take any value between 0 and 2 pi so that was the restriction we have put so far on our states also so that we can get one to one correspondence between the point on the sphere, unit sphere in R3 and the state in H2 which are valid quantum states.

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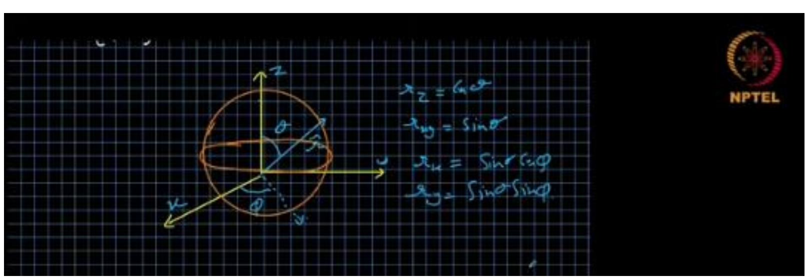

This sphere of unit radius, that has a connection with the states of a quantum system that can also be thought of as the ray space for H2, which becomes S2, the two-dimensional sphere in the three-dimensional real space. And this sphere is called Bloch sphere. Now, our state psi on this Bloch sphere is represented by theta and phi, two angles. Now, what about psi orthogonal state? Psi orthogonal, if psi is written in cos theta, sine theta form, psi orthogonal will be minus sine theta over 2, cos of theta over 2 exponential of i phi or we can write it as

sine of theta over 2 we take minus 1 common and we don't care about the overall phase, minus cos of theta over 2 exponential of i phi, which will have its own theta and phi, it's a sine of theta prime over 2 sine. We we want to write it in the canonical form so it's cos of theta prime over 2 and sine of theta prime over 2 exponential of i phi prime. So, what we are saying is psi is represented by theta and phi and psi orthogonal is represented by theta prime and phi prime. Now, what is the relation between theta and theta prime, phi and phi prime? So, we can see that theta prime over 2 should be equal to pi over 2 minus theta over 2 or we can say theta prime is pi minus theta similarly phi prime is pi plus phi. So, these transformations, this tells us that if the coordinate of psi is theta and phi, the coordinates of psi orthogonal will be pi minus theta and pi plus phi.

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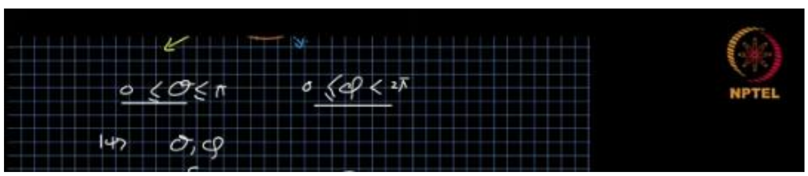


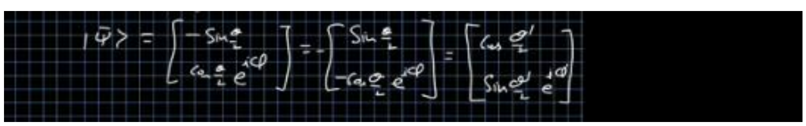

So, if we look at the three-dimensional real vector, we have sine theta cos phi, sin theta sin phi and cos theta and this is for psi and for psi orthogonal we have sin of phi minus theta, cos of pi plus phi, sine of pi minus theta sine of pi plus phi and cos of pi minus theta. Now, we recall that sine of pi minus theta is minus of sine theta. Cos of pi minus theta is minus, this is plus here, this is minus of cos theta. Sin of pi plus theta is minus sin theta, sin phi, minus sin phi and sin of cos of pi plus phi is minus cos phi.

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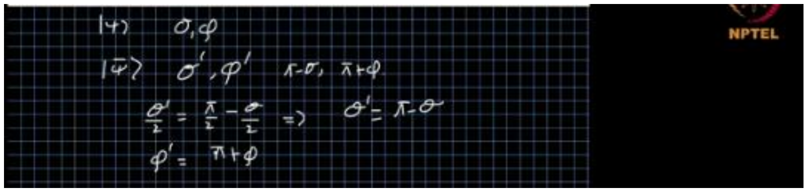


we substitute these in the in the Bloch vector corresponding to psi bar we get sine of pi minus theta is sine theta and cos of pi plus theta is minus cos of phi, sine is sine sine is again sine with minus sign and cos of pi minus theta is minus cos of theta which is nothing but minus the vector corresponding to psi. So, the r vector corresponding to psi and r vector corresponding to psi bar, they are related and that is psi bar psi r psi bar is negative of psi. What does it mean is, if a state is represented by a Bloch vector r, then its orthogonal state will be represented by a diagonally opposite point on the sphere. So, diagonally opposite point is the point which connects the origin and the point R and passes through the other side of the sphere.

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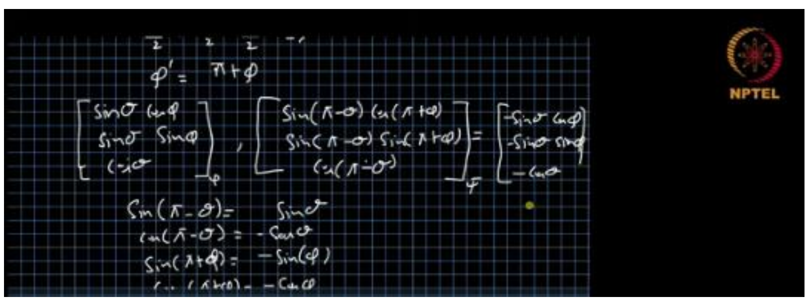


Our state is given by this point here then its diagonally opposite point will be, we extend this on the other side and it cuts somewhere here and that will represent orthogonal state even if this is psi then the diagonally opposite point will be the orthogonal state So, what we can see like if we have state psi which is written in the cos theta which is represented by theta and phi. We can calculate the density matrix corresponding to the pure state psi that will be cos of theta over 2 sin of theta over 2 exponential of i phi. And it's a dagger so cos of theta over 2 sine of theta over 2 exponential of minus i phi. When we multiply, we get cos squared theta over 2, cos theta over 2 sine theta over 2 exponential of minus i phi, sine of theta over 2 cos of theta over 2 exponential of i phi and sine square theta over 2, and if we compare it with half times identity plus r dot sigma, it turns out that r vector is nothing but sine of theta, cos of phi, sine theta, sine phi and cos theta that is the Bloch vector corresponding to the state psi.

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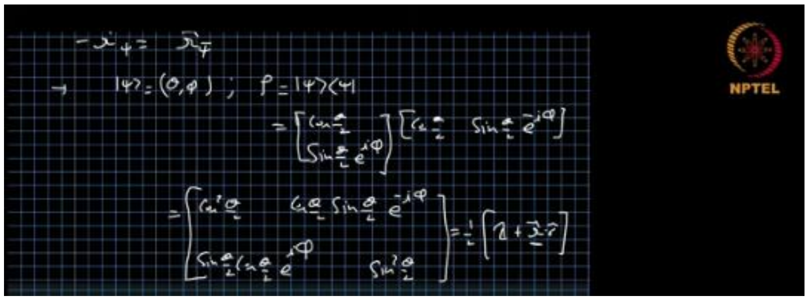



So, the and that should be obvious because we have already established that this r vector is the vector of the expectation values of sigma and the vector of expectation value of sigma is the Bloch vector. So, this vector here in the density matrix decomposition is the Bloch vector r we are interested in. Another interesting point, if we have an observable A, which can be written as sum over mu from 0 to 3, A mu sigma mu. If we have such observable, which can also be written as a 0 sigma 0 which is identity, plus sum over i from 1 to 3 or not just that, we can write it a vector dot sigma vector where a vector is the three-dimensional real vector then the eigenvector of psi of A, let us call it phi

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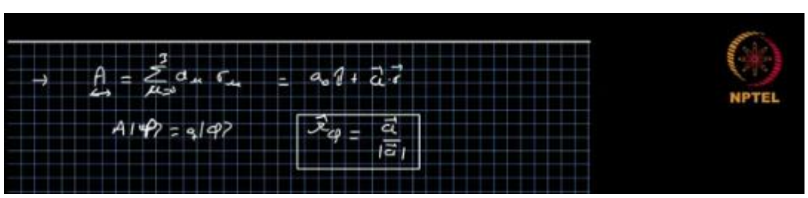


The eigenvector of A phi, such that it is with plus eigenvalue, plus A plus eigenvalue. The Bloch vector corresponding to phi is nothing but a vector divided by a magnitude, normalized vector. So that will be the Bloch vector corresponding to the eigenstate of an observable. So in that way, this makes the calculation very easy and very straightforward. And since we have a geometric picture, it makes them intuitive also.

So, in this Bloch sphere, let me draw it again here, probably little bigger this time. And we have the z-axis, y-axis and x-axis. Let us draw an equator in a different color, so this point here that is along the z-axis, z y and x, this point along the z-axis on the Bloch sphere, this represents the state 0 and its opposite represents the state 1. We can again think of it from the statement we just made that the eigenvectors of an observable are the Bloch vector corresponding to the eigenvectors are the Bloch vectors we have used in the decomposition.

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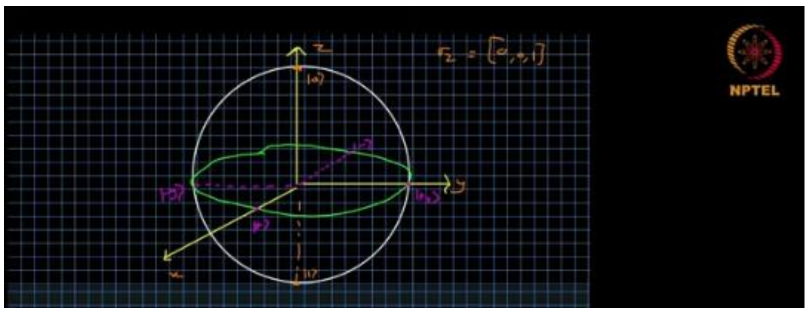


So, sigma z observable is just zero zero one Bloch vector and this is the coordinate for that and the opposite point has to be orthogonal state so it's the plus one or one state, zero and one state. Similarly, on the equator, this point is the point where x axis cuts the sphere. So, it is the plus state which is the eigenstate corresponding to plus eigenvalue and the diagonally opposite point is the minus state. Similarly, this point is the point where y axis cuts the sphere. So, it becomes plus y and this becomes minus y state.

This is how the six states, six important states in qubits are represented and all other points are other states on the Bloch sphere, the qubit space. Now we move to mixed states. By definition, the Bloch's vector is the vector of the expectation values, sigma x, sigma y, and sigma z. When we have a mixed state rho, which is given by half identity plus r dot sigma, then r vector is the Bloch vector. And the condition over r is r mode is less than or equal to 1. When r mode is 1, that represents the surface of the sphere.

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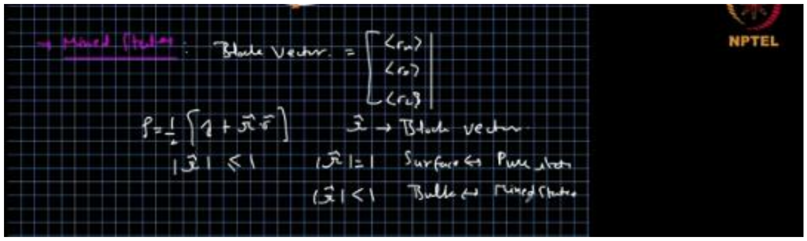


And when r mode, surface, when r mode is equal to 1, this implies pure states. When r mode is less than 1, then it is bulk. And this implies mixed states. So, in a Bloch sphere, all the states on the sphere, on the surface of the sphere, they have the magnitude of the Bloch vector to be 1 and they represent the pure states always. And every point on the sphere is a very pure state and every valid state is a point on the sphere.

Inside the sphere, the magnitude of the Bloch vector is less than 1. So, they represent the mixed state. All the mixed states will have some r vector which is less than 1 magnitude and that will be represented in the Bloch sphere and any point in the Bloch sphere will be a valid mixed state. At the center of the Bloch sphere, we have a mixed state called identity over 2 that is called maximally mixed state. Right at the center, we have the mixed state corresponding to r vector equals zero.

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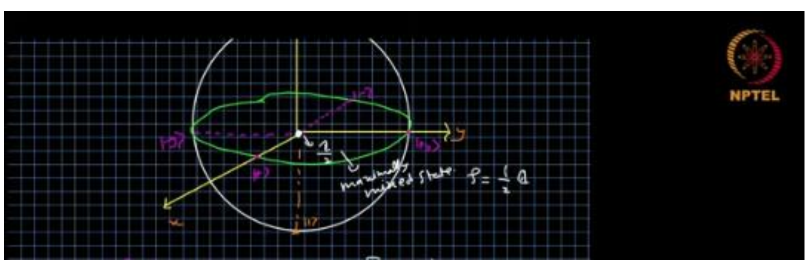


So, it is a zero vector, the magnitude of that vector is zero. So, the density matrix rho is just half times identity. And this is a maximally mixed state and it is of certain significance to us. So, now we see that r vector somehow is measuring how pure or impure the state is. So, the purity of the of a qubit state can be measured with r vector mode.

There are many definitions of many measures of purity. This is a good measure where when if r vector is equal to 1, the magnitude, then it is maximally pure. When r is 0, then it is maximally mixed state. So, all the measures can be a function of this. So, they are all related.

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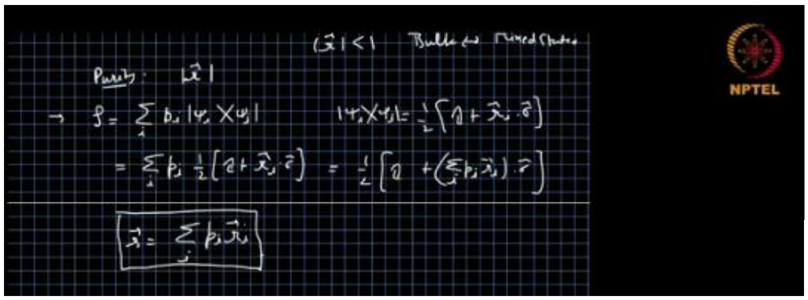


Now, if we write the density matrix rho in a preparation method, that is p i, psi i, psi i, where psi i, psi i is half identity plus r i dot sigma. So, every pure state has a unit Bloch vector r i. So, we can substitute it here we get sum over i p i half times identity, plus r i dot sigma, that is half identity plus sum over i p i r i dot sigma. And we can call it s vector dot sigma here, where s vector is nothing but sum over i p i r vector. So, the preparation method which was given in terms of the weight and the pure state we can represent it in terms of the Bloch vector that the Bloch vector of the density matrix should be a weighted sum of Bloch vectors of the pure state, corresponding pure state.

Now, this is very interesting because here we can understand the non-unique decomposition of a density matrix. For that, let me draw again a bigger, big Bloch sphere. And axes, this is the Z-axis, this is the Y-axis and this is the X-axis. And let me draw a equator. Let me show that this is a three-dimensional sphere. If we have a density matrix inside the bulk somewhere, here it's not visible like where it is but believe me it's inside the the bulk, okay, so it's a mixed state.

Now there is a Bloch vector for it and that is let us represent it by s it doesn't matter what we call it now if we find, some pure state, one here, one here, one here, one here, one here, such that the all the vectors, such that let me take the different color for pure states, such that this Bloch vector and this Bloch vector, and this Bloch vector and this Bloch vector and this Bloch vector all of them we will take some of those with proper weight it gives you the this vector then you can write the mixed state as a superposition of or mixture of all these pure states. To give you one example let us say we have this s vector was not normalized so let us extend it and it hits the pure state here and there is the opposite state here so this is some eta state and this is eta bar state the orthogonal state and the corresponding vector is n vector and minus n vector. Then s vector can be written as the n vector times p plus 1 minus p times, minus n vector that will be 2p n minus 1, 2p minus 1 n. Then s p is just 2p minus 1 times the n vector we get. So, in this way, we can write the density matrix, which is represented by the s vector by the sum of two vectors n and minus n. This is a trivial example, but it is a profound one because now n represents a pure state and minus n represents another pure state and orthogonal state to the original one.

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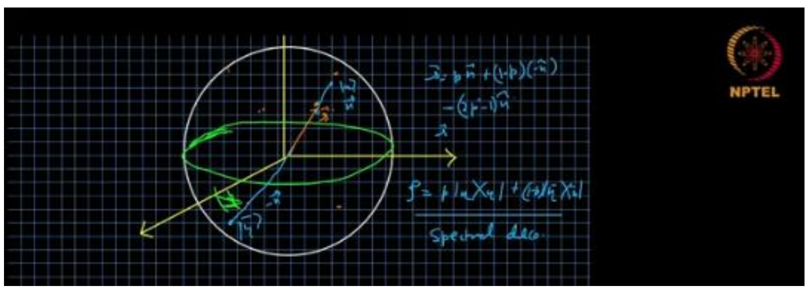


Eta and eta bar are orthogonal states. So rho can be written as a state eta plus a state eta bar with their corresponding weights p and 1 minus p. And this is precisely the spectral decomposition. So if we have a vector, we extend it and make it a pure state and take the orthogonal state. And if we decompose rho in terms of that, we get a spectral decomposition. But we could have done like we take a line passing through the vector s, the point s and hitting the sphere at two different points 1 and 2.

Let us call it psi 1 and psi 2. Since this line is not passing through the origin, the psi 1 and psi 2 are not orthogonal. But still, we can write the s vector. So, this I am running out of the colors here. Let me yeah, so this psi 1 I am representing by r1 vector Bloch vector and r2 Bloch vector psi 2.

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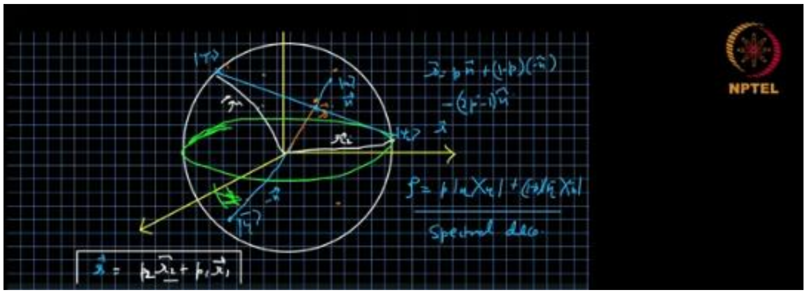


s can be written as some probability p1 times r2 and p2 r2 and p1 r1 vector. Because s falls on a line joining r1 and r2. This can be understood simply as if you have two points 0 and 1 and any point between them. Here x point can be written as the the p times point 0 plus p 1 minus p times point 1, because it's a convex combination of the two points so we can always write it as the weighted sum of the two points here the two points are represented by not 0 and 1 but r1 and r2 so their corresponding weights are p1 and p2 and we can represent the vector s as a weighted sum of the two. So, in that way we can represent the density matrix rho as a mixture of two states which are not orthogonal.

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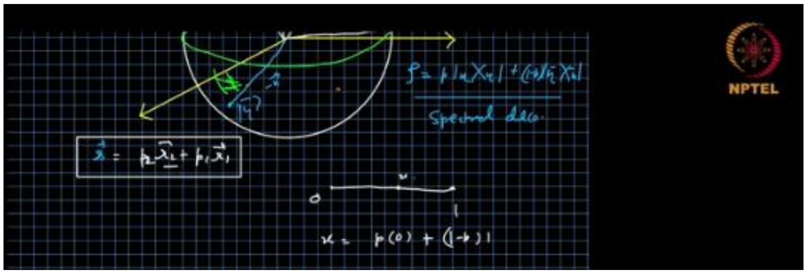


Now, how many such lines which we can find which passes through the rho in this sphere and the answer is infinitely many. So, we can draw basically any line which passes through rho and cuts the sphere at two places and we will get a decomposition for the density matrix. If this line happens to be passing through the origin also, then we get a very special case that is spectral decomposition. Now, that is the decomposition with two states. Now, we can as well think of a plane passing through the density matrix rho, a two-dimensional plane passing through the rho.

It cuts the sphere in a circle. The sphere will be cut, the circumference will be a circle. Now, the plane in which we have rho and which has, which is bounded by a circle on the Bloch sphere, we can choose a triangle. So, let me draw it here. We cut, we have a smaller circle, it will not be a radius one circle and the density matrix rho will be somewhere here in this.

So, now we can find a triangle in this such that the rho lies in this triangle. Now, the vector in this triangle in this plane can be represented by the weighted sum of the three points, three vertices of the triangle. We have to find the weights, but it is not impossible and it exists. So, we can always write the point in the circle as a weighted sum of the three vertices. And how many circles we can choose in this way?

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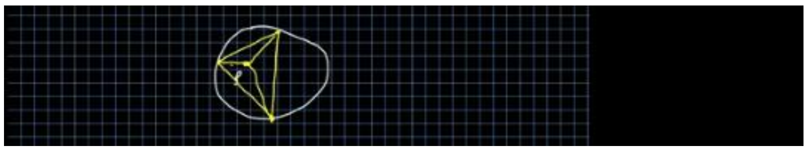

Infinitely many. And how many planes we can choose which passes through rho? Infinitely many. So, in that way, we can find the three pure state decomposition of rho by first finding the plane in which, which passes through rho and then finding a triangle in that plane. Similarly, we can have four points in this, like we could have taken four points.

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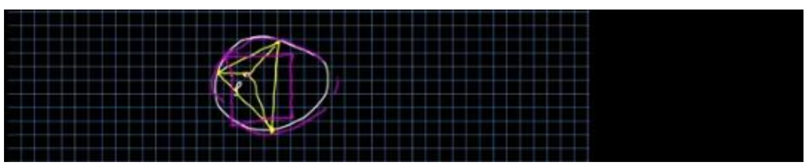

And the point is such that the rho was inside it. And then we will have four state decomposition. In fact, we can write rho here as a mixture of all these points with corresponding weights. In that way, we will get a decomposition of rho with infinitely many pure states. So, we can extend this idea and we can get more and more complicated decompositions.

But this is, I think, sufficient to embrace the idea that there are non unique decomposition of rho. We can have two state decomposition, we can have three state decomposition, we can have four, five infinitely many states decomposition, whatever we want. Of course, we cannot just take arbitrary set of states and assume a thing that it will give us a decomposition for rho, but there always exist enough decomposition for rho that we have enough freedom to do many of the things we want to do. Next is the unity transformation in the Bloch sphere. So, let us consider again U z, which is exponential of minus i gamma over 2 sigma z, which is also exponential of minus i gamma over 2 exponential of i gamma over 2.

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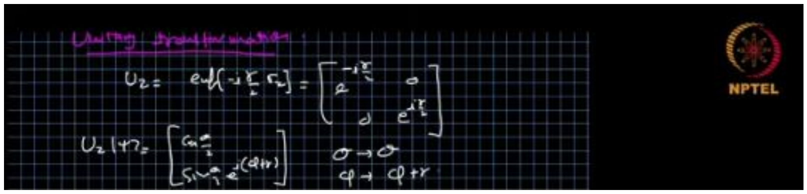

Now, if we apply U z on psi, we get after removing the phase, we get cos theta over 2 sine of theta over 2 exponential of i phi plus gamma. So, now what happened is theta remained same, but phi became phi plus gamma. Now if we look at the Bloch sphere again let me draw the Bloch sphere maybe smaller one this time, the axes is z y x. What we are interested in is a vector this angle is theta this is phi. The phi is the angle between the projection of the vector given vector r in the xy plane this is x y and z so U z rotation transformation what it did is theta remained the same but phi became phi plus gamma. It is not very difficult to see that it means we have rotation about z by an angle gamma.

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So, u z is actually a three-dimensional rotation about the axis z by an angle gamma so that phi becomes phi plus gamma. Similarly, u of y which is exponential of minus i alpha over 2 sigma y, which is cos of alpha over 2 identity minus i sine of alpha over 2 sigma y which is cos of alpha over 2 minus sine of alpha over 2, sine of alpha over 2 and cos of alpha over 2. If we apply U of y on a state which is real, which is cos of theta over 2, sine of theta over two it means we have taken phi to be zero phi zero means the state lies in the exact plane then we get cos of theta plus alpha over two and sine of theta plus alpha over two, then, theta goes to theta plus alpha and phi, well, we always have 0 phi here.

So, that will correspond to so theta like then this was theta over theta and then we get theta plus alpha.

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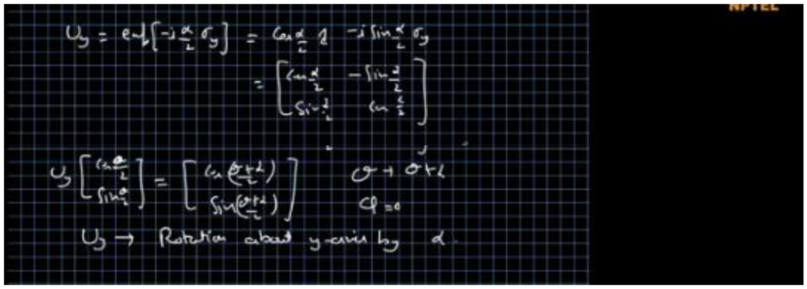


So, it becomes alpha. So, this will correspond to rotation about y axis by an amount alpha. So in that way, the rotations caused by just sigma y matrix or sigma x matrix or sigma z matrix they are the rotation about x y and z axis by the corresponding angle so we can say from here that unitary matrix which is minus i theta over 2 n cap dot sigma where n cap is a unit vector. This matrix will cause a rotation about Bloch vector n cap by angle theta. So, all these things are very interesting.

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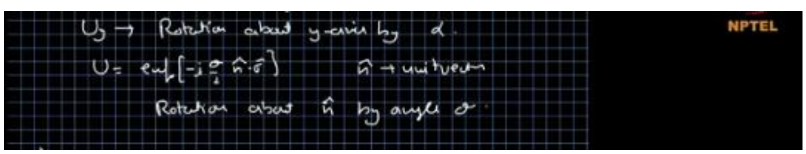


So, we can see, we can make a very general statement here that a unitary matrix U, which belongs to special unitary 2 by 2, there is a corresponding rotation in R3. So, U causes rotation on R3. H2, the two-dimensional Hilbert space and with the corresponding rotation in R3, three-dimensional real space and rotations in R3 are given by another group called SO3, special orthogonal 3 by 3 matrices. If there is it seems like there is a relation between the two so if we want to see the relation how do we see it, so if we are given a unitary U, we can apply it on rho and U dagger this is the transformation on the density matrix and we get another density matrix rho prime rho can be written as half identity plus r dot sigma, and rho prime can be written as half identity plus s dot sigma now U rho U dagger is equal to half identity plus s dot sigma here it is half identity plus U r vector sigma U dagger equals of identity plus s dot sigma.

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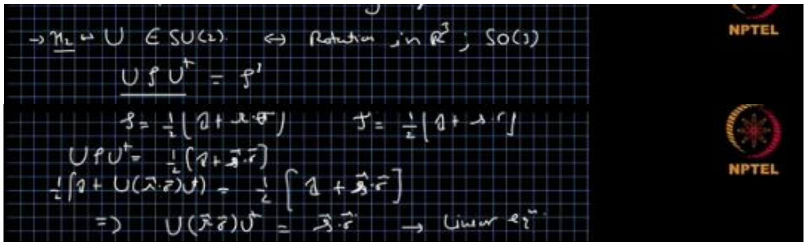


Comparing the two sides, we get U r vector sigma U dagger is s dot sigma. Now, this is a linear equation. It means we can always find a matrix R which is 3 by 3 matrix which acts on three-dimensional real vector and gives us a three-dimensional real vector. This R matrix belongs to special orthogonal 3 by 3 unitary. You can see, it can be an exercise.

See that, prove that R is orthogonal. It means R R transpose equals R transpose R is identity. R vector mode is same as s vector mode. The length of the two vectors is same.

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