**Introduction to Quantum Field Theory - II (Theory of Scalar Fields)**
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**Module - 4**
**Lecture - 9**
**LSZ Reduction Continued**

So, let us continue from where we left last time.

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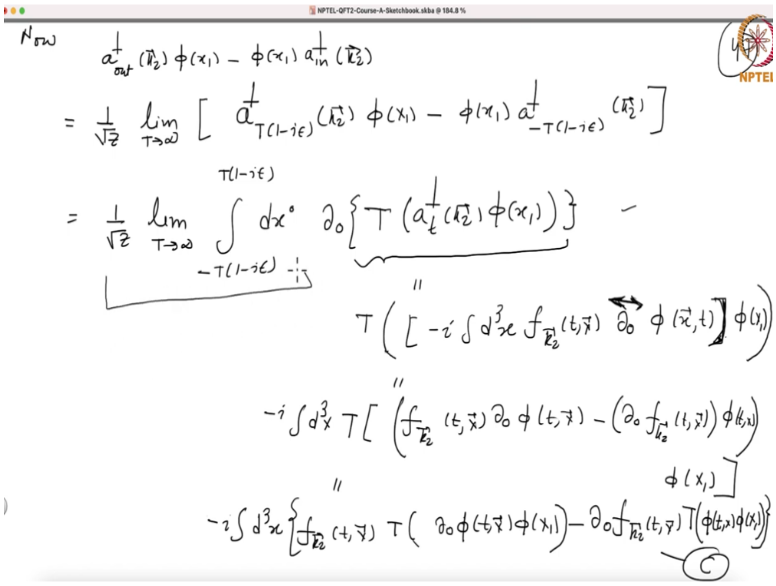


Just a second; So, we were here, but before I proceed, let us go back and see what we were doing.

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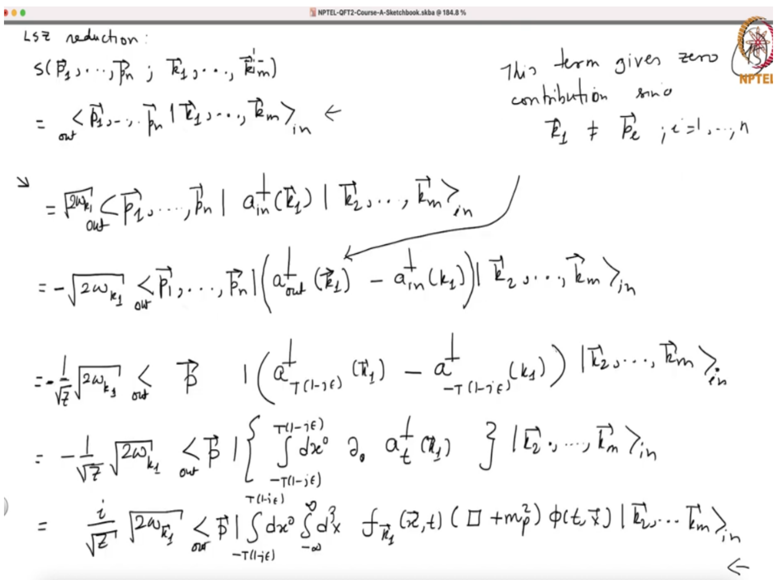

So, we were looking at these S matrix elements which are basically these objects. So, you have an in state here with m momenta and n momenta for the out state, and there are no other labels as I said last time. For example, if you had a theory which had particles with electric charges, then you will have those electric charges also appearing in this list, but here, all the particles are identical with the same mass and no electric charge because I am looking at real scalar field theory.

So, there are no other labels; and let us look at the algebra that we are doing so that we can keep track of some factors and if you make a mistake, we will hopefully catch it. So, here in this step, we get a factor of root 2 omega k 1 when we pull out or when we get rid of the momentum k 1 from the in state. So, that is coming from normalisation and that k 1 get replaced by a in dagger k 1.

And then of course, there is a minus sign coming because of this thing which we did. And when I replace a out dagger and a in dagger in terms of this a dagger T or a dagger minus T, I get a factor of 1 over square root of z. So, I get root 2 omega k 1 over square root of z when I get rid of momentum k 1. So, at least these 2 factors are going to appear. Then here in this step, you see you got a time derivative of a t dagger and that brings in a factor of -i. So, that cancels this but you get a factor of i. Let me show you. Now I have numbered the pages, so I know where it is. So, on page 20 or 21 here.

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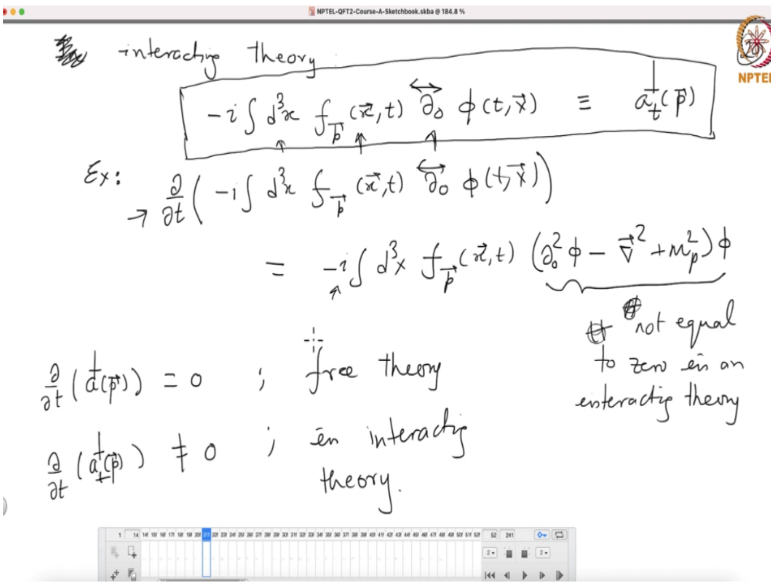


So, here we had seen long back that if you; so, this is the a t dagger which is defined like this, and if you take the time derivative of this object in the above expression, so, this is time derivative of a t dagger; it is minus; it is exactly the same thing as above; it is our integral expression. So, I am taking time derivative of a t dagger and the result is -i times this integral. So, this is a source of an i in our calculation and of course, this -i kills the minus sign we had there.

So, let us go back. So, one source of i is this one here. So, what we say is when I get rid of k 1, I get root 2 omega k 1; I get 1 over square root of z and also; move it; and also when I convert this time derivative of a t dagger in terms of the field phi, then, apart from other things, I bring a factor of -i and that minus sign gets killed to give you a plus i over square root of z.

So, you see at this stage here, what I have achieved is that k 1 is gone; there is no reference to a in or a out; the only thing we have is the field phi. So, this expression involves phi the field, and the overall factor is i over, i times root 2 omega k 1 over square root of z. And what we will see is that the same overall factor comes again when you get rid of the label k 2 which is here. So, with this in mind, let us proceed. We have to keep track of many things now at this stage. So, where is it? This is a shortened notation for the out state. I have suppressed the labels.

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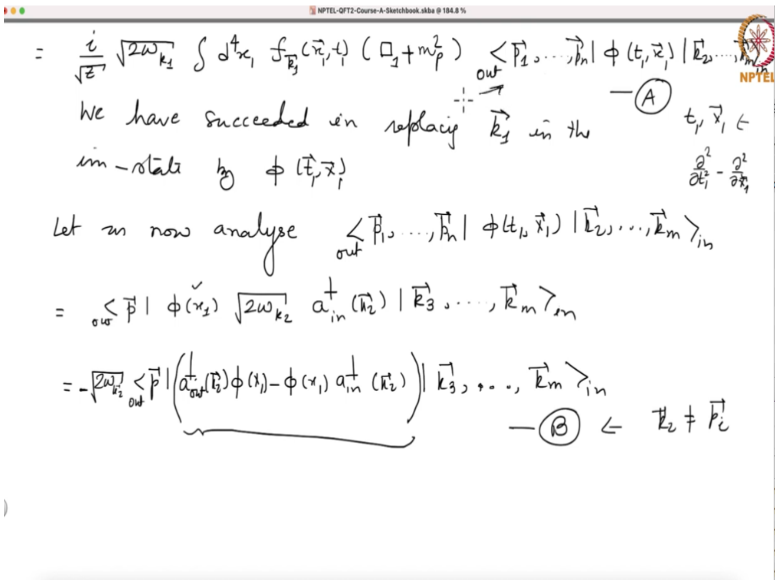


And here, so, finally I got this overall factor which I have talked about times this integral with some function, with the function f which we have seen several times, times this differential operator acting on this object, this matrix element of phi. And then we started looking at only this matrix element of phi and we repeated similar steps. So, you see again I am bringing, I am getting rid of k 2 and I have brought in a in dagger that comes with the square root of 2 omega k 2 and again this step.

So, now I am looking at only this factor; this gives you 1 over square root of z as I said before, because now you are converting a dagger out and a dagger in, in terms of a dagger of T and a dagger of -T. So, you get 1 over root z and then that is where we are. So, we have not yet taken del nought of a t dagger, so, nothing to decide here. So, we had eventually arrived at this expression -i integral d cube x f k 2 time ordered product of; some brackets are not very properly placed, let me see.

This phi x 1; so, I will just look at my notes; that will be easier. So, 1 over root z, this is fine; this is fine; T -i this is a square bracket, so, this one is closing here. And then you have, so, this entire thing in the round brackets is time ordered. I should not have missed this. Now, this is equal to; let us check; -i d cube x time ordered product of f k 2 del nought phi minus this. That is fine. And then you have -i d cube x f k 2 T del nought phi times phi x 1.

So, this bracket is missing, so, this is, these 2 operators are time ordered, minus del nought f k 2 x and then time ordered product of this, and here is the curly bracket closing. So, this is fine. Now you see here, you have in this term, second term. So, these are just functions and derivatives, but the operator content is in here; so, the time ordered product of phi of t x and phi of x 1.

Here also we have almost the same thing, time ordered product of del nought of phi t x phi of x 1. Now if I could interchange or take the del nought outside of T, then I would have time ordered product of phi times phi of x times phi of x 1 and this will also have time ordered product of phi times phi x 1, phi of x and times phi of x 1, and these 2 will be the same factors. Now let us see whether we can indeed take this del nought outside of the time ordering operator.

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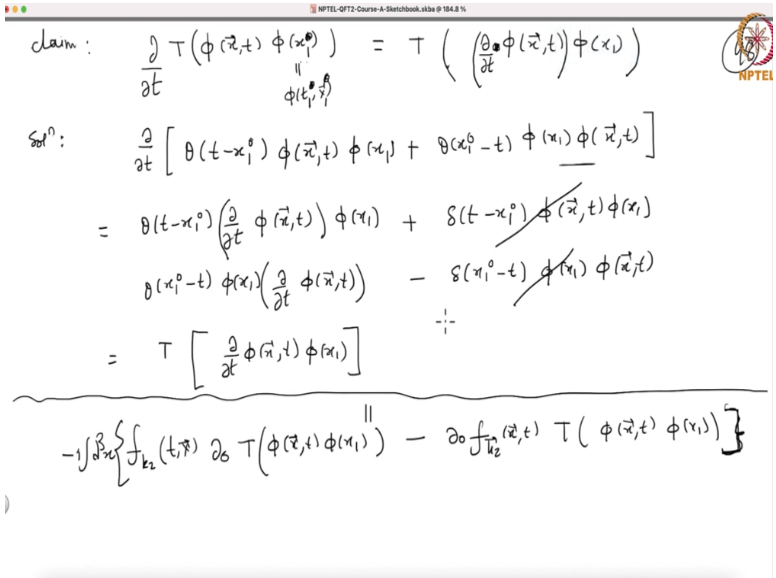


So, here is a claim; how should I go? So, here, so, I claim that time ordered product of phi x t and phi x prime, phi x prime, I mean t prime and vector x prime, this and if you take the time derivative with respect to t, this is same as time ordered product of del nought phi. Let me; this is del over del t and this is kind of obvious because if you take this time; look at the left-hand side, we have time ordered product of these 2 operators.

So, let us say x the time t is smaller than the time t prime. So, here this is phi of t prime x prime. So, let us say t prime is smaller, then you have this sitting on the right and then phi x t sitting on the left and if the other way around, then you just interchange the order. Now, when you take a time derivative with respect to t, that field phi is, the time derivative is at that time. So, del phi over del t is an object which is still defined at the time t.

So, the ordering does not get affected because of the time derivative, and this is what is the claim but nevertheless I will do some algebra to show you, but it should be clear that this is a result which we should get just because taking a time derivative of a function at which is defined at some time will not change, will give you another function which is again defined at the same time, but nevertheless let me show this to you.

So, we have del over del t. What is this object? This is theta of t minus x 1 0 that is a step function. So, only when t is greater than x 1 0, this is 1, otherwise this factor is 0; times phi x t phi x i x 1 not x prime; I am writing x 1 plus theta of x 1 nought minus t phi x 1 phi x t. That is the way you can write time ordered product of these 2 operators and then you have the time derivative. So, let us just do the differentiation.

So, time appears in this theta and in this phi. This one does not have t, so, that is just going to sit out as a constant. So, you have theta of t minus x 1 nought times del over del t of phi x t times phi x 1 plus; now I differentiate theta t minus x 1 nought and that gives you a delta function, delta of t minus x 1 nought times this function. We will do the same thing to the second term here and it gives theta of x 1 nought minus t phi of x 1 that is a constant and then I have, I take a time derivative on this one, del over del t phi of x t.

Then I get minus delta of x 1 nought minus t; this I differentiate this theta function, I get minus of this delta function. So, this is what we get, and you see that these 2 terms, they just cancel. So, this hits when t = x 1 nought. The same as here, t = x 1 nought, otherwise this is 0, the delta function vanishes. So, it makes phi of x t and again, or let us say phi of x x 1 nought and phi of x 1 x 1 nought; similarly here, and these 2 cancel; they are identical.

So, you are left with this term which is just time ordered product of del over del t phi x t; both of these terms have identical factor here; times phi x 1 and you see they are time ordered because you have these theta function or unit step functions which tell which way it is ordered. So, we have shown that indeed what we can argue simply is true by showing this algebra also. So, we will use this and I will take this and substituted in here.

So, I will pull out the time derivative and put to the left because that is what I have shown that I can interchange the order of, I can take the time derivative inside the time ordering operator or pull it outside, either way. So, this gives you what? This gives you; so, we will continue from here. So, this gives you; so, I will write, continue from that page. So, you have -i d cube x and then you have f k 2 t x; maybe I should go to my note; that will be better; f k 2 t x and you can meanwhile also write from your own notes.

So, del nought, so, I am pulling out the derivative here, del nought time ordered product of phi x t phi x 1 minus del nought f k 2 x t time ordered product of phi x t times phi x 1, curly brackets, nought; should not put this. So, that is good. Now, with this I can; f k 2, this is good. So, let us go back and see what we have done. So, here we were only manipulating this piece, the thing in the curly brackets. Now we should look at this entire thing also.

So, I have to take a time derivative of whatever I have written down here and integrate over x nought and put 1 over root z, and that is what gives you a dagger k 2, a dagger out k 2 phi minus phi times a in dagger k 2. So, maybe I should write down this entire thing so it is easier to read.

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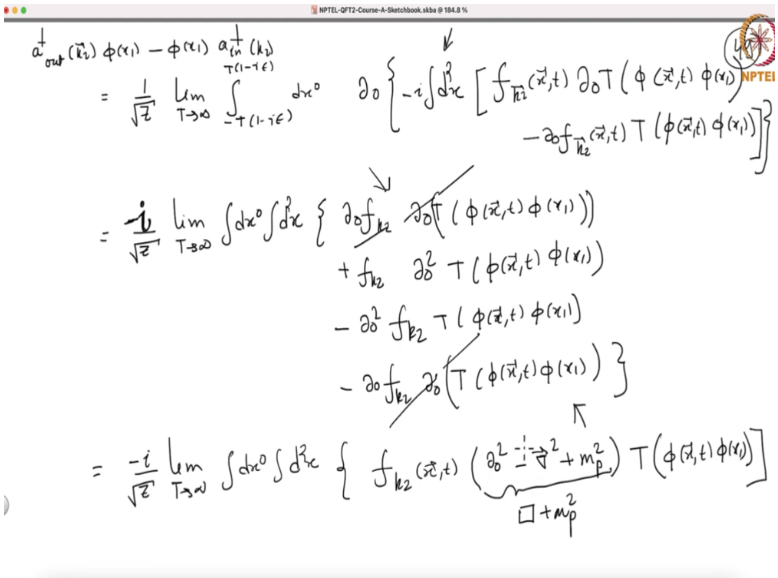

So, what I have shown now is that a out k 2, this thing is equal to 1 over root z limit capital T going to infinity -T 1 - i epsilon to T 1 - i epsilon dx 0. So, let us see, this is up to here. That is what I have written on that new sheet. And then I should take a time derivative of this object. So, let us do that. Then I should take a time derivative of this expression here, this one.

So, I am going to write that one out -i integral d cube x f k 2 x t del nought of phi x t phi x 1 minus del nought of f k 2 x t time ordered product phi x t times phi x 1. So, I have just substituted everything, but remember this also, this expression also has to be put back into here in this one but we will come to that later. So, what do we get then? So, here you have a time derivative.

This del nought is del over del t. So, it acts on this function f because you have time dependence in here and it acts on this phi or this entire object. Similarly, it acts on this and on this entire object. So, if I do that, I will get the following: 1 over square root of z limit t going to infinity; I will just write dx nought; I will not write the limits explicitly. Then you have d cube x from here and then let us collect everything else.

So, this i, -i I am going to pull out and write here -i; everything, all the factors are taken care of and then I get del nought acting on f k 2, then del nought of; I am just doing the differentiation phi x t times phi x 1; then, second time, the derivative x on this one. So, I get f k 2 times del nought square because del nought acting on del nought gives you del nought square of this operator.

Now let us look at the other term, second term, again del not x on this. So, it gives you minus del nought square acting on f k 2 times time ordered product of phi x t phi x 1 and then you get minus del nought f k 2 and then the derivative x on this time ordered product. So, let us see the first term here and the last term here, del nought of k 2, del nought of k 2; del nought of this time ordered product and here del nought of the same time ordered product.

They come with opposite signs, so, these 2 cancel. So, what are we left with is just, correct, del nought square acting on this f k 2 and then you have del nought square acting on f k 2 and del nought square acting on; so, you write down the expression of f k 2 that you already know. Take the time derivative toys and substitute in here and you will get the following. You will get the following result, f k 2 x t; remember f k 2 is an exponential function, the time dependence is exponential, so, you will get f k 2 back when you differentiate twice here.

So, that is why I am able to take it out, and you have del nought square here and this one will give you minus gradient square plus m p square. Just doing the differentiation will give you this thing and then you have time ordered product of; so, this is what we get and this is what is sometimes written as box plus m p square. This operator box is just del nought square minus gradient square.

Very nice and good. So, now I will take this and substitute in this. Where is it? In this expression so that I obtain what this factor is. So, I am going to write down now this factor. So, what we have is now; maybe I should write down.

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Then gives

$$
{}_{out}\langle \vec{p}_1,\dots,\vec{p}_n|\,\phi(x_1)\,|\vec{k}_2,\dots,\vec{k}_m\rangle_{in}
$$

$$
= \sqrt{2\omega_{k_2}}\,\frac{i}{\sqrt{Z}}\lim_{T\to\infty}\int d^4x_2\; f_{\vec{k}_2}(\vec{x}_2,t)
\times (\Box_2+m_p^2)\;{}_{out}\langle \vec{p}_1,\dots,\vec{p}_n|\,T(\phi(x_2)\phi(x_1))\,|\vec{k}_3,\dots,\vec{k}_m\rangle_{in}
$$

$$
S(\vec{p}_1,\dots,\vec{p}_n;\vec{k}_1,\dots,\vec{k}_m)
= \left(\frac{i}{\sqrt{Z}}\right)^2\sqrt{2\omega_{k_1}}\sqrt{2\omega_{k_2}}\int d^4x_1\, d^4x_2\; f_{\vec{k}_1}(x_1)\, f_{\vec{k}_2}(x_2)
\times (\Box_1+m_p^2)(\Box_2+m_p^2)
\times {}_{out}\langle \vec{p}_1,\dots,\vec{p}_n|\,T(\phi(x_1)\phi(x_2))\,|\vec{k}_3,\dots,\vec{k}_m\rangle_{in}
$$

This gives p 1 to p n phi x 1 k 2 to k m, and this is in state, that is out state. Remember, if you are getting lost in the algebra, just remember that we are one after one removing these momenta from the in state and replacing by phi's. That is what we are doing, and this is what we had got. This factor we had gotten when we had done it for the first time. When I took this thing, this inner product and when I first time replaced k 1, I had gotten this expression.

Apart from some integrals and some functions and derivative operators, I was left with a matrix element involving 1 label less in the in state and had replaced that by phi. So, I am writing result for that factor now. So, this now I have shown that this is equal to 2 omega k 2 square root i over root z, remember, because I am going to remove this k 2, I am going to get this vector root 2 omega k 2, and also remember, I get i over root z every time I repeat these steps as I had talked earlier.

So, I am going to get this times limit t going to infinity and instead of using x; here, the variable I was writing as x; I will just, because this is dummy, I will just write it as x 2. So, d 4 x 2, then you have f k 2 x 2 t, then we have box 2; the box is with respect to the label x 2; so, I am writing box 2 with a subscript 2; this, and then we have out p 1 to p n time ordered product of phi x 2 phi x 1; I would write that in the next line; times box 2 plus m p square where m p is the physical mass.

So, I have now from k 3 to k m; I have gotten rid of k 1 and k 2 and instead of those, I have got 2 fields here which are time ordered product and then these operators. So, now I will take this and put back into this expression here, expression A and that is what I am going to do now. So, S, now we are almost there at least as far as making our first important result. Explicit is concerned I have the following.

So, that is the S matrix element. I have shown that this is equal to; so, I am substituting back into this expression. So, I get i over root z when I get phi x 1 phi of x 1 and times root 2 omega k 1 and then I get again i over root z and root 2 omega k 2 when I have phi x 2. So, I will take care of those. So, I have i over square root of z into i over square root of z, twice because I have these 2 factors.

Then I have 2 omega k subscript 1, then again root 2 omega k 2; I do not know why I said subscript 1; and then I have integral d 4 x 1 d 4 x 2 f k 1 of x 1 f k 2 of x 2 times you have box 1 plus m p square box 2 plus m p square; remember, we got these operators every time, each time; and these act on this matrix element. Now you can imagine what will happen if we were to get rid of k 3, k 4 up to k m.

Each time I do that, I will bring in a factor of I over square root of z or corresponding 2 omega k; let us say next one is k 3, so, 2 root 2 omega k 3. Then you have a new integral d 4 d 4 x 3; then you will have f of x 3 f k 3 x 3; then you will have a corresponding differential operator here box 3 plus m p square, this thing and you are going to get time ordered product of phi x 1, phi x 2 and phi x 3 and then 1 label less, and then again you repeat until you hit the vacuum.

So, at the end, you will have only vacuum left and then you will have a bunch of; I mean, all these things will be correspondingly repeated. So, I am going to write down that result.

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$$
\vdots
$$

$$
\begin{aligned}
= & \left(\frac{i}{\sqrt{Z}}\right)^{m} \left(\sqrt{2\omega_{\vec{k}_1}} \times \cdots \times \sqrt{2\omega_{\vec{k}_m}}\right) \\
& \times \int d^4x_1 \cdots d^4x_m \times f_{\vec{k}_1}(x_1)\, f_{\vec{k}_2}(x_2) \cdots f_{\vec{k}_m}(x_m) \\
& \times \left(\Box_1 + m_p^2\right) \cdots \left(\Box_m + m_p^2\right) \\
& \times {}_{out}\langle \vec{p}_1, \ldots, \vec{p}_n |\, T\left(\phi(x_1)\phi(x_2)\cdots\phi(x_m)\right) |\Omega\rangle
\end{aligned}
$$

So, if you continue removing these labels k one by one, you are going to get eventually i over square root of z; how many such factors? m such factors, because there are m labels in the in state, so, each time you get this, so, you have a factor of m. Then you will get 2 omega k 1 square root of 2 omega k 1, square root of 2 omega 2 and so forth. So, you will get this. Let me write here product of i or maybe just like that way, k 1 times k m.

So, this is also a factor that you are going to get. Then, times f k 1 x 1 f k 2 x 2 to f k m x m times these differential operators and then eventually you have; time ordered product of phi x 1 phi x 2 phi x m and you have vacuum here. We have gotten rid of all the, I mean the entire in state and we have replaced it by the vacuum. Now, if we should do the same thing to the out state.