Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 4
Lecture - 8
LSZ Reduction

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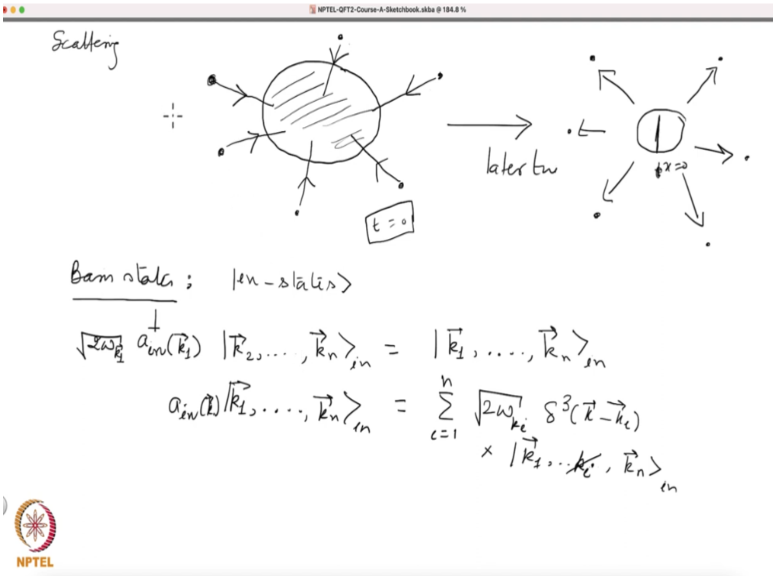

Let us recall what we were doing. Our interest is in describing scattering, where we are imagining that we have particles which are well separated in far past in time. There could be more than 2 particles. It is not necessary that you have only 2 particles which collide, but you can have n number of particles colliding together. So, I am thinking of several particles which then know proceed along some trajectories and they come close to each other; and of course, we are in dealing with interacting field theory, so, they interact with each other.

So, when they are very far apart, we have a notion of how many particles we have, but as they come closer, this notion is lost, we cannot describe any fixed number of particles. And they come close and eventually they collide and at a later time the picture is that you have from this region several particles coming out. So, here these particles were going in and later you have several particles coming out, and if you wait long enough, you will see particles which are localised in space with some different momentum or reasonably well-defined momentum and reasonably well-defined location.

So, this region where somewhere here I will define t = 0; I mean x = 0 and also some arbitrary time when they are interacting appreciably, I will call that time as t = 0; that is the picture. Now we have seen that we can define a basis. So, whatever the Hilbert space you have, the Hilbert space for this system, you can choose a set of basis states which are called in states which have the property that if you fold them with appropriate functions, you can arrive at this picture of non-interacting particles which are well separated; and we saw how to create these in states.

So, we saw that if we have an operator a in dagger and I am using a in dagger with k 1 and also we have to put this factor because of our choice of normalisations. So, if I take this operator and act on a state, in state which carries these labels k 2 to k n, then it gives me a new in state which carries this k 1 label also. That is what in effect we have learnt in the previous lectures; and of course, you can start with vacuum and then have first a single particle state or an in state carrying only 1 label and then you can repeatedly act with a in daggers to create such states.

And we also saw that if you start with a state which has these labels k 1 to k n and act with a in of k; I think this I showed you only for case of single particle state, but if you repeat the arguments, you can see that this will give you the following result i equal to 1 to n 2 omega k i again because of normalisation delta cube k - k i and there is a summation and you have to multiply with these kets k 1 to k m in.

So, all of these labels which are here, all of these labels you have to put, but a in k will remove the label k i. So, that is gone; this label will not be in here. So, for example, if you had k 1, k 2, k 3 and if you act with a in k, then if k = k 2, then this label will be gone; and if none of these labels k 1, k 2 and k m matches this label k, then this delta function will be 0 and then the right-hand side just vanishes.

So, it is just removing that; if that label is present, it removes that label and gives you a state or a sum of in states with 1 less label and if the label k is not present in this list, then it just kills that state. So, just a second; now, this is what we have seen.

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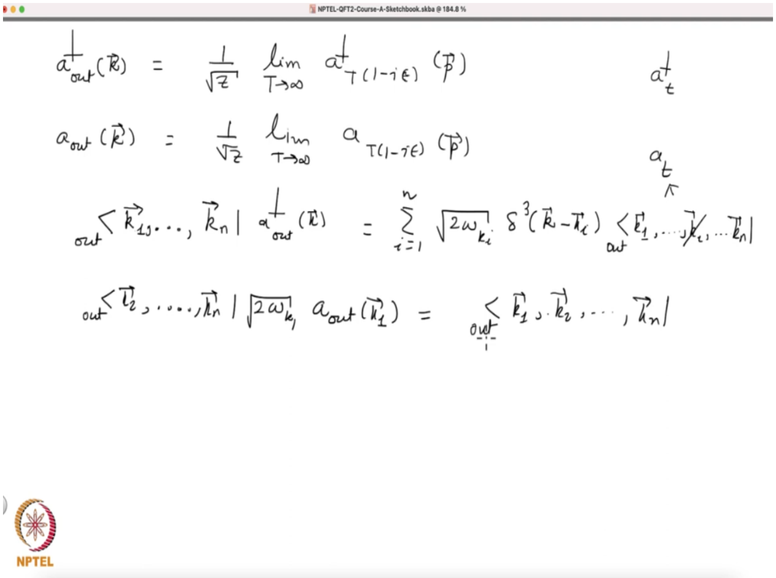

Now, you can, repeating similar arguments as we did in the last 2 videos, you can define a out k dagger which is 1 over square root of z limit T going to infinity of a dagger T 1 - i epsilon p. So, if you were to discard i epsilon, this is basically a dagger at a very large time. Remember, we had this a dagger of t, this operator and in this operator we have to put t times 1 minus i epsilon where epsilon is positive and take capital T to infinity; and that is the definition of a dagger out.

And a out also will define which is 1 over square root of z times this object, where again you have to take a t and then in that substitute for this small t, capital T times 1 minus i epsilon and take T going to infinity limit. So, these are these 2 operators which carry this subscript out and you can check that the states that you, the basis states which these create are the ones which if you were to fold them with appropriate functions, they will give you states or particles; they will give you a state which will represent well separated particles in the far future, so, where because I am talking about T going to infinity.

So, you have to repeat the similar arguments as we did for a in and convince yourself that indeed what I am claiming is true. And once you have done that, you will arrive at the following conclusion that if you take; so, I will write just in the bra form, not in the ket form because that is what we are going to use when we write down the S matrix. So, you should be able to arrive at this conclusion that if you take a out dagger k and act on this bra, then this will remove this label k.

So, if there is a label k sitting in here, and that will be removed. So, similar to the case for in states, you will get 2 omega k i square root times delta cube k - k i and then you have this state which has label k i up sent and as before if none of the labels k 1 to k m matches the label k here, then the delta function vanishes and this which means that this a dagger out is going to kill this state.

And similarly, you will have, if you take state k 2, k 3 so and so forth k n and out state and you act with a out of k and actually not this but 2 omega k 1, square root of 2 omega k 1 a out of k 1 that is coming again from normalisation. So, essentially, if you take this a out and act on this ket, this out state, it will insert this label k 1. So, the state you get carries one more label k 1. So, this is equal to this out state.

So, this is what you get and now we are going to utilise this to talk about scattering; but before I talk about scattering in a process or even in general, I should first look at the S matrix elements which I had discussed in the beginning of this course. So, let us go to the next sheet. So, what do I want? See, thing is that at t = 0, if you are thinking in terms of Schrodinger picture, then at t = 0, we are looking at some states; or if you are looking at Heisenberg picture, you can take the states at t = 0 and in Schrodinger picture as the states in the Heisenberg picture and remember states do not evolve with time in Heisenberg picture.

Anyhow, so, you take at the Hilbert space which is formed by these in states which are basically defined at t = 0 which have this interpretation which we have been saying, and if you wait for some time; maybe let me rephrase it; it is not the best way to phrase it. How should I say?

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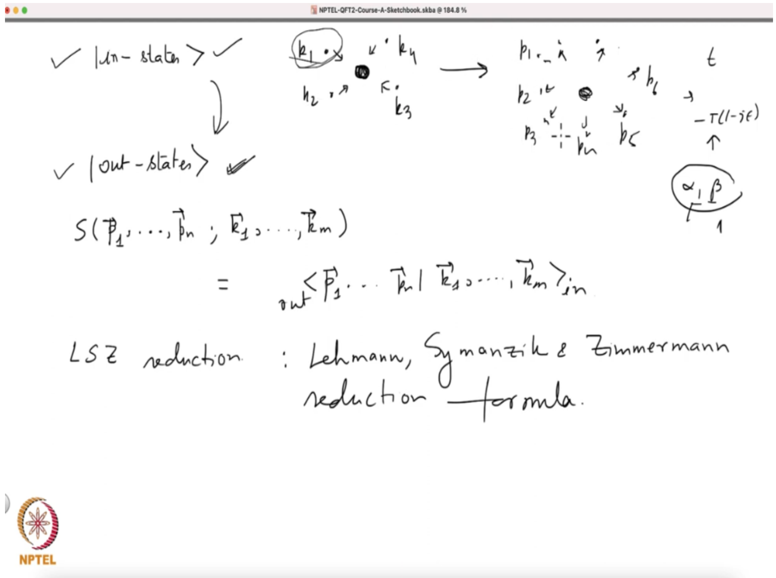

So, these in states are good basis states if you want to describe incoming particles but these are not good basis states for describing particles in the far future because you know that if you start with some incoming particles, let us say 4 of them and they come and interact here. After they have interacted and large time has passed, it might give you in the final state 20 particles going in certain directions.

And remember how we constructed in states, we took these limits where this small t went to -T times 1 minus epsilon and we took capital T to infinity and that is what was picking in states; but then, the same in states is not; so, let us say you start with a state in far past which has 4 particles, so, that state you will write in terms of these in states which carry 4 labels k 1, k 2, k, 3 and k 4 and you will have sum over all of them, all these labels are rather integral; but as time evolves, you are not going to get a state in the far future which has only 4 particles; not necessarily; in fact, in general you will not.

So, it is clear that the same basis is not going to be useful. So, you choose another basis states which are called out states, which will be good basis states for describing well localised particles in the far future; but remember, both these bases live in the same Hilbert space in just different bases, but the Hilbert space is the same for the given system; it is not that you have 2 different Hilbert spaces but these are just 2 different bases which you use.

One basis is good for writing down incoming states which concern incoming particles, and another basis is good for writing down states or constructing states which represent outgoing particles in the far future; but other than that, these are related by just a transformation that takes you from this set of basis states to these sets of basis states; and we had talked about all these things in detail in the beginning, and we had also defined what is S matrix.

I had written down in general with using notations alpha and beta, but now I am looking at 5, 4 theory and there are no other labels other than the momentum labels. So, in this case, I will just explicitly write down the S matrix to be this. So, S of; now I can be very specific that what the labels are exactly meaning what alpha and beta are because there are no other labels. So, as I defined earlier, p n; so, these are the labels coming from the out states.

These matrix elements are needed, because, if you want to express any in state in terms of out states, then you need to know these elements, these numbers or these inner products. So, that is why we are trying to find out these matrix elements, because then you can do a change of basis from in states to out states. So, we want to know these matrix, these elements which are called S matrix elements.

I am going to do something nice. The goal is that I turn these in states and out states which are right now written using this a in daggers and a out daggers into objects that involve the fields phi. Remember we are given fields phi from our action and if I could construct or write them, write these S matrix elements using those fields, then it will be of course nice, and what will be very nice is that we will see that eventually this entire S matrix is going to be turned into a calculation of correlation functions or Greens functions which we did in the previous course.

So, that is what we are going to show now, and this is what is called LSZ reduction. Let us see if I have; I hope I am not making mistake with the names, but this LSZ reduction is; let us go to the next page; that is fine; Lehmann, Symanzik and Zimmermann reduction formula. So, this reduction of these matrix elements into a form which involves Greens functions or correlation functions is what is called LSZ reduction. So, that is what I am going to show you now.

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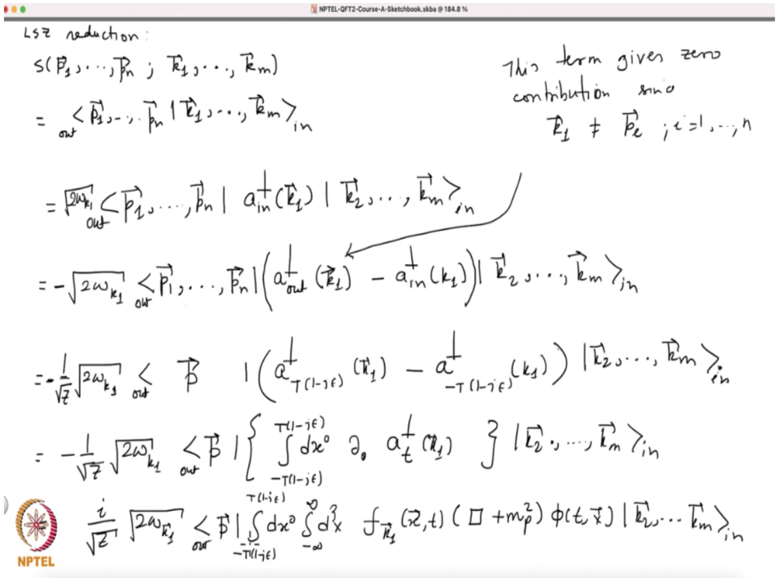

So, LSZ reduction. So, maybe I should write again. So, this is going to be long algebra but not difficult. This is what I write as; before that I should say something. So, in this scattering, this scattering which I have shown here, so, let us say these are in states I am labelling with k 1, k 2 and so forth; k 1, k 2, k 3 and k 4 and these are p 1, p 2, p 3, p 4, p 5, p 6, etcetera. So, in general, because you have an interacting theory, it is not going to happen that these particles when they come close together and then they give these particles in the future, it is not going to happen that you have this one, this guy just traveling alone without interacting with them.

Or more precisely, I am interested in only those contributions in which none of these momenta are left unaltered, meaning scattering really happens. It is not that some particles do not interact at all and they just keep going in their trajectories, meaning I am not going to find any of these k 1 or k 2 to be equal to p 1 or p 2 or any of these. So, I am assuming that I am looking at only those terms, those contributions in which interaction really happens.

So, this is something I am going to utilise. So, let us take this and I will write this as, so, using our previous results, I am going to write this as p 1 to p n; that is an out state. Now, k 1 to k m I can write as 2 omega k 1 or maybe I should write it here, 2 omega k 1 in the square root times a in dagger k 1 acting on this. So, what I am saying is that if you take k 2 to k m, these labels, this state with these labels and you act with a in dagger k 1, then that will insert the label k 1 in the in state.

Of course, you have this normalisation and this is what you had in the previous line. So, these 2 are equal. All I have used is this result, this one, first line. Now, this I will write as 2 omega k 1; this is something you can find in many places, in (()) (25:22) and Ashoke Sen's lectures and many other places, this is standard thing. So, I will write this as -a in dagger k 1, k 2 to k m in; because I multiplied a minus sign here, -1, I should multiply -1 here to take care of it.

So, this line is same as the previous line. Now, I will add another operator here, a out dagger k 1 but this I can do because a out dagger k 1 when acts on this out state, it kills it. Why? Let us go back and check. a out dagger k acting on this out state gives you this result, where it removes the label k i and gives you this delta function; but if k is not present in this list, if none of the k 1, k 2 and so forth up to k m equals k, then the delta function vanishes and it gives you 0 on the right-hand side.

So, you see that because I have said that none of these labels k 1 to k m equals; so, let us take k 1; k 1 is not equal to any of these labels p 1 to p n; k 2 is not equal to any of these labels p 1 to p n and so forth because of what I was saying here, because I am assuming that all the particles scattered, no one is left unscattered, because I am assuming that then k 1 cannot be equal to any of the p i's, any of the p 1, p 2, p n.

If that is so, then a dagger is going to kill this state. So, effectively I have not added anything; so, because this contribution is 0, so, you can imagine that this is not there, and then this line equals the previous line and that is why I am allowed to put this a dagger out with label k 1. So, I hope you agree that this is correct. Let me write down this term. This term gives 0 contribution since k 1 is not equal to any of the p i's and the reason being that we are assuming that all the particles scattered, nothing, none of these particles remain unscattered, none of these labels remain unchanged.

So, what? So, you have, this I will write as minus 2 omega k 1 and then you have this out state here. I will just drop the labels for now; it is tiring to write. And you remember what is a dagger out; that involves a dagger at time t 1 - i epsilon k 1 and you have to put a factor of 1 over square root of z. Let us go back and check. Here; you have, these are for out states but for instance also.

So, somewhere just like these ones have 1 over root z, you had 1 over root z for a ins also. For both of these, you will have 1 over root z, so, I will write minus 1 over square root of z times this minus; now what is a in dagger? a in dagger is -T 1 - i epsilon and of course a dagger here, k 1, and then you have k 2 to k m in. Now what I will do is, I will do a very simple thing, out; let me just write p here instead of leaving it blank, I will just write p.

This p stands for all these. Let me put these. Let us not clutter it. So, this, what I will do now is, I will write this as the following: dx nought or basically dt del nought which is del over del t of a t where t is small, k 1. Now, this is a derivative and then you do an integral. So, that integral will give you just a t dagger k 1 and you have to put the upper and lower limits.

So, if I put upper limit as this and lower limit as this, then you see that whatever I have here in curly brackets is same as what you have in the brackets in the above expression, because once you have done the integral, you get a t dagger and you have to put the limits and the upper limit is t 1 minus epsilon, so, that gives you this first term minus this thing with the lower limit which is this term.

So, this is identically true and I am allowed to write it like this. Now, this is useful because; let us see if I can find out. Somewhere I had; no; know whether we find it easily; probably not; a p, define a t. I had given exercise or I had shown; let us check. I am not finding it, so, I will; so, just hold on. I do not find it, so, I will just state that result that, so, this del nought of a t dagger was basically this box plus m p square acting on phi.

This is a result you should be able to find in your notes and I had written this earlier. So, let me just use that and write it down. So, you will see that if you take this expression and use what is the time derivative of a t dagger where t is small t, then it is the following. So, we will have a factor of -i; just a second, you have it somewhere. Anyway, so, i over square root of z 2 omega k 1, root 2 omega k 1 and then you have this out state which I am writing in short like this, and this is integral.

dx nought you have anyway and you are going to have dx cube once you put this expression, put the expression for this one, del nought acting on a t dagger. So, you will have dx nought d cube x and f of k 1; this is our usual f of k 1 which I have defined several times; times this box operator plus m p square. The box is this del nought square minus gradient square; this operator acting on phi of t x.

This is what you will find in your notes where these limits go from minus infinity to plus infinity, but this one goes from -T 1 - i epsilon to T 1 - i epsilon. So, make sure that you were able to get this expression. So, now what I will do is 2 things. I will write this integral dx over dx 0 and d cube x as d 4 x and I will remember that the limits for cube x is from minus infinity to plus infinity.

But for the time, the limits are different, they are not really from minus infinity to plus infinity but they are along, I mean, they have some complex components; but I will just drop those and write d 4 x but we will remember that this is the case and we will take care of this at the end.

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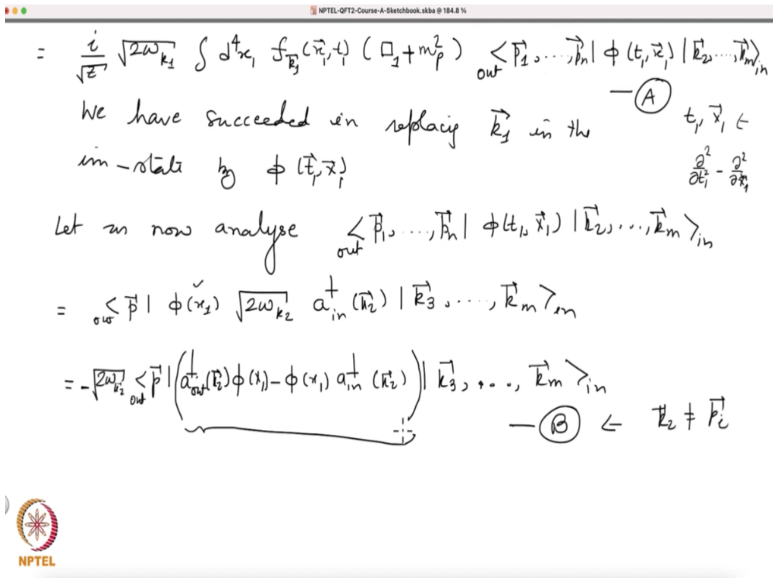

So, with this, I will get, I can write down the following. Also I will do another thing; these are just functions and simple operators, so, I mean, these are differential operators but you have the operator phi and it is sandwiched from right from this in state and from the left from this out state. So, I am just going to pull out everything to the left and let phi sit between out state and in state. So, that is all I am going to write now.

So, i over square root of z 2 omega k 1. That is correct, the square roots. Then you have integral d 4 x and we will remember what I said about the limits f k 1. Then you have this differential operator. And then that is the out state; I will write it in full this time, phi t x and then k 2 k m in. So, it is nice; I have been able to get rid of one of the labels k 1 from the in state and replaced it by an operator, by the operator phi which phi is your field operator that is present in your action and of course there are some differential operators and some simple functions which need to be integrated over, but still this is a nice result.

I am getting things which are familiar things, and now you can see almost what will happen. If I repeat the same thing which I have done just now, then I will be able to get rid of the label k 2, and in doing so, I will pull out another factor of phi and eventually I will be able to pull out or get rid of all these labels, k 2, k 3, k 4, k m and I will be left eventually with the vacuum omega here.

And every time I get rid of one of these labels, a phi will be pulled out. Going by what I have done just now, that is something apparent that is going to happen. And similarly, if I repeat similar steps for the out state with appropriate a out, then I will also be able to remove these labels p 1 to p n successively. And instead of those labels, at the end I will have vacuum on the left and I will get a bunch of phi's here.

So, what you will have eventually is at the end of this procedure, you will have vacuum, then a bunch of phi's coming from the out state, then a bunch of phi's coming from the in state, and then again vacuum. So, this is what we are expecting based on what we have seen just now, but let us see in detail what really happens, what are the factors you get and what exactly is the operator content sandwiched between the vacuum.

So, before I proceed, I will just do one thing; I will instead of calling d 4 x, I mean t and x, I will change the label, change the names; these are dummy anyway; these are integrated over. So, I can choose whatever name I wish to choose. Instead of t and x, I will call t 1 and x 1 and it will be useful to keep track of what we are doing. So, I will write here 1; I will put a subscript 1.

So, this becomes del; you understand; this is the; so, this is del over del t 1 square del 2 t over del t 1 square minus del over del x 1 square. That is the operator now, and this is this. So, I have changed these labels because; here also; because then I can remember that I have taken care of k 1. So, that k 1, that subscript 1 on k is now the subscript here on t and x. So, that way I keep track of which k I have taken care of. So, good.

Let me write down; we have succeeded in replacing this label k 1 in the in state by field phi. Now, let us forget about these other factors; we will take care of these 3 factors later and only concentrate on this matrix element. So, let us now analyse phi of t 1 x 1 and k 2 to k m. In general, m and n will be different, and anyway we want to get all the inner products, so, we have to find for all m and all n.

So, now what should we do is the following. So, this is out state. Again I will not write all these p 1 to p n but I will just write p. This is equal to out. Then you have phi x 1. I will use 4 vector, so, instead of writing t 1 and x 1, I will just write x 1. And k 2, the label k 2, I will again generate using a in dagger. So, I will write 2 omega k 2 in the square root a dagger, a in dagger k 2 acting on k 3.

I am just repeating the steps which I have already done before; same thing. So, again I do what we did before, p; something wrong. This is fine. I should take out the 2 omega k 2 outside. So, I have phi; this is an operator and a in dagger is an operator which are sandwiched between this in and out states. So, let me write it again. Just like before, I have phi x 1 from here; then you have a in dagger k 2, this one; and this is k 3, k m in.

I want to put a minus sign, and to take care of that minus sign, I have a minus sign here. So, this is right now same as the previous equation but I will insert another term which is a dagger out k 2 phi x 1. And why I am allowed to insert this because this contributes 0, and let us see why; because you have a out dagger with k 2. And this guy acting on the out state in which none of the labels p 1 to p n is equal to the label k 2 is going to give you 0.

So, a 2 dagger k 2 acting on this gives you 0. So, you have not changed the equality and of course, if you put phi x 1, that also does not damage, does not do any damage, so, I am allowed to write it this way. And I write it this way because this is useful. You can write anything else also which does not change this equality, but it has to be useful as well. So, let us see what we gain from this. Let me write.

This is allowed because k 2 is not equal to any of the p i's. I think I should have given some; this was fine; all these expressions are equal to S. Here is also S, so, let me call this as A, this one as B where B is only this part. So, A is this entire equation and this factor is what you have here and that result is B. Now I want to take only this term in the round brackets. I should go to the next page.

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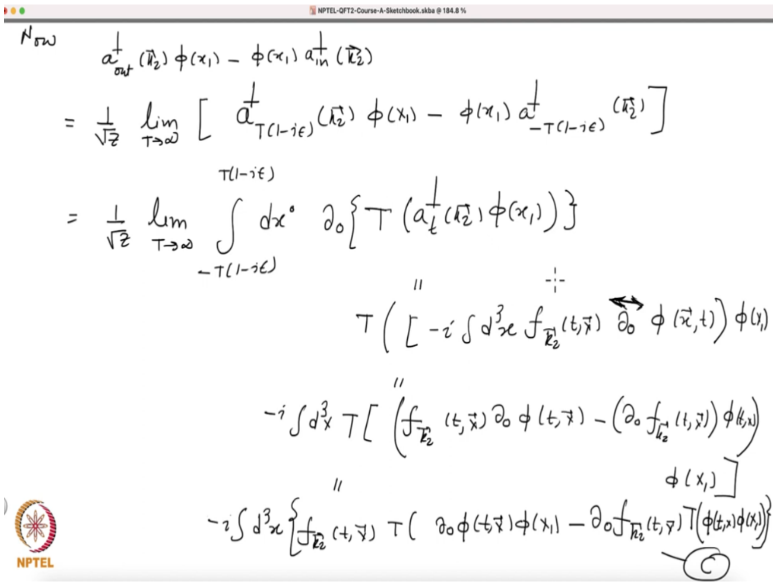

So, now a dagger, a out dagger k 2 phi x 1 minus phi x 1 a in dagger k 2; let us take this which is exactly what you had here, and now I will substitute the expression of a out and a in in terms of A's at those times, at capital T. So, you get 1 over square root of z limit T going to infinity. What is a out dagger? a out dagger is a T 1 - i epsilon with a dagger here k 2, then you have phi of x 1 minus phi x 1 and a dagger minus t; remember that in states are at, they involve -T.

So, this is fine and again the trick as before, integral dx nought del nought and I will put this entire thing, but this time I have to be a little more careful. Let me first write the result and then I will tell you. So, in the previous case, here; where was it? Here, I just had the time derivative and then integrated over the time so that I just, I would get just this and put the limits, but here I have 2 operators and you see the way they are ordered here in these two terms is; so, x 1, so, you have time as t 1, some time t 1.

So, the field or the operator that carries time t 1 is sitting to the right and then the operator which carries time capital T, and you remember T is going to infinity, so, this operator is defined at a time which is larger than the time here because this t is going to infinity, so, it is larger than any other time. So, this operator which is defined at a large positive time is sitting to the left.

So, in this term, in the first term, the lower the operators which are defined at lower, at smaller values of time are sitting to the right, and as you go to the left, the time increases. So, this one has a larger time. Let us look at this one, the same story here. This operator a dagger is defined at - t. It is a time which is smaller than the time here phi of t 1 x 1 because this is some finite time. This operator is defined at some finite time, some t 1.

Whatever that is is larger than minus infinity, time t equal to minus infinity. So, both the terms have operators which are defined at a lower time are sitting to the right and the operators which are defined at a higher time or a later time are sitting to the left. So, you see that these both terms are time ordered. We have talked about time ordering earlier also. So, both the terms are time ordered. So, we have to take care of the time ordering.

So, that is what I am doing here. So, I again do the same trick of taking derivative and then killing the derivative by the integral, but then I ensured that this time order is taken care of and I put T which is the time ordering operator which tells you that keep the lower times, the operators at lower times to the right and operators at larger times to the left; that is all it tells you; and then you take the integral and put these limits.

So, when this integral is done, you put the limit. It will automatically place the operators to the right end or left depending on which one is at a later time and which one is at an earlier time. So, this equality holds. This is an expression which is same as the above equation. So, this is good. And now again you use the; let me write this down. So, this is same as time order product of; if you look at the what happens if you take a time derivative, we have seen earlier, k 2 t x and then your operator phi x 1.

So, this is, you can write it in this form. So, this del nought, this is time derivative which acts on both this phi and f. So, this you can verify that this result is, the above factor is same as this one. And with this now I can write as, I can write -i integral d cube x time ordered product of f k 2 t x del nought phi t x minus del nought f k 2 t x; it acts on this, and then you have phi t x; not here; and then you have another here, closes here.

And then you have another factor of phi x 1 and then this square bracket closes, this one. And this one is same as; it is a feeble statement but let me write it anyway just for ease of writing later; f k 2 t x and this vector and of course this is time ordering, so, it acts only on the operators, not on functions. So, here you have 2 operators, del nought phi and phi x 1. So, time ordering operator acts on del nought phi and phi x 1.

So, that is what I should write. And make sure that you verify that whatever I am writing is correct; t x and then you have time ordered product of phi t x times phi x 1; let me try to; times time ordered product of phi t x and phi x 1 and then curly brackets close here. Let us call this C. Now, let me see what happened. Time ordered product of; just a second; del nought phi times phi x 1; this is fine; del nought t.

And there is something which I am bit unsure, something simple actually. So, maybe what I will do is, I will continue in the next video after I have checked in my note if I am making something wrong. So, we will continue this in the next video and you can probably already see that we are going to get not just set of phi's just when I pull out, when I get rid of these labels, I am not just going to get a set of phi's but rather time ordered product of a set of phi's. That is what it is going to lead to. There is something very minor which I am just saying. I will check and then I will record the next one. See you in the next video.