Introduction to Quantum Field Theory - II (Theory of Scalar Fields)

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Module - 3
Lecture - 7
Creating Multiparticle States

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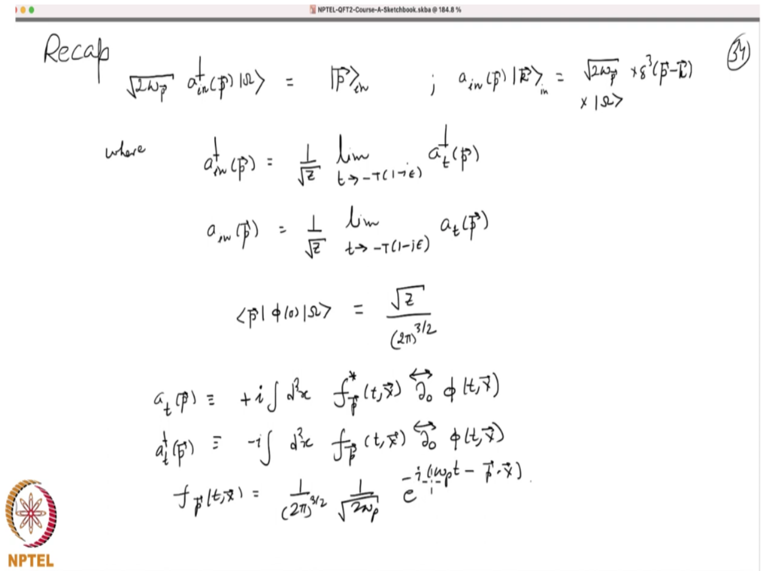

So, let us start with a quick recap of what we have done so far. We have learnt that we can create single particle states in an interacting quantum field theory by acting on with a in dagger on vacuum; and this is normalisation and this gives us instead p. And also we saw that if we take a in and act on single particle state; you can put an in here; then this will kill this particle and give you vacuum. These are annihilation numbers, some functions.

What is important here is that you get vacuum. That is the state you produce, where a in dagger p is 1 over square root of z limit t going to -T 1 - i epsilon a with a subscript t, a dagger with a subscript t. This is an in dagger and a in p is the following, where this z was defined to be the following. So, this matrix element of phi in vacuum and single particle state, this is defined to be square root of z with a factor of 2 pi 3 halves in the denominator.

And what was a t of p? That was +i d cube x f p t x. And a t dagger p, that is this expression, where f p t x; this is all we have done so far. Remember, in the manner I am doing, I am not being super careful, but if you want to be very careful, then you should define these in states carefully and by folding them with appropriate functions so that when you take that state and evolve backwards in time with Schrodinger equation, you end up with a particle which is fairly localised in space at some point x which you choose and has a fairly well-defined momentum.

But I would proceed without those folding functions and the conclusions will be the same; but if you want to be very careful, you can put in those functions. So, the recap is over.

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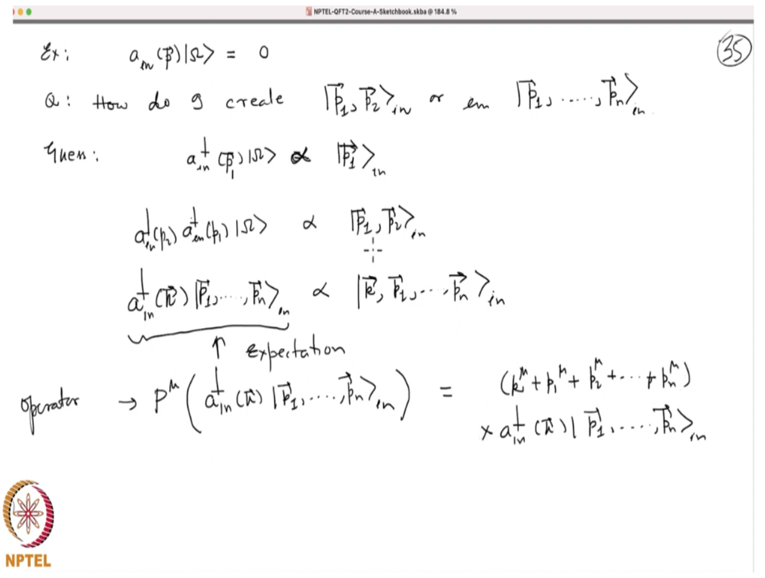

I will give you an exercise, which should be easy to do, based on what we have done so far; is that if you take a in meaning this a in which removes single particle and gives you vacuum, so, if you were to take this operator and act not on a single particle state but on vacuum, then it will kill the vacuum. It will annihilate it. So, you will get 0. This is something you should be able to argue.

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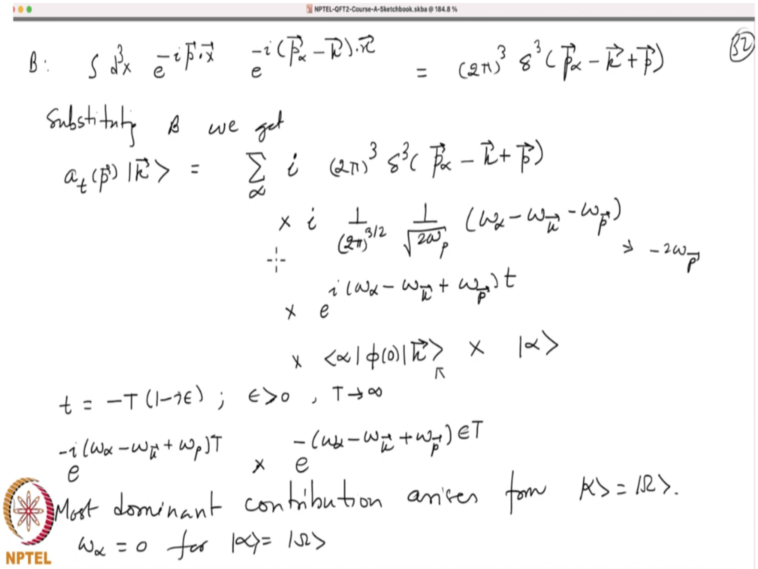

So, what you should do is, you should; exactly this thing. Instead of ket k here, you put ket omega. So, you correspondingly change these expressions and then try to argue what you are going to get and see that indeed you get 0 on the right-hand side. So, that is an exercise you should do. Now I know how to create a single particle state by acting with a dagger on omega.

Now I want to know how to create a 2 particle in state or a 3 particle in state or a state with any number of particles. How I am going to do that? Now, you might already be able to guess that since we have arranged everything in a manner that it follows or it mimics free field theory, our naive guess would be that having multiple a daggers acting on vacuum will give you a state with several particles, because that is what happens in free theory.

So, let us try that and see whether indeed that happens. So, question is, how do I create, let us say this state, or in general, a state of this kind? So, that is the question. And again, if you want to be very careful, you should put the folding functions which I am not going to do. So, our guess which is a good guess is that since a in dagger p acting on vacuum creates; let me put a subscript 1; gives you a state which is proportional to a single particle state; for the time being, I am not worried about the normalisations; then we guess that a in dagger p 1, a in dagger p 2, that would be something proportional to this state with 2 labels.

So, that is our guess. Or, in general, if I am given a state which has these labels p 1 to p n and if I act on this state with a in dagger of k, then I am expecting based on my experience from free field theory and because I am mimicking free field theory by defining a in and a in daggers, that this thing should be proportional to a state which carries these labels, meaning it just inserts a k in here.

So, that is our expectation and let us see that whether this expectation is right. Now, if what I am expecting is correct, then at the very least this state, when I act with this a dagger on this state, this whatever you get here; this we have to show that this is proportional to this state; but if I take this thing, then if my expectation is right that indeed I am going to get what is on the right-hand side, then at least if I act on this, this subject, this state with operator p mu; remember these are operators; operator p mu which is the operator which gives you the full momentum of that state, and act on this state, then what I should get?

I should get; this is my expectation; I should get that this is an eigenstate of the full momentum operator, meaning it is something proportional to again the same state. So, it should be proportional to the same state, meaning it is an eigenstate. And what should be the proportionality? It will be the eigenvalue of this operator which should be k mu plus p 1 mu plus p 2 mu p n mu. That is what I should expect to get if indeed this is going to work. So, let us first verify this expectation whether it is correct or not.

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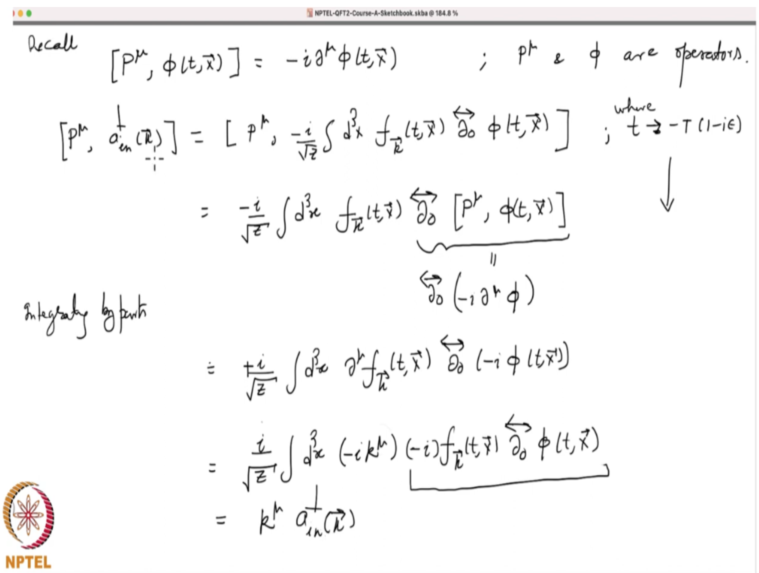

So, recall that p mu is the generator of translations. That is what we have learnt in the previous course, first course on quantum field theory. So, p mu is the generator of translations. So, if I take commutator of p mu with this field phi where phi is in the field of the interacting theory, then it should generate translations, which is this. So, what I am doing here is, see, this is the translation, because if you take phi and do a Taylor expansion, then the first term would be proportional to del mu phi times the translation.

So, this is this object and this you already have seen in the previous course. So, remember, here p mu and phi are both operators at phi is a field operator and p is the momentum operator. So, if you wish, you can put hats on this. So, p mu and phi are operators and that is why this commutator makes sense, otherwise it will not make sense. Now, given this result, I can calculate the following object p mu a in dagger k.

So, I take this operator and take a in dagger and find out the commutator. Now, that is easy because here, a in dagger is just a dagger t and a dagger t is in terms of phi and some derivatives. So, we are going to substitute this. That is the reason why I listed down everything here. So, what will be that? It will be p mu minus i over square root of z integral d cube x, where I should put t = -T 1 - i epsilon. Why? Because I am using a in dagger here.

a in dagger involves not time t but time; t should be replaced by -T times 1 minus epsilon. So, that is why I have put here or you can put here. Where? Now, this -i over root z, these functions, these derivatives, these are all ordinary functions and ordinary derivatives; they are not operators. So, this commutator really acts between p mu and phi, not on these objects. So, I will pull that out.

So, -i over square root of z integral d cube x f k t x, then this term derivative which acts on both sides, and then you have p mu from here, commutator with phi t x, where this condition has to be supplied after you have taken the time derivative. So, what then? Now, this object is just del nought; and p mu commutator with phi, I will put -i del mu phi. That is good; del nought phi.

Now you have on this phi, 2 derivatives acting; one is del mu and other is del nought. So, what I will do is that, I will take this derivative del mu and put it on f of k by integration by parts. So, you remember, if I do an integration by parts, derivative will get shifted from this function to the other function, to f k; but in doing so, you will pick up a minus sign. And then, I can write here, integrating by parts, I will get -i over square root of z; or because I am going to integrate by parts, I will put that minus to plus, that plus, that sign.

Then you have integral d cube x. The derivative del mu gets shifted on f, so you get del mu f k. Then you have this time derivative and then -i phi t x. Now, this is same as i over square root of z. This derivative will give you -i k mu times f again. And there is a -i here, so, that -i I will bring out. So, this derivative generates -i k mu times f; so, these two are here. This -i, I am keeping here. And then, what remains is, this derivative acting on phi.

So, let us go back and see what a dagger t p was. It was -i d cube x f derivative phi, which is same as what you have here, -i f derivative phi and this integral. So, what you have got is the following. You have, so, these integrals will be all absorbed into a in dagger p and what will be left is, and of course, this one, 1 over square root of z is also. So, let us see again; -i; so, I have to keep -i also. So, this entire thing; square root of z; this, there is also a factor of 1 over square root of z because I should, I want a in, not a t dagger.

So, these are factor of square root of z, 1 over square root of z, that is here. And then what is left is k mu which comes out of the integral, k mu, i times -i, which is 1. So, this i times this -i is 1 and it gives you k mu. And the remaining things make up a in dagger of k; or something has gone wrong in between; no, it is just; no, everything is fine. It is just my notes; I made a mistake somewhere; let me correct. Good. So, what did we get? We got the following, that this commutator p mu acting on p mu with a in dagger is same as k mu a in dagger.

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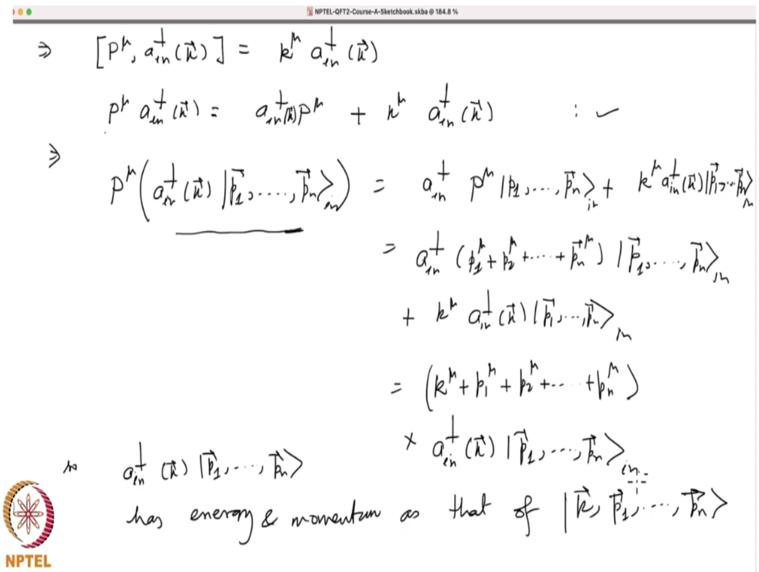

So, let me write down, a in dagger k, this commutator is equal to k mu. Remember, on the left, you have index; you have an up index, so, on the right also, you should have up index, you cannot put it down. And then we got a in dagger k. So, this is nice. Why this is nice? Because it says that p mu a in dagger k; I am just opening up the expression for this commutator; is equal to a in dagger p mu; let me put k; plus k mu a in dagger k, and this is useful.

This is telling you immediately that if you were to take this state, so, let us take a in dagger k and act on p 1 to p n, this in state, and then ask what is the momentum of this, it may or may not be an eigenstate of this operator but we are expecting that to happen. So, let us see whether indeed it is true. So, what I will do is, I will use this result which I have just proved. So, p mu a in dagger acting on this will be; I will use vacuum side; a in dagger p mu; and then you have this state; p 1 p n plus k mu a in dagger k; and then again you have p 1 to p n.

So, what is that in side, a in dagger; now, this is already assumed to be an eigenstate of momentum, so, this will be just p 1 mu p 2 mu mu times again the same thing; let me write; I am writing too much; plus this object, and the same thing here. So, you see that you can factor out a in dagger acting on this state. Both these terms are the same thing. And then you just sum up k mu and these things and these ps.

So, you get k mu plus p 1 mu times; so, our expectation was right indeed. This state here is, this entire thing is an eigenstate of p mu with eigenvalues being just the sum of k and p. So, a in dagger has energy and momentum as that of the state; at least this is established that this state has energy and momentum same as that of this state, but we have not established that this is indeed this state, but at least as far as energy and momentum are conserved, we have shown that these 2 states have the same energy and momentum.

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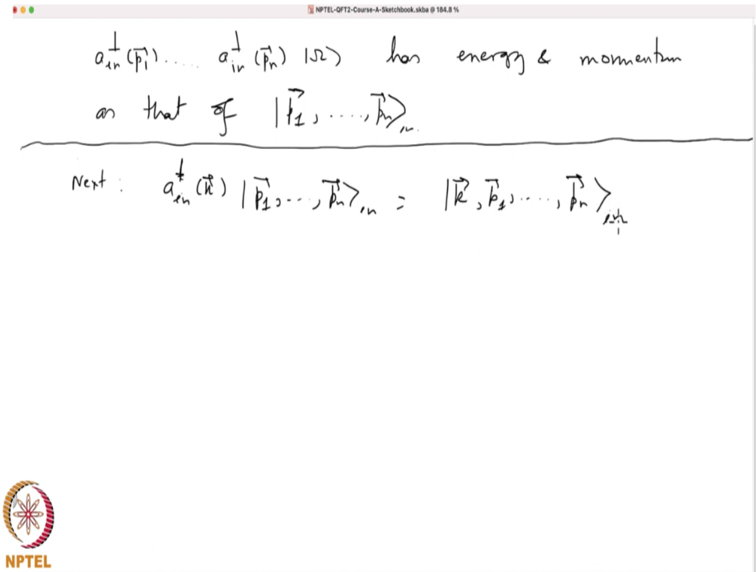

And if you follow the argument, this also means using this; this result gone; this result; because you can start with vacuum and then act on a in dagger and create a state p. Now, this state has momentum p but then you can act again with a dagger and that creates a new state which has momentum let us say; so, you have first one; first a dagger is with p 1; second a in dagger is with p 2; and the state created will have momentum p 1 plus p 2 and so forth, which means that what we have argued is a in dagger p 1, a in dagger p n acting on vacuum has energy and momentum as that of this state, this one.

That is what we have shown. Now, I will show you; something wrong happened. Now, we will show you that indeed; next let us show that indeed a in dagger k acting on an in state gives you this. Before this, I have not shown; all I have done is, I have talked about the energy and momentum being same for both these states, on the left-hand side and the right-hand side; but now I am going to show you that indeed they are same states. Let me do it on a new page.

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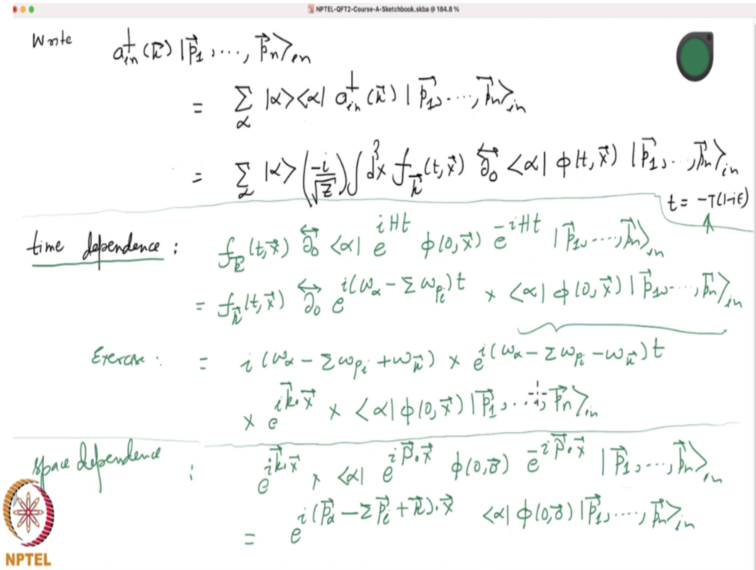

So, we start in our well repeated manner and write the following. So, let us look at a in dagger k and you know what I will do; I will just insert a complete set of basis states, the same thing which we have been doing over and over again. So, this is same as; so, here are our basis states, complete set of basis states, a in dagger k and p 1 to p n. So, let us evaluate this factor. I will carry along this alpha, not a; just a second.

So, this is summation over alpha. Now, this other remaining factor I can write as -i over square root of z. I am substituting the definition of a in dagger and we will have integral d cube x. In fact, I should have; I could have done this, -i over square root of z integral d cube x f of k t x and we have this derivative acting on alpha phi and p 1 to p n, where I have to put this time to be -T 1 - i epsilon. So, now, we have this time derivative operator.

So, let us find out all the places where time dependence is. So, time dependence is of course not here because ket alpha is in Heisenberg picture; so, there is no time dependence here. So, time dependence is in f k; that is a simple time dependence, and then there is phi. Or, the time dependence of phi is easy; I will just write down; and also I will have to worry about the space dependence, so that I can integrate out over x.

Again f k has a very simple exponential space dependence and phi also has a very simple space dependence. So, let us look at that. So, as far as time dependence is concerned, you have f of k. Maybe I will do one thing, make it a bit easier to read. We will use some other colour. Which? Let us try this. So, time dependence is f of k t x del nought. Then you have ket alpha as above picture; so, there is no time dependence.

And time dependence of this, you remember we have already done this thing before. I can pull i, pull out by writing to the iHt where H is the Hamiltonian and then writing phi 0 x e to the -iHt; and then you have; so, time dependence is completely given by these factors. phi is an operator, so, it evolves with time according to this expression. So, good, and what does that give?

This gives f k t x del nought, and e to the iHt acting on this bra alpha and then e to the minus iHt acting on this ket with these ps will give you e to the i energy of the state alpha, omega alpha minus summation over all the ps, which is omega p i. This is a summation over i, which I am not writing, times t. So, this is minus omega p 1, minus omega p 2, minus omega p 3 and so forth and times alpha phi 0 x; and again I should remember that this is the time I should put.

And here is a simple trivial exercise. Show that this expression is equal to; so, basically, I am in this; I am not going to do anything to this factor; exercise is about this first one that you get the following: This will be i times omega alpha minus summation omega p i plus omega k times e to the i omega alpha minus omega p i minus omega k t times e to the i k dot x times this remaining factor. That is what you will get.

So, here, instead of writing f, I have written the full thing; so, that x dependence is here. So, we have talked about time dependence and you see the time dependence is very easy, it is very simple, it is just, so, this exponential factor the way we have been getting earlier. And how about this space dependence? That is even simpler. So, you have one space dependence coming from e to the i k dot x here and the remaining is hidden in this phi 0 x; but that is also easy because you can pull out the; you know how to relate phi at origin to phi at some other point x.

So, we will do that. So, x dependence is completely contained in e to the i k dot x times this piece which I will write as alpha e to the i p dot x, where this is the momentum operator, phi at origin and then e to the -i p dot x acting on this state which is equal to; now that this is a; so, this p will act on bra alpha and this p will act on this ket and it will give you the following: p alpha coming from here minus summation over p i coming from here plus k and that is from here, times the constant, alpha phi 0 and this.

So, now I have x dependence also, and there was the time dependence. So, now what? I substitute in here, in this expression, both, everything and I carry out an integral over x. And that integral over x, this is the only place where x is, this exponential, and that gives a delta function. It will give you a 2 pi cube times delta p alpha minus summation p i plus k and then these constant ps and these time dependent parts.

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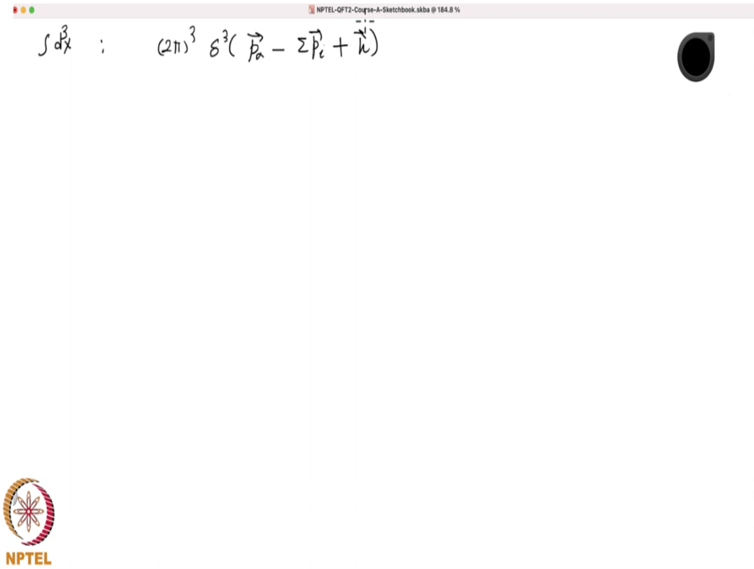

So, I will get; let me write down, e to the; so, I will get from the d cube x integral, will give me a 2 pi cube. We can go back to black. So, integral d cube x, that is going to give me 2 pi cube times delta cube of p alpha minus p i - k. Let us check; plus k; how come? Plus k, yes. Something is wrong; plus k.

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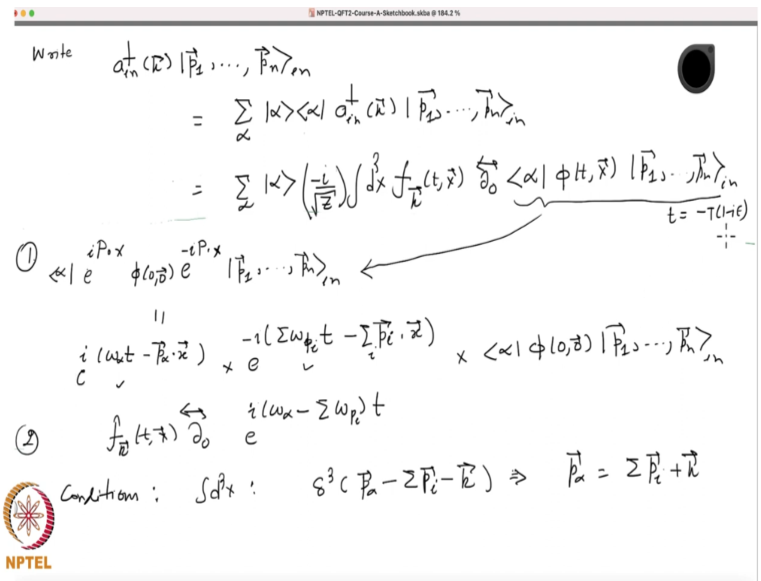

So, we will repeat the steps that we have earlier done. So, I will pull out the space time dependence from phi and then everything will be easy. So, we will use the result that alpha phi t of x, I can write as e to the i p dot x. So, it is 4 dot product of 4 vectors and then phi at 0 and then e to the; by the 0, let me make it more explicit. And then you have e to the -i p dot x p mu dot, p mu x mu, that is what I mean. So, this is this factor.

Now, this will be e to the i omega alpha t minus p alpha dot x and this will give you e to the -i summation over omega p i times t minus times; let us see whether everything is fine. Looks all right. This looks fine. So, that is some space and time dependence in here. And then this derivative has to act. This del nought will act on this time dependent part and also on f of k. So, you will get; so, that is one point and then the other point is f of k, this time derivative acting on this piece, e to the i omega alpha minus summation over omega p i t.

So, this, I just took the time dependent parts from here; and then, this derivative will act on these 2 pieces So, fine, now you will get the following 2 simple conditions. So, you evaluate this; do the calculation, it is easy; you have done this before already. And then, that will give you a space dependence which will be e to the i k dot x from here; and then you have simple space dependences coming from here and here, and you integrate over d cube x.

So, integrating over d cube x, we will give you the following constraint that you will get delta cube p alpha minus summation over p i - k, which means that you get the constraint that p alpha should be equal to sum over all pis plus k; that is the constraint that you will get, just similar to what we had encountered earlier in, when you were looking at a in dagger acting on vacuum. And the second constraint will come when we take t going to -t times 1 - i epsilon and here is that constraint.

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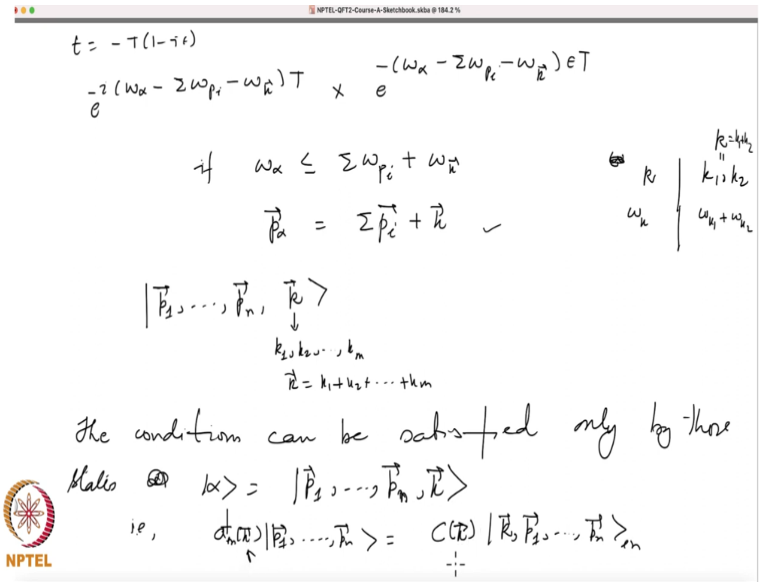

So, when you put t equal to -T times 1 - i epsilon, you get e to the -i omega alpha minus summation omega p i minus omega k t times e to the minus omega alpha minus summation omega p i minus omega k epsilon T, where epsilon is positive and fixed small number, and T we are going to take to infinity. So, you get a damping from this. So, and these are summation over alpha of course, outside, sitting outside coming from here, this one.

So, you see that if omega alpha is less than or equal to summation of omega pis and omega k, then this will contribute, otherwise that gives you a damping, a factor that goes to 0 in the t going to infinity limit; but if omega alpha is lower than this sum, then you get a non-vanishing contribution. So, that is why we are saying that these such states alpha will contribute for which this is satisfied.

And another condition I had already written which is p alpha is equal to summation over p i that plus k, that came from the delta function. So, all those states alpha which satisfy these 2 constraints will contribute to the sum, to this sum over alpha, others will not. So, that is the conclusion. Now, you see the p alpha, for the state p alpha which has momentum; so, let us look at this p 1 to p n; so, you take this state.

And then you have other label k, but this k can be split into many parts. So, let us say k 1 is really, k is really not 1 label but many labels; k m; where all these together make k. So, k is k 1 plus k 2 plus so and so forth k m. Now if I take such a state instead, then it will have a momentum p 1 plus p 2 plus p n plus k 1 plus k 2 plus k m, where this k 1 plus k m makes k. So, this condition is satisfied but how about this condition?

Now, if the energy of this state has to satisfy this constraint, then there is only 1 possibility that your k is not; this is not 1 label; I am saying suppose you have many labels; instead of this, you have p 1 to p n, then k 1 to k m; then this k 1 to k m, you should not have m of them but only one of them, only one, meaning you should have only 1 label k. If that is the case, then this condition will be satisfied; both will be satisfied simultaneously, not otherwise.

And you know why, because, if you have; we saw this earlier that if you; let us say you have omega, suppose you have momentum k and there is k 1 and k 2 which make k, so, k = k 1 plus k 2. So, I am looking at in this case a single particle state and in that case 2 particles of momentum k 1 and k 2 such that the sum of the momentum is same as k. So, in this case, the energy omega k, that is what you will get; and here, the energy will be omega k 1 plus omega k 2.

And you know that omega k 1 plus omega k 2 will be greater than omega k unless this is not a 2 particle state. If it is a single particle state, then of course they are equal, but otherwise, sum of energies will always be higher compared to a single particle state. So, you see this constraint that omega alpha should be less than equal to this omega p i; that is anyway fixed; there is nothing can do; plus omega k together with this constraint that the momentum conservation.

This can be satisfied only if you do not have so many labels k 1 to k m but only 1 label corresponding to a single particle k, and then the condition will be satisfied by the equality. So, this can be satisfied; the conditions can be satisfied only by those states ket alpha which is; so, there is only 1 solution or unique solution which is this, this k is 1 label, not many labels. So, only this one contributes.

That is, I have shown that a in dagger k acting on p 1 to p n, this ket is equal to k; I can put this label first; it does not matter; but there is a provision for proportionality constant; so, it does not have to be equal to this, but it could be some number times this or some function times this. And that function cannot depend on how many ps are here and it cannot depend on these values of ps; it can only possibly depend on k, k which is here.

So, that is the constant. So, now we are there; we have almost achieved our goal except for fixing this c k; and that is easy to fix because it is independent of how many momenta you have, what is the value of n.

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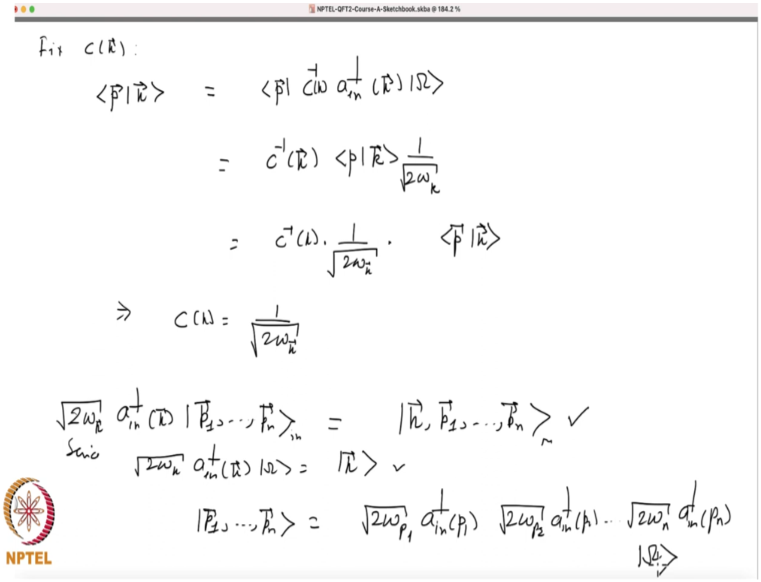

So, I can just take a single particle state and then fix this constant. So, let us check c of k. So, how do I do that? So, let us take a single particle state and its overlapped with another single particle state. Now this is, how do I create k? I create a k by a in dagger k acting on vacuum with some coefficient here. So, I am using this formula actually. And you see, you will have to multiply with c k inverse.

Now, this is same as; I will pull out c inverse k and you have bra p; then you have a in dagger k acting on omega and that we know what it is. Here somewhere; a in dagger acting on omega is 1 over root 2 omega p times p ket p. So, I get k times 1 over 2 omega p or k does not matter, because of the delta function. So, what should I write? k. So, this is c inverse k times 1 over 2 omega k times; now, both left-hand side and right-hand side of this factor which means this should be unity and which implies that c of k is 1 over 2 omega k.

So, what is the final result? Final result is a in dagger k acting on a state with n labels p 1 to p n gives you; this thing gives you k. So, it inserts a label k in your original state. So, that is a nice result, and already we know that a in dagger acting on vacuum gives you a single particle state. We conclude that since 2 omega k or root 2 omega k times a in dagger k acting on vacuum gives you this state.

And then we have shown this is true. This implies that any state p 1 to p n, this state with labels p 1 to p n can be generated by repeated application of these a in daggers acting on vacuum. So, this is nice and this is an important result. Now we know how to create single particle states or in states starting with this operator a in dagger acting on vacuum and we also know how to kill states or remove states by acting with a in. So, we will continue with the discussion in the next video and we will slowly build up towards scattering.