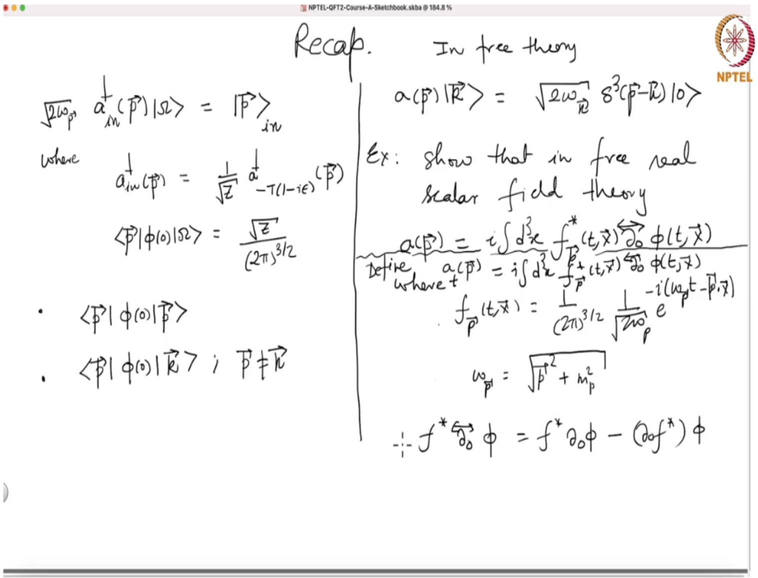
Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
Prof. Anurag Tripathi
Department of Physics
Indian Institute of Technology - Hyderabad

Module - 3
Lecture - 6
Annihilating Single Particle States

(Refer Slide Time: 00:15)



So, let us recap what we have done so far. So, we saw last time that we can create particles in interacting quantum field theory by operating with this operator on the vacuum. You can get some normalisation here and that gives us single particle state p. And these were in states; it does not matter so much for single particle states, but you can keep this if you wish; and where a in dagger p is equal to 1 over square root of z.

We will talk more about this factor z later, but for now, I will not say much. So, that was the definition of a in dagger. And just to remind you, this quantity is what we called as square root of z over 2 pi 3 halves. So, one more thing we had argued that this subject is a constant, so, you can go back and see how we had argued this thing and then you can ask some simple questions similar to this one.

So, you could make some questions for yourself. You could ask what you can say about this quantity. So, instead of omega here, meaning the vacuum here, suppose I take both the bra and the ket to be this single particle state, then what can you say about this matrix elements making a similar analysis as we did for this one; and you could also ask for this object. Just for fun you should do this because you could run similar arguments and try to arrive at some conclusions.

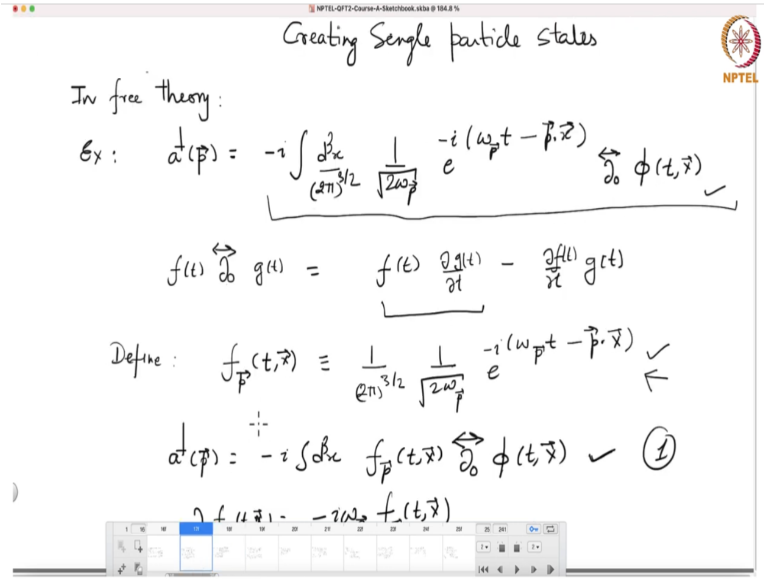
So, after this recap, let us start something new. So, recall in free field theory, not only that we had an operator a dagger which acting on vacuum created single particle states, we also had an operator a, a of k with some momentum here, with which if you acted on a single particle state of momentum k; so, here; I want to write p and this one I will keep k. So, if you take a single particle state with momentum k, act with an operator a p and I am right now in free theory, then you recall that that would kill that particle and give you vacuum.

So, recall that these are annihilation operators, so, they remove particles. So, if you recall, you will get delta cube p - k meaning it will kill the particle only if the momentum p is same as momentum. So, it will kill a single particle state and it will leave behind the vacuum. And this is the coefficient which hits only when p = k and there was this normalisation factor. So, just a second; now we would like to have such an operator in our interacting theory also.

So, we have learnt how to create particles; we know how to create particles out of vacuum, but now we also want to kill the particles just like we did in free theory. So, let us repeat what we did for the case of a in dagger. So, we take our inspiration from free theory and then we have mimicked the steps and then we will eventually arrive at an operator a in; that is the goal which will kill the particles. So, here is a simple exercise.

You should show that in free real scalar field theory, a p is given by this expression; del nought; so, this is a time derivative which acts both ways, the way I had defined earlier in the previous or previous to previous lecture, and here you have a complex conjugation. You can show this explicitly or you can also convince yourself that this is; just a second; correct because; where is it? Here; no, not this one; this one.

(Refer Slide Time: 07:34)

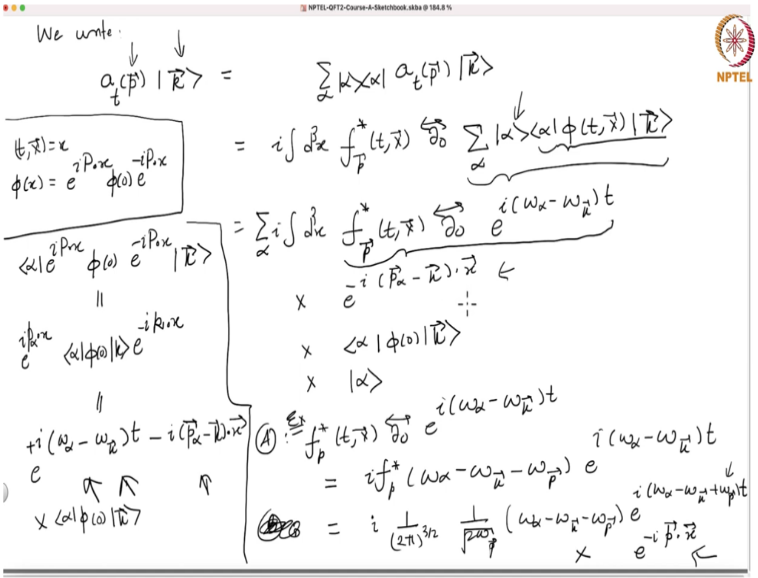


So, you see, you have in free theory a dagger of p is given by this expression. So, if I take a dagger on both sides, that will give you a of p; that -i will become +i, because you have to do a complex conjugation; f p will become f p star; and of course, that has not changed, this del nought; and phi dagger is phi, is because it is a real scalar field. So, remember phi is an operator, so, phi dagger, when you put a dagger on both sides, you will get a phi dagger, but because it is a real scalar field, you will get phi dagger equal to phi.

So, the only change will be this sign will change, it will become +i and there will be a complex conjugation here; and that is what I am claiming that this expression you are going to get; but nevertheless you can try doing it explicitly if you wish; and where, let me write this expression; 2 pi 3 halves 1 over 2 omega p and then e to the -i omega p t - p dot x. And as always, omega p is p square plus m p square; that is the physical mass.

And this f star del nought phi is f star del nought phi minus del nought of f star times phi. Now we are going to make use of this. So, again we will proceed in a fashion similar to that for a dagger and we will write;

(Refer Slide Time: 10:15)



I think I should have; I jumped a little bit. So, you show that this is true in free theory and then we will define also; how shall I do it? So, sorry, this is not going to be very nice looking. So, now we will define a p with a subscript t. You remember that when we went to interacting theory, the same object, the a dagger in that case which was similar here except for this star and sign of i; that object depended on time explicitly.

It was not time independent as it was for free theory. So, the same thing is going to happen here as well. In this case, you will find that if you look at this expression i d cube x f p star t x and this double sided time derivative operator acting on phi, then in this, in interacting theory, this will turn out to be time dependent. That is why I have put a subscript t on a where all these f p and omega p and these things are the same.

So, that is something you can convince yourself based on what we had done in previous case, and now we proceed in the similar fashion and take the following. So, I will take a p with subscript t and act this with this on a single particle state of momentum k where k is really k in but it does not matter; k in is same as k out as far as single particle states are concerned. Now, this, again I will insert a complete set of bases.

These are basically in states which I am going to insert. So, this is same as left-hand side, and now I am inserting a complete set of bases. These are all in states which includes ground and single particle states and multiparticle states. So, what happens to this object? This I can write as; so, now I will take this expression of a t p from here and I will put in here. So, I get i times integral d cube x f p t x star, this double sided time derivative and this. Good.

So, again as before, I will use the following that phi of x, where this x stands for this. When I write x without an arrow symbol, it means full vector. This is really this. So, phi of x is equal to e to the i p dot x, where p is momentum operator; phi 0 e to the -i p dot x. So, using that I can write i integral d cube x f p star. What should I write? No, I should not be writing; it is fine. So, let us look at this object, this. What is that?

It will become e to the i p dot x phi 0 e to the -i p dot x ket k. Earlier, in the case of a dagger, we had vacuum instead of a single particle state here. That is the only difference we see at this stage. So, what is this object? This is e to the -i. This acting on ket k will give you momentum k dot x. And this will give you e to the i p alpha dot x. These are complex numbers and you get alpha phi 0 k, which is same as e to the -i; I will keep a +i; e to the i.

So, p alpha, it will give you omega alpha; that is the energy of the state alpha minus this one that is omega of k; that is the energy of this single particle state of momentum k; t - i p alpha minus k dot x. So, now if I substitute this, I get a t acting on k is equal to summation over alpha i d cube x f p star t x del nought acting both sides, that is here. And now, here is the time dependent part of this quantity.

For this quantity, the entire time dependence is contained this exponential factor here. So, that is what I am writing here, e to the i omega alpha minus omega k t. Then I will write this part. Then I should write; I have missed writing this factor here. Then I have this factor alpha phi 0 ket k and then finally this ket alpha; that is all. So, now we should evaluate this and it will be quite easy to show that this quantity.

Let us look at this quantity that this f p star del nought and this exponential, it will give you; let me write it down what it will give; f p star t x that is an exercise you should do, e to the i omega alpha minus omega k t, and it is simple to; there is nothing special, nothing difficult in this. So, you will be able to arrive at the following result. So, f p star omega alpha minus omega k minus omega p e to the i omega alpha minus omega k t.

Let me remind you again, omega p is the energy of; omega p is p square plus m p square where p is coming from the argument of this a; omega k is the energy of the state k, ket k; and omega alpha is the energy of state ket alpha over which we are summing. So, that is something we should be able to show. And then, see, so, that is about the time dependence. The entire time dependence is in here now, in this expression; f p also, f also depends on time.

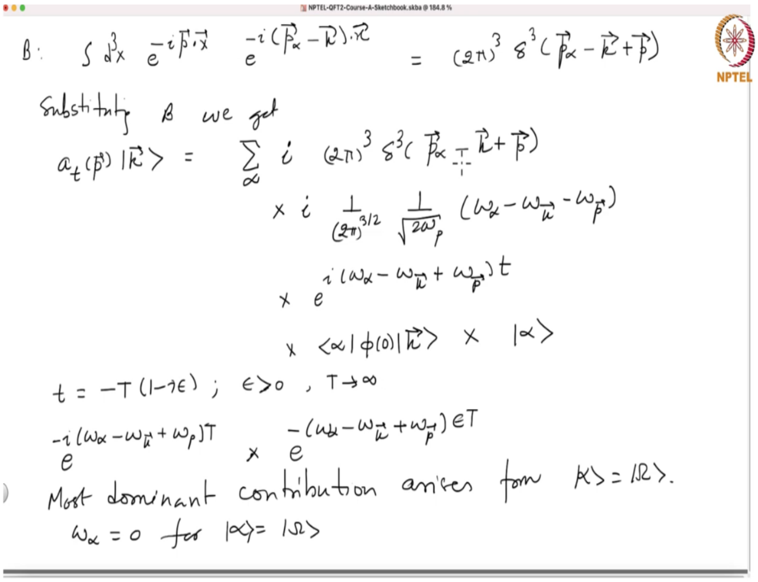
Now, let us look at the x dependence. The special dependence is quite simple. You have an exponential function here, so, that contributes to space dependence. And then, other space dependence comes from this f p star, but that space dependence is also trivial. It is again an exponential. So, let me show you here; it is just e to the +i p dot x. So, that exponential and this exponential here will combine and you can integrate over this d cube x and that will give you a Dirac delta function.

So, that is what I want to collect. Just a second; I want to add one more step before that, so, it will be easier. So, here, this is i over; so, I am just writing the expression of f p now; i; so that I can make the space dependence explicit here; omega alpha minus omega k minus omega p, then this exponential, and so there will be a time dependence coming from f of p star which will give you e to the i omega alpha minus omega k plus omega p t.

It is e to the i omega p t. Why? Because, here f p has e to the -i omega p t. So, if you take f star, it will become e to the i, +i omega p t. And that is why I have written e to the +i omega p t. And then you still have the space dependence, which is e to the -i p dot x. Let us go back. Here you have e to the +i p dot x because -i times minus is +i. And when you take a complex conjugate, it becomes e to the -i p dot x.

And now I can do an integral over x. So, this is one place where you have x and that is the other place. So, let us integrate over it. So, if I combine these 2 exponentials, I will get p e to the -i p alpha minus k plus p because there is a -i here, I will be pulling out -i. So, it will be p alpha minus k plus p, and when I integrate this, I will get a delta function.

(Refer Slide Time: 23:08)



So, let me write the delta function, integral d cube x e to the -i p dot x times e to the -i p alpha minus k dot x. This will give you 2 pi cube. It is an integral over 3 volume, so, it will give you a 2 pi cube times delta cube p alpha minus k plus p. Good. So, I have taken care of the x integral and now I will just put this result and write down the expression of a t acting on k. Let us see what do we get.

So, substituting B, we get a t p acting on ket k is equal to summation over alpha i times 2 pi cube delta cube p alpha minus k plus p. Remember, delta functions are the easiest things to integrate over. They make everything very easy. And exponentials are also very easy to integrate over, because they give you Dirac delta functions. So, check that you are going to get this. Let us see.

So, till now, I am just; so, you have; I have taken care of delta x dependence that gives me a delta function and now I am writing the remaining factors. And here is the time dependence. So, as in the previous case, we again have a sum over all possible in states which form the basis. Now look at this exponential. We will do the same trick as before. We will take t equal to minus capital T times 1 - i epsilon, where epsilon is positive, a small number but positive, and I will take T going to infinity.

So, that is what I am going to do. So, what happens to this exponential? This exponential becomes e to the i omega alpha minus omega k plus omega p; and because of this minus sign, you get a minus here; times e to the; so, this is minus times minus; that is a plus; so, it is a i epsilon t. Now, i epsilon t times this i will give you -1, minus capital T, minus epsilon t times this factor. That is correct.

So, this is what you are going to get. So, you have T going to infinity; epsilon is positive; there is a minus sign here. Now, this object, this exponential, the most dominant term in this sum, sum over alpha will arise from that alpha for which omega alpha takes the least value, because, if omega alpha takes large values because omega alpha is always positive and if it takes large values, then this becomes even larger positive number.

So, the damping becomes even larger. So, if omega alpha is large, the damping is large. So, those omega alpha or those alpha which have larger energies will contribute even lesser or rather they will damp out even faster in T going to infinity limit. So, which one will give you the most dominant contribution? The one which has the least energy; and the state that has least energy is the vacuum; it has energy equal to 0.

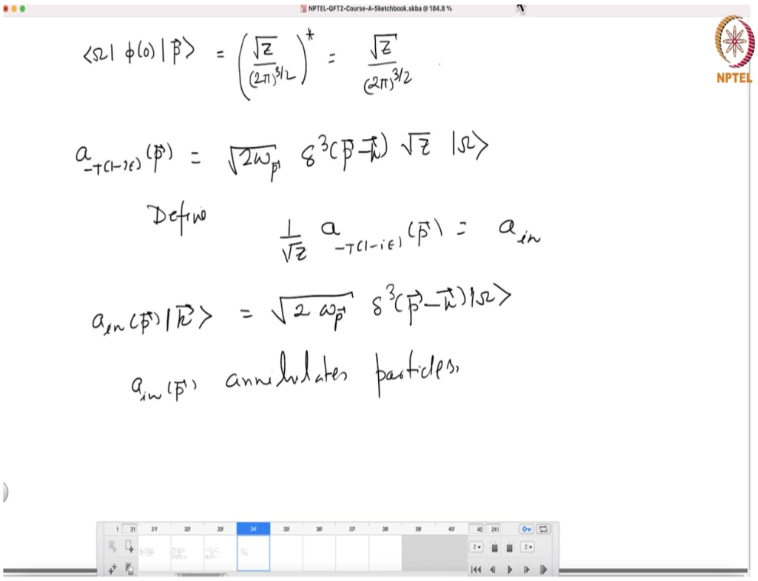
So, the most dominant contribution to this sum will arise from vacuum. So, most dominant contribution arises from, equal to vacuum; why because omega alpha is equal to 0 for the vacuum state. So, all others will decay faster when you take T to be large. So, we should focus then only on the vacuum and not worry about other terms because they are going to be irrelevant in T going to infinity limit, because this is the most dominant one.

So, let us work with the most dominant one. So, I am not going to sum over all other states; I am just going to look at alpha equal to omega; but if alpha is equal to omega, then p alpha is also 0, the momentum is 0 for the vacuum state. So, you get delta cube p - k. And here you get, omega alpha is 0, so you get minus omega k minus omega p, but delta cube p - k will force k and p to be equal, which means that this will contribute -2 omega p.

This is 0; these 2 will together contribute -2 omega p because of; that you can do because you have a delta function, because delta function will hit only when k = p. So, you can identify k = p here. And here, the omega alpha is 0; omega p; and minus omega k, you can replace by omega p, again because for the same reason, you have a delta function. So, this becomes e to the 0 because this thing in these round brackets vanishes.

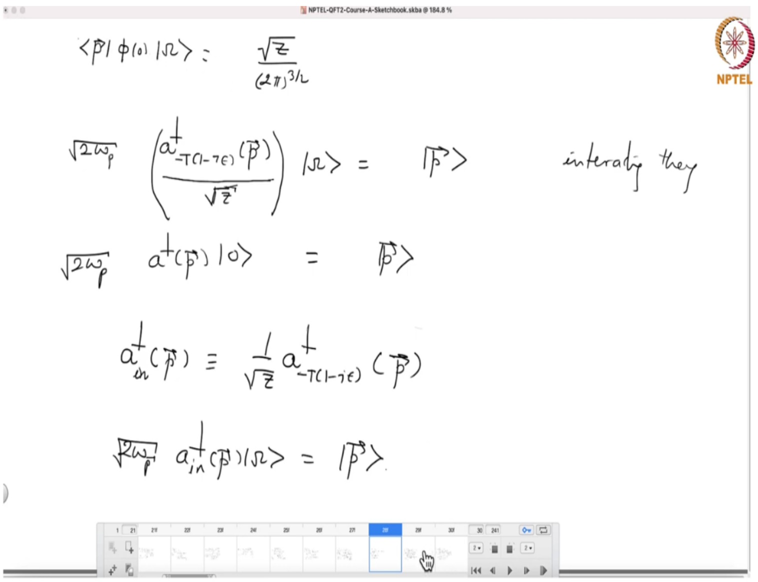
This is ensured by delta function. So, omega p minus omega p is 0, so, e to the 0 that contributes a factor of 1. This gives you a factor of -2 omega p that is delta cube of p - k. And something else; and this is of course vacuum; and this k will also be forced to be p because of this delta function. So, if I put all these things together, I will get; let me write it down; another thing that I have to use is; so, this is what I will use.

(Refer Slide Time: 32:02)



And the other thing that I will use is the following: omega phi 0 p; that is what is in here; this thing is from root z over 2 pi 3 halves, complex conjugate of it. Earlier we had shown I think; let us see; where is the z? Let us look at z.

(Refer Slide Time: 32:54)



So, here, the vacuum was on the right and bra p was on the left; so, vacuum on the right. Let us see what do we have now. Now, vacuum is on the left. So, this is a complex conjugate of the previous thing. So, that is why I have put complex conjugate, but we will see later that z is real. So, I can write simply this thing. So, now if I take this and substitute in here together with all the other things that I have mentioned, I get the following.

Another thing let us see here. This factor gives you a factor of 1 over 2 pi 3 halves. So, 1 2 pi 3 halves coming from there, from this factor. Then you have 1 over 2 pi 3 halves coming from here. So, these 2 together will make 1 over 2 pi cube and that will cancel this 2 pi cube. So, there will be no factors of 2 pi cube. So, no factors of 2 pi cube. This i times i will give you -1; i square is -1, but then you have a minus sign coming from here; because this is -2, this will give you -2 omega p.

So, that minus sign gets cancelled, so, there is no minus sign and there is no factor of 2 pi, 2 pi cubes. So, those 2 are gone and you have a factor of 1 over 2 omega p here and a factor of 2 omega p here. This is square root. So, that will give you a factor of square root of 2 omega p. So, now I can write the final result, a of p with; so, no factors of i or 2 pi and straight forward way.

So, the only things are left is 2 omega p; I just explained why you will get square root of 2 omega p; and there will be a delta function, and then of course square root of z coming from here, and then your vacuum. Now we will define 1 over square root of z; I am taking this square root of z to the left; times this operator as a in. And with this, we get a in; so, I am taking z to the left; and using this definition a in of p acting on a single particle state ket k, this gives me 2 omega p times delta cube p - k vacuum.

So, you see, we have found an operator that kills a single particle state. So, that kills particles and creates vacuum; and this is same as what you have found in the case of free field theory. So, a in p annihilates particles. We will continue our discussion in the next video and see how to create states with more than one particle and also given a state with more than one particle, how to remove those particles using a in operators. So, see you in the next video.