Introduction to Quantum Field Theory (Theory of Scalar Fields) - Part 4
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Lecture No # 05
Module No # 01
Creating Single Particle State – 2

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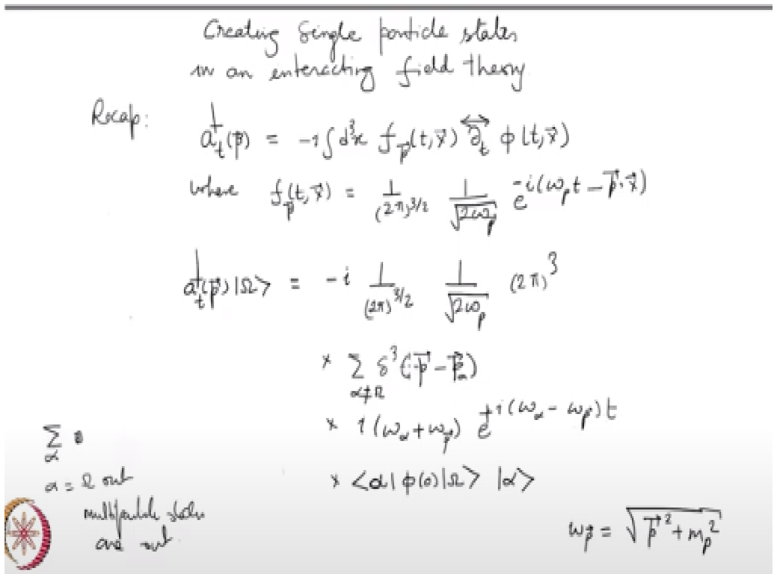

So we have been trying to create single particle states in an interacting quantum field theory to be more specific in real scalar field theory. But whatever we do here can be applied to fields of other kind as well so actually we are very close to achieving our goal. So let me quickly recap what we have done so far? So we have been trying to just take the same steps that we take in free field theory and see that if we arrive somewhere.

And in that direction we had defined this subject a dagger p with a subscript t because this object depended on time explicitly that is something I had asked you to check no I think I have shown you. This f and then you have this time derivative operator which acts both the ways which are defined last time; where f is defined as following some factors.

Remember this omega p when you write it is it will be is omega p square = p square + m p square where m p is the physical mass not the mass parameter in the theory but the physical mass of the particles which you will get in this theory. This was the definition of this and we had found explicitly that if you take this operator and act on the vacuum of the interacting theory you create a linear sum of all possible states.

And let me write down all these factors for easy reference it was a 2 pi cube and then we had sum over all the states single particle states, vacuum everything else. So this summation over alpha and you had this delta cube and then there were some other factors omega p t times alpha. Or maybe I will change the order of this compared to last time I had alpha here then field phi at 0 and then this omega.

So this matrix element of phi in these 2 states and then you had keto alpha. So apart from all these other things you see that you have to sum over keto alpha again I had also argued that vacuum does not contribute. So out of this alpha you can drop the vacuum because we argued that this delta function does not hit for vacuum because p of the momentum of vacuum is 0 and since p is arbitrary that delta function does not contribute.

So from this keto alpha get omega drops out so you are left with some over states which are single particle states and multi particle states. So here let me write not equal to omega not a great notation but this is what it is?

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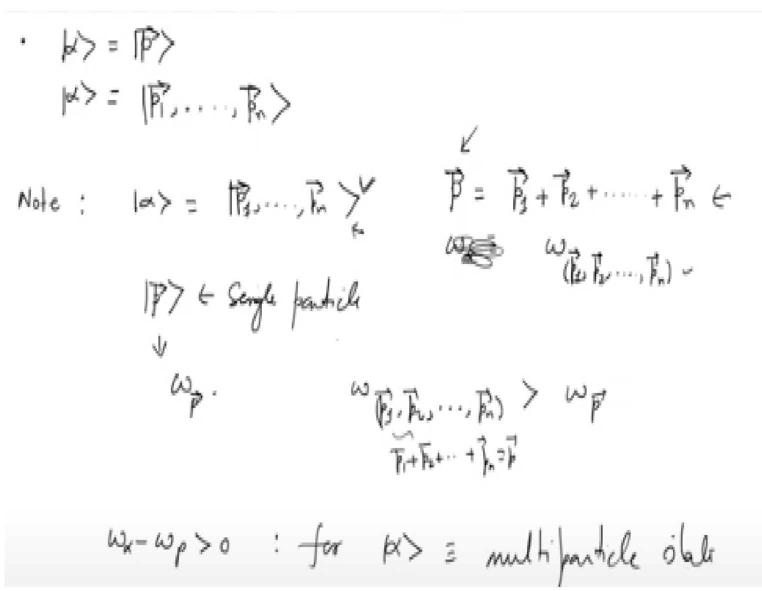

So good now we have to look at let me remove these also how do I remove and this also. So recap is over now and vacuum is also out of the equation now we have to look at the contribution of single particle states and multi-particle states. Where when I am saying multi-particle states I mean states if you are thinking in terms of stranger picture then states at time t = – infinity.

Where we have assumed that; these states represent particles localized in space far apart from each other so now I will make a note here which will be useful. So let us start with multiple particles multi-particle states then I will come back to single particle state. So let us first deal with multi-particle states. So suppose I am looking at this state then the total momentum or the sorry the momentum of this state.

And what is the momentum is the generator of the translation so if you look at the generator of translations in your interactive on the field theory and take that operator and act on the state you will find that this is an Eigen state this is how we had constructed this. And the momentum p will be equal to this we have been doing for quite some time so hold on. So that is the momentum of the state and it will have some energy which I will call as omega alpha.

I will leave it as alpha just but alpha means this not the best notation but Uranus what I mean so energy will be omega alpha where alpha is a state. Now you might already be aware of but even if not there is no worry that energy of a multi-particle state with total momentum p which is this will always be larger than the energy of a single particle state with the same momentum. So if you had if you had another state which is this, which is a single particle state and let us say its corresponding energy is omega p and this one let me write slightly so by this side.

So let us say for this multi-particle state the energy is omega a temporary notation just to help me in telling what I am saying. So by putting all this it is clear that I am not talking about even better it will be I think this is still confusing let me write this we this way. So even though the energy of this state sorry the momentum of this state is p which is sum of these. And for the single particle state I am again I am taking on the momentum to be the same as here.

So for both the states for this multi-particle state and the single particle state both of them the momentum is p nevertheless the energy of the multi particle state which is this one will be higher than omega p. Let us claim I am making it might be intuitively clear even if not then I will show you even if it is I will show you not even if not. This where so; this I will show you but for now I will utilize this result to do something very nice.

So the point is energy of a multi particle state and a single particle state the difference is positive for same momentum total momentum of these 2 steps. So how can I utilize that well let us see here now you look at this exponential and we have played such games before also long before in the previous course. So here you have this exponential so as far as the states which are multi-particle states are concerned it is clear that omega alpha is greater than omega p.

Because you are given some p and omega p is what omega p is right that is the energy of a single particle set so that is omega p corresponds to energy of a single particle state with momentum p. Now you have omega alpha here and I am right now looking only at multi-particle states then omega alpha will always be greater than omega p so this object omega alpha - omega p is positive that is what I want to use.

Now I will show it later but that is what I want to use. Now omega alpha - omega p is greater than 0 for this a multi particle state. So let us see how to utilize this but before I can do that I have to make one correction so there is a mistake somewhere here this sign is not correct it should be plus so make this correction.

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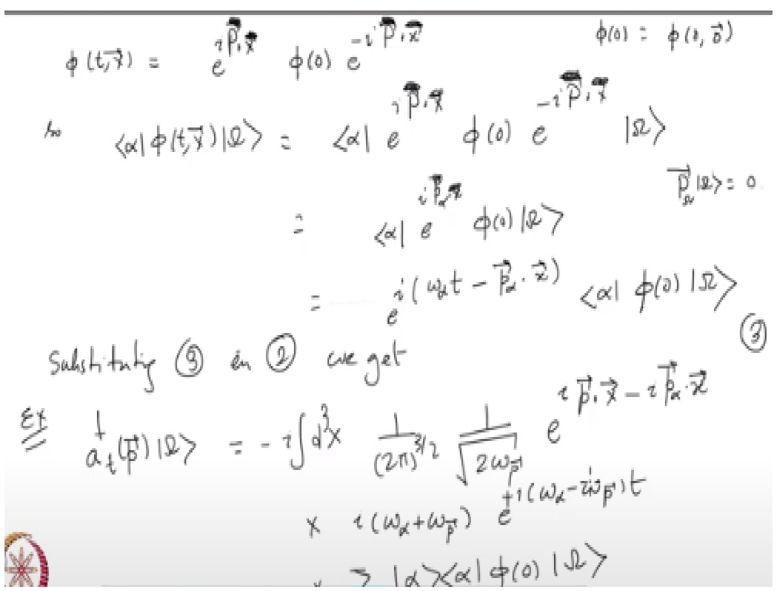

Let us see whether you can find out where it was four and this is you see it had a e to the i omega alpha t there was a plus sign here. So omega alpha came with the + i so it should be coming with a + i not with a - i. So that that was a mistake and that was creating trouble for me now this should be fine + i n. Good so this being corrected so we have omega alpha - omega p very good.

So here we have this term now I want to get rid of all the multi-particle states and actually I can do so by playing with this t so I will choose a t such that this exponential factor provides the damping and it kills the entire contribution. So for that what I will do is I will take t = - t where my t is a T is a large time so it is a large negative time, 1 - i epsilon. Where epsilon is a positive number small and a positive number but a fixed one you choose whatever you like but keep it fixed.

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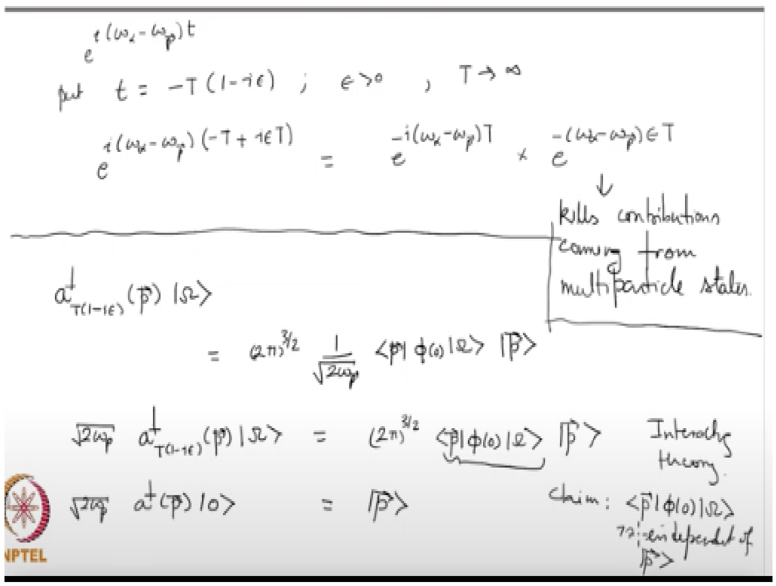

So if you do so then what happens let us look at the exponential e to the, i omega alpha - omega p t. And now put t = - T 1 - i epsilon where epsilon is greater than 0 and t i will eventually take to infinity. So if i do this then i get e to the, i omega alpha - omega p times - T + i epsilon t. Which is same as e to the - i omega alpha - omega p T times i from here and i from there will give you - 1 i square is -1 so - omega alpha - omega p epsilon T.

Now understand why i made this choice for this t if I choose some if i choose a + i epsilon then it will not help because then I will get a plus sign here and you will see in a moment why i need a minus sign here. So epsilon is fixed some small number but fixed and now I am going to take t going to infinity. Now if T goes to infinity this provides a damping factor and gives you 0 and kills the entire contribution.

Why? Because epsilon is positive T is positive omega alpha - p is also positive and that is crucial so this entire thing here is positive except for the minus sign. So it is an exponential of a large negative number and which is a very small number so in the limit T goes to infinity this exponential goes to 0. So this damps out the contributions coming from multi-particle states or rather kills. This is very nice because now only vacuum is out multiple states are out and only single particle states are left.

So let us go back here so in this sigma alpha now I have killed the vacuum this is out and all multi particle states are out. What is left is only single particle states so let us look at them so for a single particle state omega alpha will be equal to omega p. Why because alpha p alpha will be forced to p by this delta function I mean that was true earlier also but earlier energy was energy of a multi-particle state was different from a single particle state actually it was larger than a single particle state.

But now because omega alpha refers to a single particle state with momentum p then the energy of omega alpha will be same as omega p because the momentum of alpha is same as p. So this exponential provides you a factor of 1 because e to the 0 is 1 so there is no damping and you get contribution only from the single particle states if you take t to be - t 1 - i epsilon. So, good this is nice moment because we now know how to create single particle states.

So let me write this down this nice thing, so what I have shown you now is that if you take this operator a dagger of p with a dagger t 1 - epsilon p and act it or hit it on the vacuum then I get the following. What do I get? I get I will just write down the answer I will I going to cancel out few factors of omega p that you can also verify it is so I am not going to do the algebra it is trivial.

So you should get this, times 1 over 2 omega p and the square roots and then you have p phi 0 omega times this single particle state. So everything else is some number these are numbers complex numbers but then you have a I mean in principle complex numbers but then you have a single particle state here. So the goal has been achieved I will write it in a more familiar form so I will write it as I will take this root 2 omega p to the left hand side.

So 2 omega p in the square roots times a dagger of t 1 - i epsilon with an argument p acting on omega gives you gives you 2 pi 3 halves times this vector times p. This is for your interacting theory interacting scale of field theory and real scalar field theory. And now let us compare this with what you had in the case of free theory in free theory we had the following so remember a dagger in that case did not depend on time at all. So there is no subscript here and when it hits the vacuum you get a single particle state.

So it is exactly the same thing except for these factors. Good so now I make a claim that this factor actually does not depend on p at all so I claim that this is independent of p. No matter what the momentum you what is the momentum of the single particle state that that object remains the same so if that is a constant independent of p means it is a constant.

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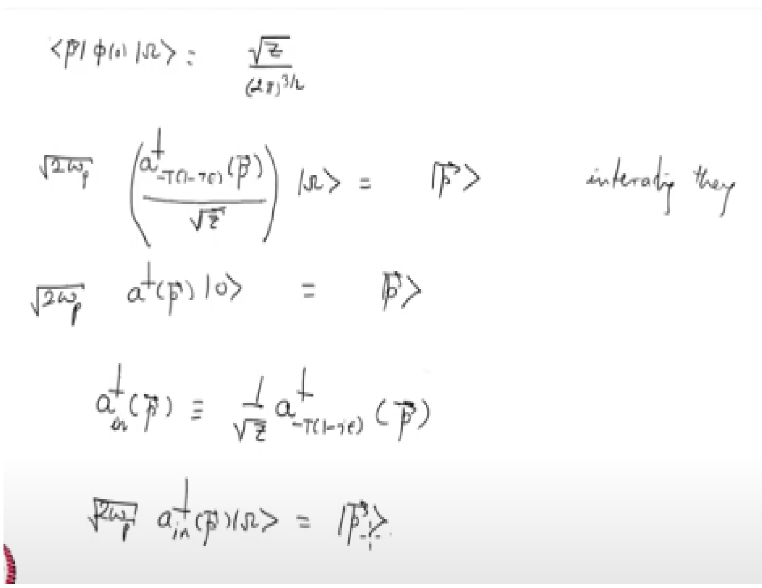

So I will call it the following so it is a constant and I define that constant to be this root z. And also divided by 2 pi 3 halves it is a constant so I am defining that constant to be equal to this I am calling root z also divided by 2 pi 3 halves and the reason i put 2 pi 3 halves is this vector. Because now I have so this is equal to root z divided by 2 pi 3 halves so this 2 pi 3 halves will cancel this 2 pi 3 half and you will be left with root z times this single particle state.

And if I take that root z to the left hand side and combine with a dagger so you will have a dagger divided by root z. And then you have exactly this form as in the case of single particle state. So let me write it down so you will get 2 omega p a dagger t 1 - t did i put a t no - t - t1 -

epsilon p and then over root z. Where; z is a constant we will talk in a little bit more details about z later times.

What was here vacuum this is equal to a single particle state so this is for your interacting theory let me write the equivalent thing for free theory excellent. So what we have basically done is we have see here when I wrote these summation over alpha all, these states keto alpha. This keto alpha were assumed to be the basis states which were in states you can choose out states or in states so here they were assumed to be in states so all this keto alpha they were in states.

So what we have done here is we have created a single particle in state and how did we achieve that here how did we achieve that by operating with this operator? So I will give it a name I will call it a n dagger p and what is that that is just a dagger of p with - T 1 - i epsilon not nice divided by root z that is this. So we have learned that we can create single particle states by acting with a in dagger on the vacuum of interacting theory and get these single particle states so our goal is achieved.

And now we are in business but I should provide you with 2 proofs that are still pending that energy of a multi particle state is higher than the energy of a single particle state if both the states carry the same momentum. And also that this object is indeed a is a constant it is independent of p so let us do that now.

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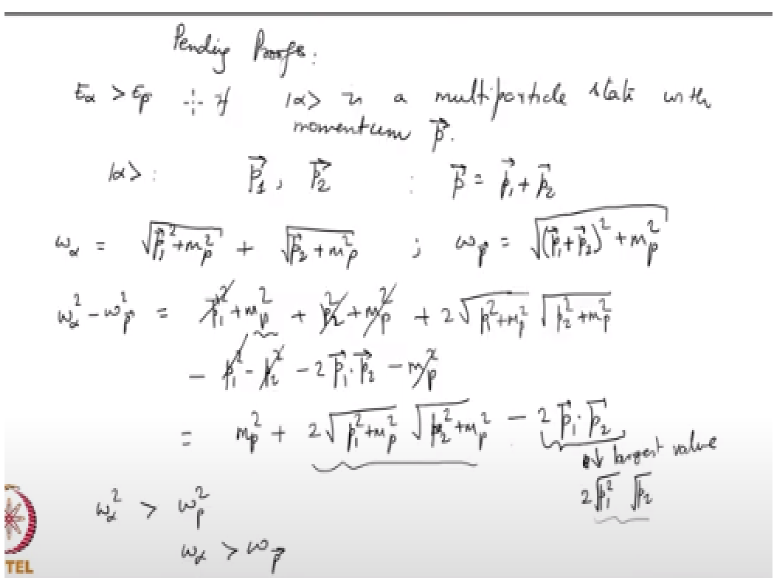

So first is that e alpha is greater than e p if keto alpha is a multi-particle state with momentum p. So I will show this to you for 2 particle state and from that you can you will be able to convince yourself that this is true in general. So let us take keto alpha and I am saying it is 2 particle state so one of them has momentum p 1 and another particle has momentum p 2 and the sum of the 2 is what is p?

And the particles have masses m p physical masses. So what is energy of the state alpha omega alpha is equal to so remember these particles the energy and momentum of these states are same as that of 2 particles which are non-interacting with each other. So there is no interaction energy and only kinetic energy associated with their momenta. So it will be for particle carry momentum p1 it will be this and for particle carrying momentum p 2 it will be p 2 square + m p square and I should add to get the total energy.

And the single particle state with momentum p will have momentum p square + m p square root but p is same as p 1 + p 2 right. Because both the states single particle and multi-particle states have the same value I have the same momenta and p is this so I will write it in this form this p is for physical this is not the momentum it is subscript.

So since both omega alpha and omega p are positive numbers I will just look at this this difference rather than looking at omega alpha - omega p it will be slightly easier. So what is this omega alpha square is this square + that square + 2 times - square of this. And square of this will be this square + d square so this is p 1 square – p 2 square - 2 p 1 dot p 2 – m p square. So this says this object m p square + 2 - 2 p 1 dot p2 good.

So let us see so this is positive anyway the only thing is we have to worry about this difference. So here the largest value this takes is 2 p 1 square in the square root or magnitude of p 1 times magnitude of p 2. Where this is not equal to this is the largest value it can take because this is a dot product and cos theta will be 1 when both are in the same direction and in that case you will get this.

So this is the maximum you can subtract out of these 2 terms and you say even then this term is larger than this because you have this m p square added. Which means that omega alpha square is greater than omega p square because this right hand side is positive. And since both are positive omega alpha and omega p we conclude that omega alpha is greater than omega p for 2 particle states.

And that is what we had utilized in here when we wanted to damp out all the multi particle state contributions and also note that here t is a large negative time so it is really in the far past. So this operator a dagger is creating I mean if you are looking in Schrodinger picture then it is creating single particle states in the far past so that one of the pending proof's is done.

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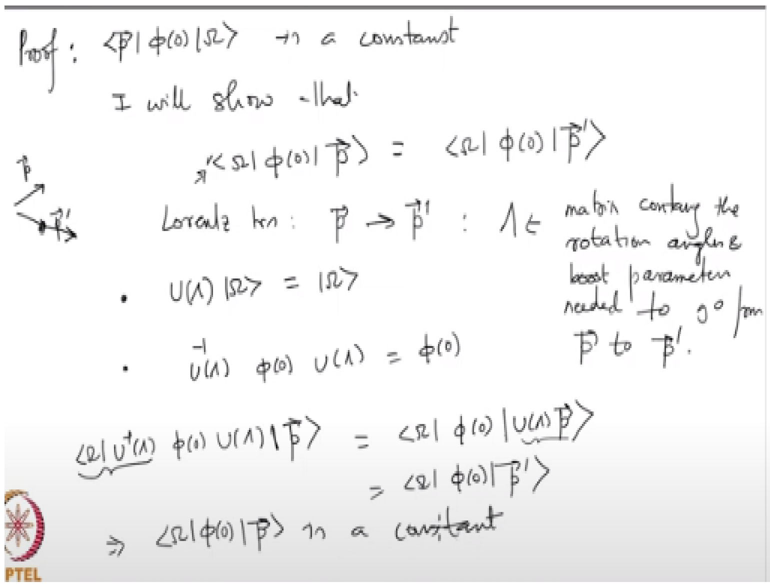

Let us go to the next pending proof also that one is also easy so I am going to prove this now. So I will show that if you take omega phi 0 p this object and that object they are just complex conjugates of each other because phi is real. So it does not matter whether I show this is a constant or that is a constant and whatever I am going to do you can do it for this absolutely no difference.

So I am going to show that this is equal to this some p some other p prime. Now given a single momentum p you can arrive at another momentum p prime by doing a Lorentz transformation. So suppose you have a momentum p like this and momentum p prime is in that direction. So suppose you were looking at a particle single particle state which has momentum p and that p is in that direction. And suppose you have another single particle state which has momentum p prime and p prime is in this direction.

So you can and suppose the magnitudes are the same then you can arrive from this one to this one to; p prime by doing a rotation. You just have to do a rotation about an excess perpendicular to this to the screen of your laptop or whatever screen it is. Else if let us say another possibility is that this p prime has a larger magnitude so in addition to a rotation you will also require a boost. So once you have done a rotation then you do a boost and by doing so you will be able to increase the velocity or equivalently the magnitude of the momentum.

So you can reach from p to p prime by a Lorentz transformation and that is what I am going to use. So let us say the Lorentz transformation that is needed to go from p to p prime is parameterized by the parameters lambda. So lambda is a matrix whose elements give you the angles of rotation and the; and the boost parameters so lambda contains all those all those parameters.

Because that is all you need so rotation will change the direction of momentum and boost will change the direction of the magnitude. So lambda let me write matrix containing the rotation angles and boost parameters needed to go from p to p prime. Good so that is fine now I am going to utilize the fact that under Lorentz transformation the vacuum remains in variance.

So here is the vacuum of the interacting theory and the operator remember we are doing quantum mechanics so the operator u that carries out this Lorentz transformation lambda so u of lambda. When x on vacuum does not change the vacuum so that is it remains invariant that we have already assumed so that will be one input. And second input will be the fact that phi is a scalar field so if you take phi of 0 where 0 is for t and space both.

Then doing a Lorentz transformation will give you again the same thing back it does not change by a scalar we mean an object which does not change under the transformation. So how do field operators transform again quantum mechanics they will transform like this? So what do we have now? We want to look at this object let us look at this object so omega what am I doing correct omega and then instead of phi 0 I will write this left hand side because they are the same things.

So this is equal to this but then you acting on vacuum gives you vacuum so this remains unchanged so this is vacuum. Then you have phi 0 and then you have you acting on the keto p which is u lambda acting on momentum so that is the states. All I have done is I have instead of writing this operator acting on the kit I have now written the kit. The kit is the one which carries momentum you the momentum you get by changing p to lambda p so that is the notation here so that is the kit and which is what we call p prime.

So you see that I have arrived at the right hand side and thus the result is independent of the p because no matter what p prime you choose you can always show that omega phi naught p same as omega phi naught p prime so that is a constant. This implies that used earlier when we where was it and we wrote that this is a constant. Excellent so now we know how to create single particle states in an interacting theory.

And the next thing you should be wondering about is how to create multi-particle states and that should not be very difficult once we know how to create a single particle state and then also you should ask how to destroy particles. Remember we had in free theory operators a and a dagger created single created single particle states and a would remove particles from the state so we would also like to have those kind of states sorry those kinds of operators.

And once we know how to create multi-particle states we would love to throw them at each other so that they can scatter and produce something else in the final state. And then we can ask what is the probability of creating so and so final state when you create when you; collide so and so particles in the initial state. So these will be our next goals and we will meet in the next video.