Introduction to Quantum Field Theory (Theory of Scalar Fields) - Part 4
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Module No # 01
Lecture No # 04
Creating Single Particle States -I

Let us resume our discussion and try to create single particle States in an interacting quantum field theory.

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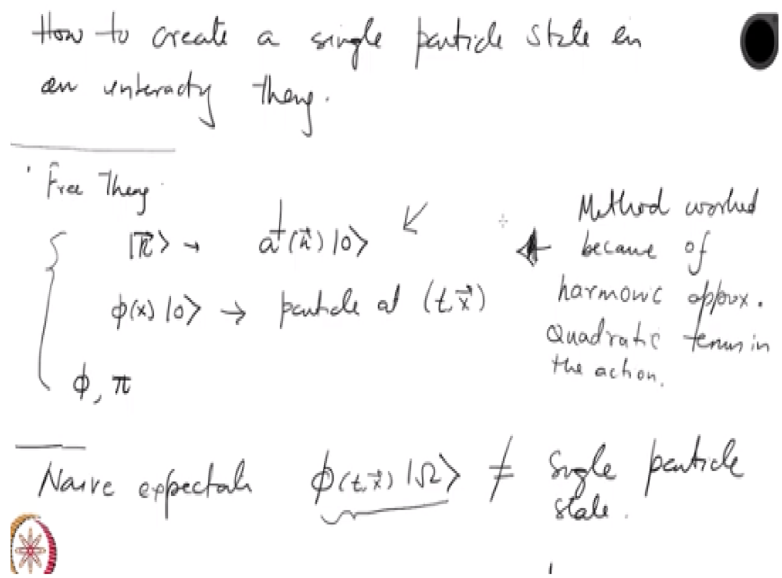

We have already talked about this a little while ago in this previous video. And the goal is to now find equivalent or find a way so that I can create these single particle states of different moment k. And we are in search of an operator equivalent to that of this a dagger in free theory; which when I activate on the vacuum gives me a state of this kind. I am sorry there are 2 cats which are making a lot of noise it might get captured in the audio but there is nothing I can do about it because they are not in my house.

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$$|\vec{p}\rangle = \underbrace{\sum |\vec{k}\rangle\langle\vec{k}|}_{\mathbb{1}}\vec{p}\rangle$$

$$= \sum |\vec{k}\rangle\, 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{p})$$

$$= \underbrace{\sum 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{p})}\,|\vec{p}\rangle$$

$$= \boxed{\int \frac{d^3k}{2\omega_p}}\, 2\omega_p\, \delta^3(\vec{k}-\vec{p})\, |\vec{p}\rangle$$

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \int \frac{d^3k}{2\omega_k} |\vec{k}\rangle\langle\vec{k}| + |\text{bound states}\rangle\langle\text{bound states}|$$

+ contribution from "in" states with more than one label.

Before we proceed I noticed that last time here I had written d cube k over 2 omega p. So I have corrected it to k of course it cannot be p here when you use the delta function the p will be I mean you have either p or k you cannot have a mixture of this. So it should be this. So let us go back to creating single particle states. So how should we proceed? So let us go to field theory and I go to field theory because the interacting theory is going to behave as a free theory as far as these particles are concerned in distant past.

Because when you take T times going to minus infinity all these particles are located far away from each other and they do not interact with each other. So I should be working in that limit and try to find out a field Phi in that limit what that field will be in that limit or equivalently I try to construct an a dagger in that limit. Which will behave as free field operators so that is roughly what we want to do so let us get our inspiration from a free field theory?

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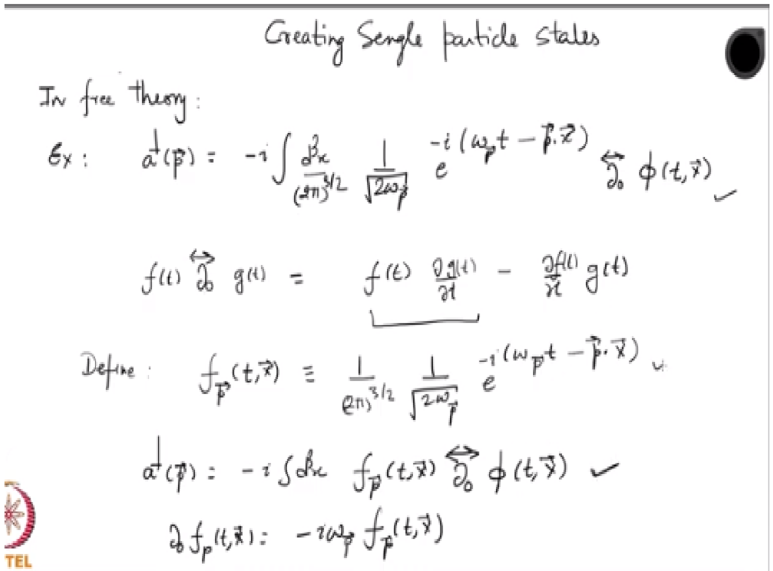

So in free theory we have you can show this so that is an exercise that a dagger of P this is the one which when you act on vacuum gives you single particle states apart from some normalization is this. 1 over 2 Omega p and then you have e to the minus i omega p T -k dot -p dot x. And then you have this operator which I will explain what it is phi x I have been writing phi t, x. So this is an exercise which in fact I am going to do now but let us first tell what is del naught is?

So suppose you have been given 2 functions of time f of t and g of t. Then this object is defined as f of t del g over del t - del f over del t times g of t. So that is the definition of this thing so this is just a time derivative but it acts both ways. So if you take this del naught with this double arrow and if it is and this object this operator the way it access it acts in an ordinary way when it is acting on the right so g of t.

So this is the first term so you would have got an f t times del naught g t so this is this. Then you have a minus sign and then in the second term it is acting on f -del f over del t g t nothing deep here it is a simple definition of this operator. So this is what you have to show that this is indeed true so let us I want to do that algebra but let me say a few things no let us do that. So let us recall what Phi is because Phi has a time dependence on which this does not will act and also the time dependencies here in this omega t factor on which again del naught will act.

So let me do a little bit cleanup of the notation and define so this entire except for this d cube x entire all the other things here this part that I have defined as f of p f t of x with the subscript p. So that a dagger becomes - i integral d cube x f p t of x f p t x and then you have this time derivative operator acting on both the sides. Also let me record here that if you calculate del naught of, f p t x then of course differentiating this exponential will bring - i omega p and again you get back that exponential and these factors are anyway constants.

So you will get - i omega p f p t x. So now we need to substitute what Phi is so that I can differentiate with respect to time because we need to act on with this del naught and try to get the result. So let us start with the right hand side what is the right hand side right hand side is this object this which is same as this here where f p is defined in this equation. But before I look at right hand side I should give you the expressions for Phi.

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$$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a(\vec{k})\, e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + a^{\dagger}(\vec{k})\, e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right]$$

$$\partial_0\phi = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{i}{\sqrt{2\omega_{\vec{k}}}} \left[\omega_{\vec{k}}\, a^{\dagger}(\vec{k})\, e^{i\omega_{\vec{k}} t - \vec{k}\cdot\vec{x}} - \omega_{\vec{k}}\, a(\vec{k})\, e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} \right]$$

R.H.S

$$= -i\int d^3x \left[f_{\vec{p}}\, \partial_0\phi - (\partial_0 f_{\vec{p}})\, \phi \right]$$

$$= -i\int d^3x \left[f_{\vec{p}}\, \partial_0\phi + i\omega_{\vec{p}} f_{\vec{p}}\, \phi \right]$$

$$= -i\int d^3x\, f_{\vec{p}} \left[\partial_0\phi + i\omega_{\vec{p}}\, \phi \right]$$

So phi of t x is d cube k to Pi 3 halves and this vector a k to the - i omega k t + i k dot x that is in the exponent plus a dagger of k e to the i omega k t -i k dot x. And of course del naught Phi will be d cube k over 2 pi 3 halves this one has the dagger comes with a + i Omega. So let me write that one first I have pulled out the vector I here and you have omega k that is what you get after differentiating then e dagger of k then again the same factor e to the i omega k t - k dot x.

And this one will bring - i so that is here so I am left to the minus sign. So that is fine now when you substitute these all these things in the right hand side of this expression let us call it 1. Then you will get let me write it down right hand side is equal to - i integral d cube x this thing then you have f p del naught Phi -del naught f p phi. That is what you get that follows from the definition of this del naught with this double-sided arrow.

So now this is - i integral d cube x and f p let me suppress the arguments del naught phi minus now del naught f was - i omega p times F so that makes a + i omega p times f of p times phi. Is that fine yes that is good now things look a bit cleaner if I pull out f. Now you can substitute phi and del naught phi and you will get the following.

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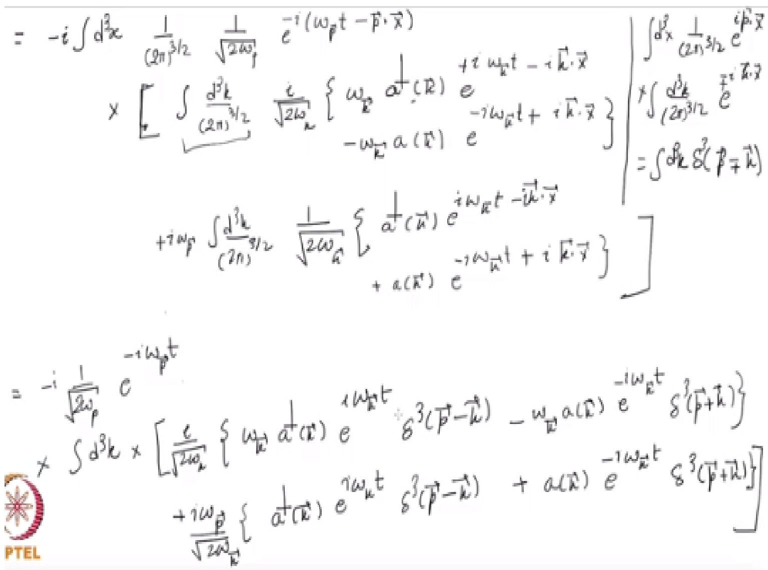

I think I am writing too big so I will write here the expression that you will get – i. Let me decrease the size and brush palate this might be better. So it is the right hand side is now - i integral d cube x 1 over 3 by halves 2 omega t to the - i omega p t -p dot x times integral d cube k over by 3 halves by substituting phi of phi over 2 omega k root omega k omega k a dagger k e to the -r. So e to the + i omega k t -i k dot x - where should I write minus omega k e to the - i omega k t + i k dot x that is one term and then the other term is + i omega p integral d cube k over 2 phi 3 halves 1 over 2 omega k.

And then you have a dagger of k e to i omega k t -k dot x + a of k e to the - i omega k t +k i k dot x and this square bracket closes here or maybe I should put it here. Now we can see something here you have an integral over x. So you have a d cube x. And the next x dependence is quite simple you have here + i p dot x and then you have here these factors of x sitting in the exponent.

So doing x integral is fairly easy doing paint k or k integral is not easy because you have k appearing in all these places omega k dagger k in addition to the exponentials.

So we will do the integral over x first and of course you are going to get d log delta function. Because that is what you get when you integrate the exponentials so let me write here we are going to use d cube x 1 over 2 pi 3 halves that is from here. And then you have e to the minus I times – x + i p dot x and this multiplies either this piece which is d cube k over to 3 half so I am going to collect this also.

And then you have either e to i k dot x or you are going to get - i k dot x these are the 2 possibilities which you have. And this is so when you do the integral over k Sorry over X it will give you 2 phi q which will cancel sorry 2 pi cube times delta function with p plus minus K. But the 2 pi q will cancel these 2 factors of 2 3 halves so I am going to get d cube k delta cube or delta of p - plus that is what we are going to get.

And that is what I am going to substitute in this expression here so what it becomes is the following I am left with an integral over k times let me do it no first we have this factor e to the – i omega p t 1 over 2 omega p. These factors of 2 phi 3 halves have been taken care of so let us write this 1 over and there is a factor of - i also here so we have - i 1 over 2 omega p in the square root then e to the – o omega p t.

That factor is taken care of all the 2 packs we halves have been taken care of integral over x I am going to do now and put the delta functions and what I am left with is integral over k. So let us write integral over k in the next line integral d cube k times first this term and then this term. So first is i over 2 omega k the square root then we have omega k then we have a dagger k then this piece will stay here e to the i omega k t.

But this one when combined with this and integrated over d cube x will give you a delta cube of p - k let us say p. So that is + p and that is - k so that is correct this is p – k. And then you have other term - omega k a of k e to the - i omega k t and delta cube p +k why p + k because again plus P you have here and that is a + k. So delta q p + k and then let us look at this term here that is + i omega p over root 2 omega k and then these vectors you get a dagger of k e to the i omega k t delta cube p - k then you have + a of k e to the - i omega k t delta cube p + k.

Now whenever you have delta function things become very easy so we are now going to integrate over d cube k and utilize the presence of these delta functions and you can check that this is what you are going to get.

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$$
\begin{aligned}
= & -i \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p t} \\
& \times \left[\frac{i}{\sqrt{2\omega_p}} \left\{ \omega_p a^\dagger(\vec{p}) e^{i\omega_p t} - \cancel{\omega_p a(-\vec{p}) e^{-i\omega_p t}} \right\} \right. \\
& \left. + \frac{i}{\sqrt{2\omega_p}} \left\{ \omega_p a^\dagger(\vec{p}) e^{i\omega_p t} + \cancel{\omega_p a(-\vec{p}) e^{-i\omega_p t}} \right\} \right] \\
= & -i e^{-i\omega_p t} \frac{i}{\cancel{2\omega_p}} \cdot \cancel{2\omega_p} a^\dagger(\vec{p}) e^{i\omega_p t} \\
=
\end{aligned}
$$

You will get - i 1 over 2 omega p e to the - i omega p t that is this first vector. So let us see what this gives this k this delta function which will turn the k into p. So this will become e to i omega p t this will become omega p this will become 1 over 2 root 2 omega p. So I will still keep it as i over root 2 omega p and turn all these into all this case into p. So you get i over 2 omega p omega p a dagger p e to i omega p t and I will continue and write down the other terms also a, of – p, e to the - i omega p t.

So what has happened here is? Because this is delta q p + k it hits when k = - p so it will turn all the case into - p so it will be omega of - p here but omega of - p same as omega p because k enters quadratic ally in the expression of omega. So that leaves it as omega of p but here it makes it omega of - p and that is why I have put omega of p omega of p but this is omega a of -p. And similarly for the other term I will write down 1 over 2 omega p omega p a dagger p e to the i omega p t + omega of p a - p to the - i omega p t.

That is what you get and let us see - i omega p these 2 terms are identical and the signs are opposite so they cancel. And it gives you finally these 2 add up they are also identical so they add up gives you a factor of 2 actually and you have 2 one so you have one over root 2 omega P coming from here and these 2 also have one over root 2 omega p so that makes 1 over 2 omega p so let me write down - i from this one then e to the - i omega p t this one.

Now let us take care of these factors this will leave us with i over 2 omega p then they add up factor of 2 so 2 omega p a dagger p e to the i omega e t. Let us see whether I got it right these 2 cancel i times - i is 1 e to the - i omega p t is canceling e to i omega p t gives you 1 so you have a dagger of p and that is what we set out to prove that. This objects where it is this right hand side is equal to a dagger of p. So what we have now is an expression of the creation operator in free theory in terms of Phi that is what we have proved.

But our goal is to go to interacting theory but this will be useful because I will use it to walk along and arrive at interacting theory. So let us see what we should do next good so our first thing should be that we define and it an object just like this. So this creation operator is this in terms of free field where Phi is free field phi this phi is not the phi of our interacting theory this phi is the phi for free theory it evolves according to the Klein Gordon equation.

Free real scalar free theory but now no one stops me from looking at this object for the case of interacting theory. So let us go and do that and hope that it will help but we know that it is not going to help directly because we are not working in free theory. But nevertheless let us go ahead and write it down so I am going to use this definition f of p and this thing so a dagger of p is - i integral d cube x f p del naught phi.

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interacting theory

$$\boxed{-i\int d^3x\, f_{\vec{p}}(\vec{x},t)\,\overleftrightarrow{\partial_0}\,\phi(t,\vec{x}) \equiv a_t^\dagger(\vec{p})}$$

Ex: $$\frac{\partial}{\partial t}\left(-i\int d^3x\, f_{\vec{p}}(\vec{x},t)\,\overleftrightarrow{\partial_0}\,\phi(t,\vec{x})\right)$$

$$= -i\int d^3x\, f_{\vec{p}}(\vec{x},t)\,\underbrace{\left(\partial_0^2\phi - \vec{\nabla}^2 + m_p^2\right)\phi}_{\text{not equal to zero in an interacting theory}}$$

$\frac{\partial}{\partial t}\left(a_t^\dagger(\vec{p})\right) = 0$; free theory

$\frac{\partial}{\partial t}\left(a_t^\dagger(\vec{p})\right) \neq 0$; in interacting theory.

Well del naught is interval errors and I will define this and I look at this object in interacting theory. So now we are in interacting theory what was that object? It was - i integral d cube x f where this phi is the phi of interacting theory. It does not evolve according to Klein Gordon equation that does not evolve according to that. So let us look at this and I leave it as an exercise to show that if you take this object and take a time derivative forward.

The reason why I am asking you this is if you let us forget that let us take a time derivative of this object. And show that this is equal to - i integral d cube x f p then del over delta that is del naught square minus this plus m p square where m p is the physical mass of particles in this theory. So we have assumed that the theory has a single particle states and those particles have mass physical mass m p and remember p is not necessarily the mass parameter that appears in the Lagrangian.

So m p will in general differ from those parameters the parameter m but this is what you should show now if the theory were to be free. If it were a free theory then phi will evolve according to Klein Gordon equation and this is exactly what you that operator you have in Klein Gordon equation. So that would vanish so del naught square phi minus gradient square + m p Square acting on free field would be 0 if phi what a free field. But since we are looking at interacting theory we find out that this object this derivative of this object is not 0 right.

So this is not equal to how should I write not equal to 0 in an interacting theory but this will be 0 if theory was interacting and in that case of course the physical mass m p would be same as the mass parameter in the free theory. So what we have seen is that del over del t of let us make first statement about this thing being this object belonging to free theory. So in that case this was given a name a of p that is what a p dagger this is equal to 0 right this is 0 for free theory.

But del over del t of the same object is not 0 in interacting theory so I am going to call this as a dagger p and I will put a subscript t because this does depend on time otherwise this derivative would have been 0. So that is why I attach a subscript t and call it a dagger p with the subscript t and this says not equal to 0 in interacting theory. So that is the definition of this object and because in free theory this same object creates single particle states with different moment we are interested in working with this. So what should we do?

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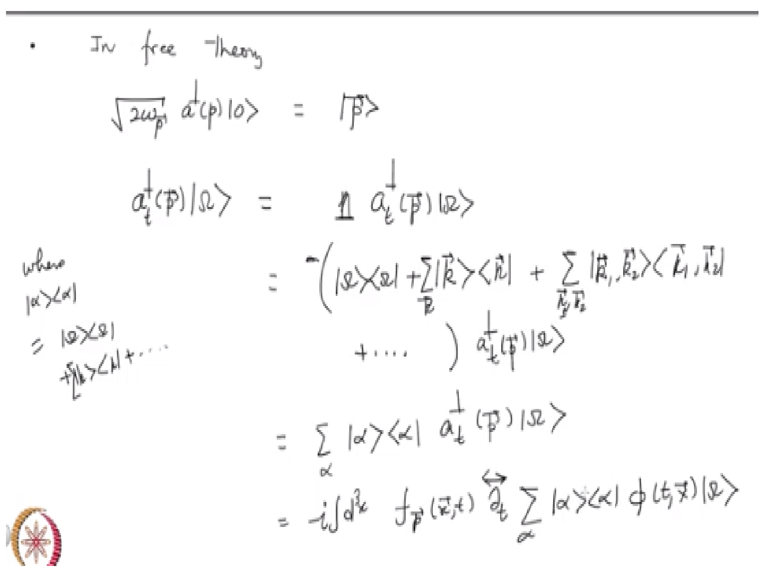

Let us do the following first recall what we are after so in free theory we had a dagger p acting on vacuum that gave us a single particle state and the normalization was that you should multiply with this vector that was for the normalization. Now we have a t dagger p that is this object equivalent of this object in interacting theory and I want to hit on vacuum and want to get this which of course it is not going to come out but never nevertheless let us proceed.

So what I do is I take this and multiply this with an identity multiplying identity does not change anything. Now this identity is the following object we have already seen this so that is the vacuum and then you have these states which carry single labels single particle states and then you have of course there is a some overall the k which really is an integral but for now I will not worry. And then you have states which carry 2 labels and you should sum over k 1 and k 2.

And then you have other states and 80 dagger p omega any of my pen is almost gone. So this is what we have which I will write in short as so instead of writing all these things I am going to write in short is get alpha for this of this operator; where alpha takes values omega k, k 1, k 2 like this. Now this is what this is - i see what I am doing is now I am substituting this expression a dagger of p the subscript t is this object.

So d cube x f p and then time derivative acting on phi so when I substitute that here I get - i integral d cube x and we have f p t then you have a del over del naught or deliver del t same thing. Let us write I will put t instead of 0 and then we have Phi and then finally omega is that fine alpha phi omega that is that is current. So here this is this is this object then you have omega that is here and a dagger contains d cube x f and this double derivative sorry this derivative acting on both sides on Phi.

So that is what it is here see these alphas do not contain any t they do not depend on time you are working in Heisenberg picture. So states do not evolve with time so there is no time dependence in here so this del this time derivative is going to act only in on f and phi. So the time dependence of phi is not very complicated it is easy to pull out and those we can do using the following relation.

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$$\phi(t,\vec{x}) = e^{i\vec{P}\cdot\vec{x}}\,\phi(0)\,e^{-i\vec{P}\cdot\vec{x}} \qquad \phi(0) = \phi(0,\vec{0})$$

So

$$\langle\alpha|\phi(t,\vec{x})|\Omega\rangle = \langle\alpha|\,e^{i\vec{P}\cdot\vec{x}}\,\phi(0)\,e^{-i\vec{P}\cdot\vec{x}}\,|\Omega\rangle \qquad \vec{P}_\Omega|\Omega\rangle = 0$$

$$= \langle\alpha|\,e^{i\vec{P}_\alpha\cdot\vec{x}}\,\phi(0)|\Omega\rangle$$

$$= e^{i(\omega_\alpha t - \vec{p}_\alpha\cdot\vec{x})}\langle\alpha|\phi(0)|\Omega\rangle \qquad (3)$$

Substituting (3) in (2) we get

Ex

$$a_t^\dagger(\vec{p})|\Omega\rangle = -i\int d^3x\,\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\,e^{i\vec{p}\cdot\vec{x} - i\vec{p}_\alpha\cdot\vec{x}}$$
$$\times\; i(\omega_\alpha+\omega_{\vec{p}})\,e^{-i(\omega_\alpha-\omega_{\vec{p}})t}$$
$$\times \sum_\alpha |\alpha\rangle\langle\alpha|\phi(0)|\Omega\rangle$$

So phi T X you remember that it is this where p is the moment operator of that theory. So with this if you want to calculate this object phi t x this then because this is what is there in your expression alpha phi omega. Where alpha and omega they are both independent of time because you are working in Heisenberg picture. So you have alpha phi omega is equal to alpha e to i omega operator p dot x phi at the origin e to the - i p dot x and you have omega here.

So let us look at first this part so you have phi 0 I am going to write this a little while later this when x on see this is exponential right. You can expand in the series and each term electron omega an omega is an Eigen state of the moment operator with value 0 the vacuum does not have any moment it has moment 0 so each of the of the terms will give you 0 meaning the exponential will have e to the - i 0 which is one so this gives you omega o k.

Why because P of omega = 0 now let us look at this part this will give you alpha e to i p because now it is not operator it is I am writing what you get when this operator acts on alpha it is i p alpha for dot x. This is not an operator that is just the Eigen values so that I can pull out and get e to i let me expand it no just realized here. I should have written not see this is a mistake because you have both t and x right I am this is 4 moment operator because I have put t and x both equal to 0 here.

So what phi of 0 is this is phi of 0, 0 like this one and these were meant to be 4 operate for moment and fourth position vector. So this is all ugly this should be removed this should also be removed gone so you have e to the i p alpha dot x that is omega alpha that is the energy of the state alpha t - p alpha 3 memento x i have just expanded times alpha phi of 0 omega.

Good so now I have pulled out the time dependence here explicitly and these are of course some matrix elements which are left. But then I can act with this time derivative operator on the time dependence contained in here. So let us do that so what do I find? So we substitute this result in this equation let us call it 2. So substitute and let us call it 3 so sub t 3 into we get what do we get we get?

We get the following that is simple so I will leave it as an exercise that you show that a dagger t p omega it is an exercise you should do fairly easy - i integral d cube x 1 over 2 phi 3 halves 2 omega p to the i p dot x that is the p here. That is the p which is coming from the argument of a dagger of t - i p alpha dot x. And where that p 1 phi is coming from this alpha these are the moments of all these states.

Then times i omega alpha + omega p, e to the - i omega alpha - omega p t times summation; over the entire alpha phi 0 omega that is what you should be able to show. Now again you can do the x integral because x dependence is fairly simple it is only sitting in the exponential and that will give you a direct delta function. So you end up with finally the following.

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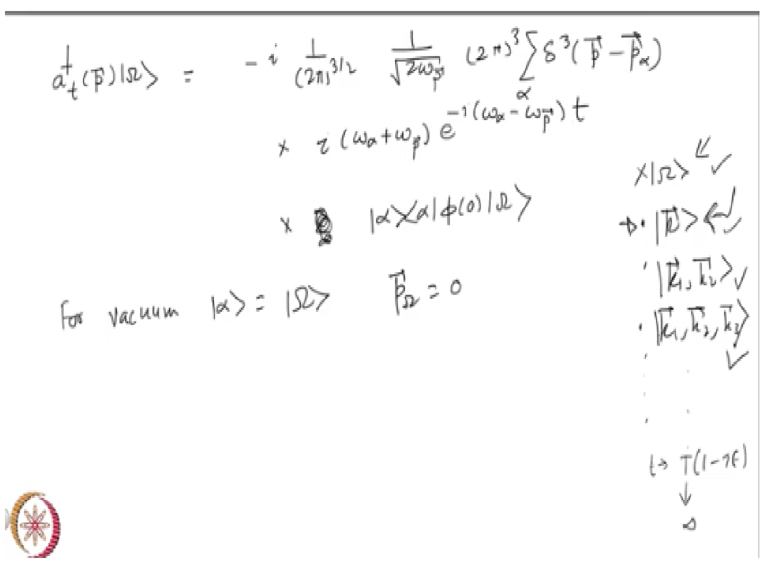

Remember this is what we have been looking for a dagger on vacuum that created single particle state in free theory now we are what we see is that we get the following. We get apart from some factors which are these foreign things in on which we want to worry about but nevertheless we should carry them this is important so delta q p - p alpha. I should there is a summation over alpha and then you have within that summation i omega alpha + omega p e to the - i omega alpha - omega p t.

Now that I have already written here then you have phi 0 omega let us see this do not worry about all these factors. So what we have eventually is there is a summation over all; alpha and then you have some factors some functions. And then here is the real thing forget about this also that is some factor you have this object which you in free theory which gave to you a single particle state.

Now the same thing or the equivalent of the same thing in the case of interacting theory is giving you a states is giving us some over all possible states. So it is so for example alpha takes get alpha takes such values right it becomes it is omega then it is also this then it is also they sum over all of these. So what we were searching is that this acting on omega will give you only this only pick out this as in the case of free theory that is what we want to do we want to create a single particle state.

But what we see here is that we instead get a some over all possible states which is some over vacuum and then single particle states then states which have interpretation of 2 particle states in the far past of states which has interpretation of 3 particles in the far past and so forth. So it did not give you what you wanted but that's not surprising. So let us look at first at the contribution coming from vacuum now contribution coming from vacuum and alpha takes the value omega or k t l phi scat omega that is simple that does not contribute because you see p alpha is 0 when alpha so for vacuum that is when this is get omega p omega is 0 right.

So here you have 0 and p is completely arbitrary this is dagger p acting on omega where p you have is completely orbited see in your hands you choose. So then if this is our this p is arbitrary then this delta function is not going to give you anything it is going to vanish because an arbitrary p - 0 that does not hit that delta function does not hit so this delta function vanishes so it gives you a 0 contribution.

So you get 0 contribution from get omega which means your alpha runs not over this but only over these days and the remaining ones. So this looks bad that we did not get only single particle states but remember we should also not expect this what we should expect is that we should get single particle states in the far past. There it should work there is no requirement that this should help us at any time t but only at t going to minus infinity.

And what we are going to see in the next video is apart from other things how to pick out what limit you should take that this only picks out single particle states get k and not others. And that will not be difficult and it will be a repetition of what we have done already in the first course. Where we did things like limit where we took t going to where T goes to infinity things like this. Because you see there is an exponential here and I can play with the exponential and create a damping a damping term so that it picks out only the relevant things.

But I will show you in detail how it works and once I have done that we would have created a single particle state in interacting theory meaning we would have found the right operator. Which create single particle states and then once we have done that creating states with more than one particle will not be difficult; so see you in the next video.