Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 12
Lecture - 35
Renormalisation Group Equations 2

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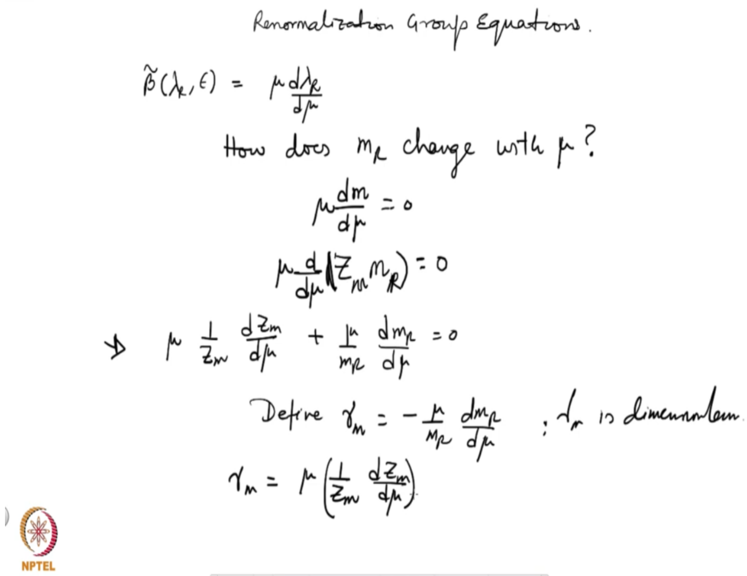

So, let us resume our discussion on Renormalisation Group Equations, and till now we have seen how the coupling constant changes as you change the renormalisation scale mu. That is something we saw earlier and we had defined beta tilde which is a function of lambda R and epsilon only, when you are away from 4 dimensions. So, we took d = 4 - 2 epsilon; and in MS bar scheme, it depends only on lambda R and epsilon; in general scheme, it will depend also on m R and mu also explicitly; but in this case, the mu dependency is implicit, in this scheme, and this was defined to be d lambda R over d mu.

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$$f = -2\lambda_R \frac{dA_1}{d\lambda_R}$$

In MS Scheme

$$\tilde{\beta} = -2\epsilon\lambda_R - \lambda_R\left(-2\lambda_R \frac{dA_1}{d\lambda_R}\right)$$

$$\boxed{\tilde{\beta} = -2\epsilon\lambda_R + 2\lambda_R^2 \frac{dA_1}{d\lambda_R}}$$

Eg: $\tilde{\beta}(\lambda_R, \epsilon) = -2\epsilon\lambda_R + \frac{3}{16\pi^2}\lambda_R^2$

Let us see what else we had done earlier; and also we saw that the contribution to beta tilde or the beta function comes from only the single pole terms. So, let us see what this A 1 is; A 1 is here.

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$$Z_\lambda = 1 + \frac{1}{\epsilon}\left(\lambda_R a_{1,1} + \lambda_R^2 a_{2,1} + \cdots + \lambda_R^n a_{n,1} + \cdots\right) + \frac{1}{\epsilon^2}(\cdots) + \cdots$$

$$Z_\lambda = 1 + \frac{A_1}{\epsilon} + \frac{A_2}{\epsilon^2} + \frac{A_3}{\epsilon^3} + \cdots$$

$$Z_m = 1 + \frac{B_1}{\epsilon} + \frac{B_2}{\epsilon^2} + \frac{B_3}{\epsilon^3} + \cdots$$

Now we will see the $\frac{1}{Z_\lambda}\left(\mu \frac{dZ_\lambda}{d\mu}\right)$ does not depend on μ explicitly.

$$\frac{1}{Z_\lambda}\left(\mu \frac{dZ_\lambda}{d\mu}\right) = \frac{1}{Z_\lambda}\left[\underbrace{\mu \frac{\partial}{\partial \mu}}_{=0} + \tilde{\beta}\frac{\partial}{\partial \lambda_R} + \underbrace{\mu \frac{dm_R}{d\mu}\frac{\partial}{\partial m_R}\Big] Z_\lambda}_{=0}$$

$$= \tilde{\beta}\frac{1}{Z_\lambda}\frac{\partial Z_\lambda}{\partial \lambda_R}$$

So, this Z lambda which is the renormalisation constant for the coupling constant, here I have written not in terms of, not as an expansion in lambda R but rather I have collected terms according to the order of poles. So, A 1 has all the terms which give you a single pole. So, this A 1 gets contributions from all orders in perturbation theory. So, when you do a 2 loop calculation, you also generate double pole and a single pole.

So, that is also included in here. So, there are lambda R terms, lambda R square terms, lambda R cube terms and so forth; same for this. And we saw that because of this relation, the beta function gets contributions only from the single pole terms and the higher order pole terms, their coefficients do not contribute. Now I want to look at how the mass parameter should be changed when you change the scale mu and I should remind again that mu is not a physical parameter in your theory because you see that is not something present in the bare Lagrangian.

So, that cannot be a parameter of your theory and it cannot be related to any of the physical scales in the theory or physical scales in some experiment. If you are doing some scattering, mu cannot be related to any of the scales in the scattering experiment, the incoming momenta, outgoing momenta, whatever because that is simply not present. And what we are trying to learn here is that if we were to change mu or how I should change the coupling constants, the renormalised coupling constants such that the theory remains unchanged meaning the change should be such that in the renormalised couplings and the normalised mass parameters such that the bare parameters do not change.

So, now, before I move on to look at how the renormalised mass should change with the scale mu, I want to make a correction. There is a mistake I have done earlier which is giving a wrong result for the beta function by a factor of 2. I will just tell you what that mistake is.

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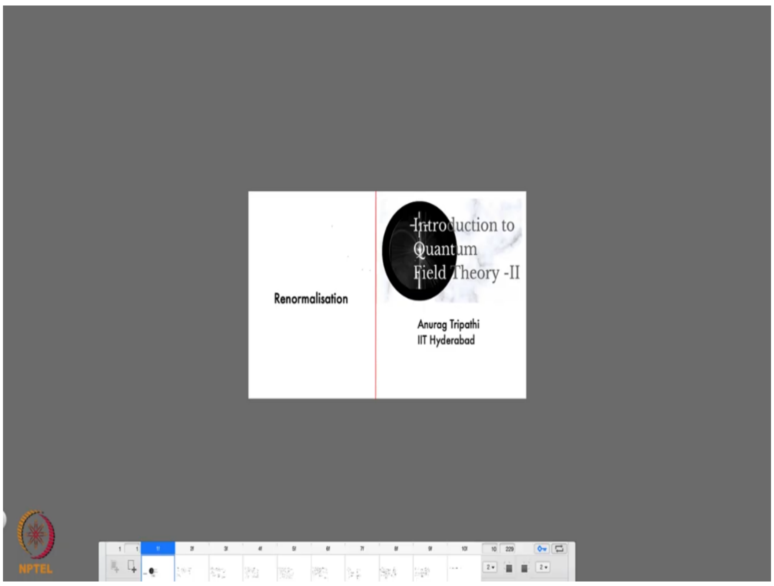

So, let us go to this other notebook. So, I should go to where I was doing renormalisation of phi 4 theory, here.

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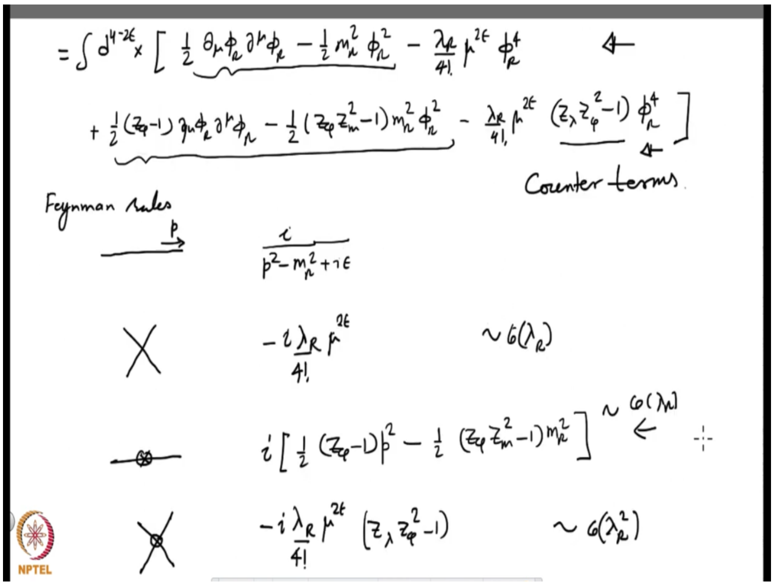

So, now here you see, this vertex is of order lambda R square because when you have done the, written the action in terms of renormalised fields and parameters, this is a term which you produce. That has an explicit lambda R here and because of these z's you get another factor of lambda R, and it starts at order lambda R, this piece, so, that is why it is lambda R square. So, we have this vertex. Now, we started calculating these 2-point function and 4-point function and renormalising them and somewhere here it went wrong; here. I hope I am not using this.

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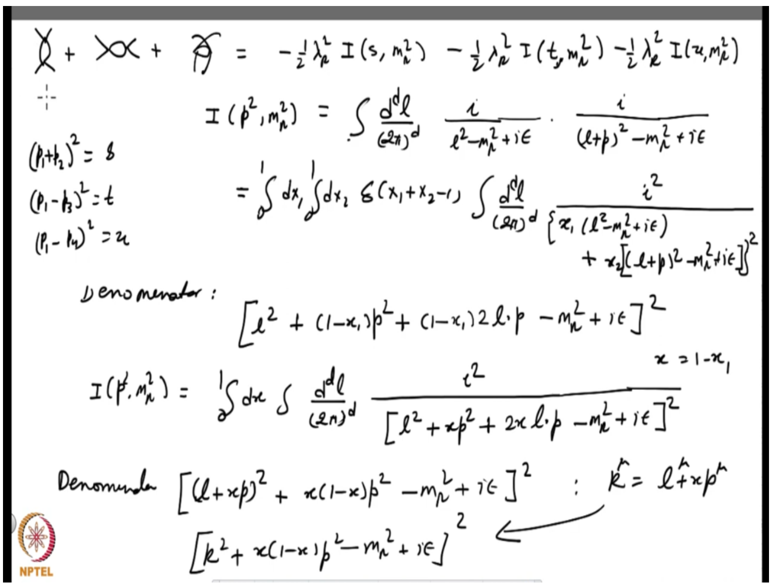

So, you see, I am looking at this renormalisation of 4-point function here.

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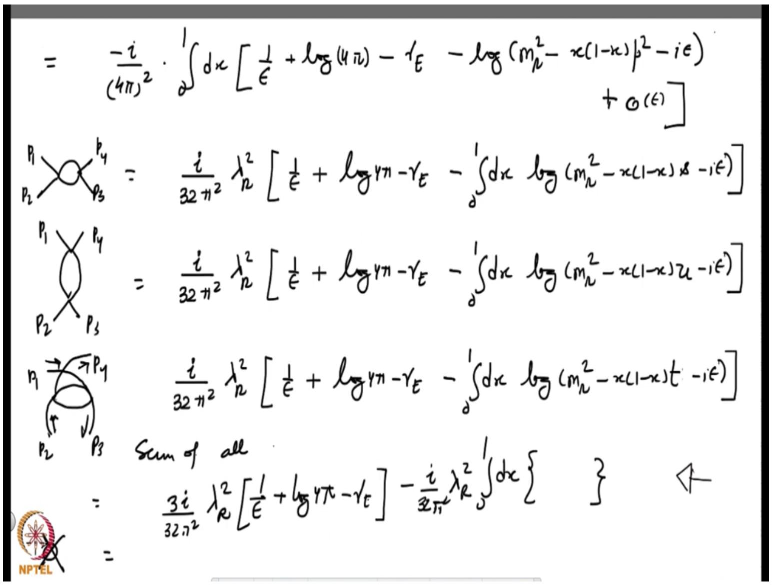

So, I calculated all these 3 diagrams at one loop and these had some poles here which we wanted to cancel and for that I need to adjust the z's, the z that appears in here, in this vertex.

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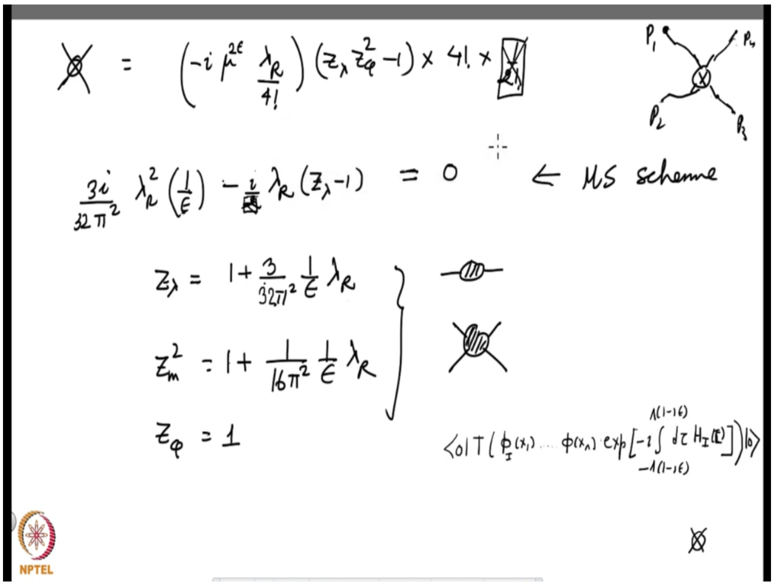

So, I wrote the expression of this 4-point function where I am using the counter term. And you see, after writing all these factors, after counting all this, I multiplied by 1 over 2 factorial, and the reason why I multiplied by this 1 2 factorial was the following. So, if you recall how our Green's functions are written, I am just writing the numerator part; We had learnt this in the first course.

So, you have, I am just writing greens functions in the momenta coordinate space. Here in this case, it will be 4 but I am just writing generic 1; tau. This is what you get in the numerator and it is basically this which you are expanding here and I am writing. This is what gave you the Feynman diagrams. So, now here you see our H interaction or H I contains this term which you had in the action. Where is that? It is way behind, here.

So, H I will contain all these terms starting from here to this last one because these all are having coupling constants in them. So, that is what enters. Now, the mistake that I made, it was a slip of mind, was the following. Where did that go? Last track, here. So, when I put H I here, that all those terms which I was telling, when they are here and I expand, so, when I expand the exponential, the first term is 1.

Then, the next term will be this, exactly this thing in the argument. Then the next one will be the square of this times 1 over 2 factorial. Then the third term would be cube of this times 1 over 3 factorial. And this is, that 1 over 2 factorial and 1 over 3 factorial, that is what comes here usually. That is what I was saying, but note that when I expand to the first order, it will be 1 over 1 factorial, not 2 factorial, 1 over 1 factorial times this piece, but this piece includes this vertex which is already order lambda R square.

So, even though I have a term which is order lambda R square, you do not get that 1 over 2 factorial. This would have been true if none of the terms in H I would be a lambda R square. If they were all linear in lambda R, then only when you go to the next order term, you get lambda R square and you bring 1 over 2 factorial. So, that is the mistake. I should not have included this one. So, this should go away.

If this goes away, then this should go away; and then, z lambda should become 3 over 32 pi square. So, let me try to make the change. So, this should become 32 pi square. Why does it not go away? So, this is a correction that I wanted to make. Now let us return to our discussion on how m R changes with mu in phi 4 theory; the discussion will be general but let us see. So, back to business.

So, our question is, did I give an expression for this one? No. So, exercise: you should check that you get beta tilde lambda R epsilon in phi 4 theory to be - 2 epsilon lambda R; this is this term; +2; maybe I will just give you the answer; +3 over 16 pi square lambda R square. So, all you have to do is take the derivative of A 1, and that derivative of A 1 you can find from the z of lambda that I have just corrected.

So, that is the beta function in phi 4 theory. Let us go to the question of how m R changes with mu and you already know why we are doing this. So, as before, we want to keep the bare mass parameter fixed as I change mu, meaning this derivative should be 0 and again it is customary to put a mu here which is same as writing mu d over d mu Z m times m R; that is hopeless; and this should be equal to 0.

So, we take this derivative and should be 0 which is just which gives you a mu; after taking the derivative, I am going to divide by Z m and m R. So, that gives you mu over Z m d Z m over d mu plus mu over m R d m R over d mu equal to 0. Now I will define gamma m as minus mu over m R d m R over d mu. See, this gamma m is dimensionless because m R has mass dimension 1, mu has mass dimension 1, so, this ratio is dimensionless and this ratio is also dimensionless and that is why I have divided by m R unlike the case in the beta function, so as to make this gamma m dimensionless.

So, gamma m is dimensionless and this gives us gamma m. So, I will use this equation and this gives us gamma m is mu 1 over Z m d Z m over d mu. Now again repeating very similar steps as we did for the beta function let us; so, I will take this equation which is derivative with respect to mu but I know that Z m, these renormalisation constants in the production theory in MS scheme, they do not depend on mu explicitly, they depend only on lambda R. So, let me instead of using total derivative with respect to mu, convert it to a derivative with respect to lambda R. That is what I will do now.

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$$\mu\frac{d}{d\mu} = \underset{\searrow 0}{\mu\frac{\partial}{\partial\mu}} + \tilde{\beta}\frac{\partial}{\partial\lambda_R} + \mu\frac{dm_R}{d\mu}\underset{\searrow \partial}{\frac{\partial}{\partial m_R}}$$

$$\gamma_m = \frac{\tilde{\beta}}{Z_m}\frac{\partial Z_m}{\partial\lambda_R} \qquad Z_m = 1 + \frac{B_1}{\epsilon} + \frac{B_2}{\epsilon^2} + \dots$$

$$\gamma_m Z_m = \tilde{\beta}\left(\frac{dB_1}{d\lambda_R}\frac{1}{\epsilon} + \frac{dB_2}{d\lambda_R}\frac{1}{\epsilon^2} + \dots\right)$$

$$\boxed{\gamma_m = -2\lambda_R\frac{dB_1}{d\lambda_R}}$$

So, you as before, you get mu d over d mu is equal to mu del over del mu plus del over del lambda R and then del lambda R over del d lambda R over d mu times mu that is beta function that is beta tilde; I can write it closer; plus del over del m R mu times d m R over d mu and these will give you 0 when you act this thing on Z m because again for the same reason, no explicit dependence on mu, no explicit dependence on m R when you are working in MS bar scheme or MS scheme.

So, what do we get? We get gamma m. So, here, I am now substituting that. So, mu d Z over d mu, I will replace by beta tilde times del over del lambda R and then I get gamma m is beta tilde over Z m del Z m over del lambda R. So, now I can take the derivative easily because everything is in the z, it is just a function of lambda R and epsilon, so, I can take a derivative with lambda.

You can replace this by a total derivative because Z m depends only on lambda R, there is no other variable. So, now let us again do what we did earlier. So, I will write gamma m times Z m is equal to beta tilde times; remember Z m we had written as; this is again not writing, this is horrible; 1 plus B 1 over epsilon plus B 2 over epsilon square and so forth. I had collected all the single pole terms, double pole terms and etcetera.

So, this I will write as, when I take the derivative, that 1 goes away and it gives you d B 1 over d lambda R times 1 over epsilon plus d B 2 over d lambda R times 1 over epsilon square and other terms. Now we should put expression of beta tilde and Z m here and exactly as before, you compare the powers of epsilon to the n on both sides. So, here you write Z m; Z m as like this, and here you export the expression of beta tilde which is already I wrote just now a little while ago; here this expression.

You substitute it there and then you can compare order by order and you will see that the expression for gamma m is -2 lambda R d B 1 over d lambda R. That is what you get and now you see that the gamma m; gamma m is the thing which controls how m R is going to change as you change mu. That is controls the behaviour of m R with mu. That is the derivative of m R and derivative of m R is controlled by gamma m.

And now we have seen that gamma m depends only on B 1; it does not depend on B 2 or B 3 meaning it is controlled by the coefficients of single pole terms only. Very naively one would think that all the coefficients whether it is coming from single pole or double pole or triple pole, they will contribute. So, for example, a 2 loop, you have both double pole and single pole; at 3 loop, you will have triple pole meaning pole of degree 3 and degree 2 and degree 1.

So, naively one would think that they all should contribute but we see here explicitly that only B 1 contributes only the questions of single poles contribute. So, that is a generic result. We have not used the fact that we are using phi 4 theory. This is true for any theory. Now I will write result specifically for phi 4 theory and of course you can do similarly for whatever theory you are interested in.

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For ϕ^4-theory

$$Z_m = 1 + \frac{1}{32\pi^2}\frac{1}{\epsilon}\lambda_R + O(\lambda_R^2)$$

$$B_1 = \frac{1}{32\pi^2}\lambda_R$$

$$\gamma_m = -\frac{1}{16\pi^2}\lambda_R.$$

$$\mu\frac{d\lambda_R}{d\mu} = \tilde{\beta}$$

- $\tilde{\beta}$ is +ive, then $\lambda_R(\mu)$ increases as you increase μ.
- $\tilde{\beta}$ is −ive, " " decreases as you increase μ.

So, for phi to the 4 theory, we have already seen that Z m is 1 plus 1 over 32 pi square 1 over epsilon lambda R plus order lambda R square terms. So, right now I am going to give an expression which is true only up to order lambda R because other single pole terms from order lambda R square, order lambda R cube, I have not calculated, so, I do not have them here. So, I will be using only this one. So, Z m is this. So, what is B 1?

B 1 is this part, 1 over 32 pi square times lambda R. So, gamma m is -2 times lambda R times this factor which makes it minus 1 over 16 pi square lambda R. So, we have also an explicit result for gamma m in this phi 4 theory. Now let me make some remarks about the beta function. So, here we had mu d lambda R over d mu as beta tilde and remember that we are doing perturbation theory. So, suppose that in a theory, whichever theory you are looking at, this beta function turns out to be positive, meaning the rate of change of lambda with mu is positive, so, as you increase mu, because beta tilde is positive, lambda R will increase.

So, if beta tilde is positive, then lambda R mu increases as you increase mu and if beta tilde is negative, then this decreases as you increase mu and that is a very useful piece of information because that can, you will see why it is important and actually in a particular set of theories, you get beta tilde to be negative and a Nobel prize was also awarded for showing that beta function is negative in those theories.

So, anyhow, one of the important reasons why one is interested in the sign of beta function is precisely how these coupling constraints behave as you change the scale mu. And actually, we will see later in detail why this sigh of beta function would be interesting for us, when we analyse in more detail about the renormalisation group equations when we analyse how Green's functions behave as you change scales and how they behave when you scale up the physical momenta.

Right now I am talking only about the mu which is non-physical; it is not a parameter really in your theory; things should be independent of mu but later I will relate things, the behaviour of Green's functions to the change of scale of physical momenta in the theory and I will be interested in high energy behaviour. So, anyway we will see all that in the next video or we will start going towards that in the next video and I hope you have an understanding of what these beta functions and what this gamma m are doing and how they can be calculated in perturbation theory.

Also one remark that suppose you calculate beta function and it turns out to be positive like here in this case it is positive, so, beta 1 is positive; so, as you increase the scale mu, this lambda starts growing but then that expression you cannot use after a while because once the lambda R has become large, sufficiently large or of the order 1, then you cannot use the perturbation theory anymore because having a perturbation theory means that the parameter with which you are doing perturbation theory is small meaning it makes sense to drop terms of higher orders in perturbation theory and that you do because lambda R would be small or the parameter, perturbation parameter would be small.

But as we see, if you are changing mu and it is increasing, then beyond a certain point, lambda R would not be small anymore. So, any conclusions that you draw from there when lambda R is not really small would not be really true and you will need non-perturbative analysis for making any statements; but as long as you are in small lambda R region you can use such relations and conclude whatever you wish to like, whether lambda R is increasing with mu or not.

So, one has to be careful in not stretching perturbative arguments where they are not applicable, especially when coupling start becoming of the order 1. We will meet in the next video and see what we do next.