Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Lecture - 34
Renormalization Group Equation 1

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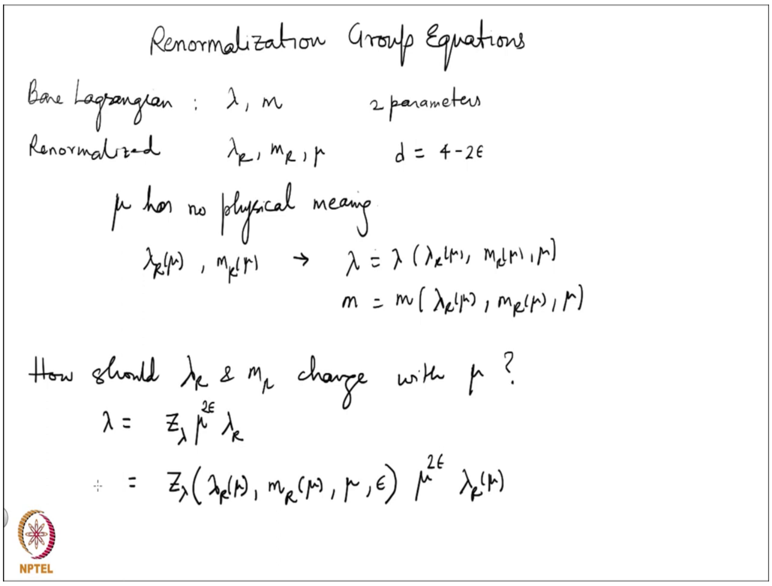

So, today we will start renormalization group equations which are differential equations that govern how Greens functions are asymmetrics elements or some coupling constants evolve as you change the renormalization parameter mu. So, that is what falls under this heading and I am going to first start by talking about the simplest thing which is the coupling constant. How can we will do it in some detail and let us see what we learn from there.

So, let me remind you what we have done so far. So, we have seen that the bear lagrangian density is written in terms of 2 parameters Lambda and M these are bear parameters but if you write the renormalize lagrangian density then instead of 2 you have three Lambda R, M R and mu where mu is the renormalization scale and you remember that we introduced this because the coupling constant became dimension full in d dimensions where these four - 2 Epsilon.

Let me also remind you that we are using or - 2 Epsilon and good now if you calculate any physical quantities in principle you can keep Epsilon fixed. So, that nothing is divergent. So, you can think of a theory being studied in d dimensions and then Lambda and M are finite they are singular rule in the abstract going to zero limit but to make our arguments it is better to think in terms of Epsilon finite non-zero.

So, these are constants. So, any observable that you calculate using this original lagrangian in terms of Lambda and M will not depend on mu right because it depends only on Lambda and M but if you calculate the same objects using these three observables now these are these are these three variables Lambda R M R at sorry using these three then it looks like you have an additional parameter.

So, instead of 2 you have three and as I said last time this allows you to change the value of mu because mu is not fixed by any physical scale in the theory even when you are scattering particles and let us say you have certain incoming momenta and certain outgoing momenta there is no relation of mu to any of those momenta this is completely arbitrary this has no physical relevance no physical meaning.

So, let me write the down mu has no physical meaning. So, the way you work is you fix some value of mu choose some value of Lambda R and M R and that will fix the values of Lambda and M and that defines your theory. So, all physical observables are then defined once you have made a choice of mu M R and Lambda R because that fixes M and Lambda for you for example the physical masses are or the probabilities of certain scatterings or any schedules.

And as we are this is last time that I can choose to change the value of mu without altering the theory provided I also change Lambda R and M R such that it compensates the changes in mu. So, we said that Lambda R should be written as a function of mu and R should be written as a function of mu says that we keep these parameters fixed. So, this fixes the theory. So, Lambda the bear the bear parameter here is a function of Lambda R of mu M R of mu and mu it could depend explicitly on mu.

And also implicitly through the coupling constant and the renamolized mass and similarly the where parameter M would depend on the renormalized coupling constant renormalize mass parameter and mu the dependence will be typically both explicit and implicit through these variables. So, now let us look at the coupling constant first NC what this requirement implies for the coupling constant meaning how Lambda R changes when you change mu that is the question we want to ask.

So, how should Lambda R and M R change with mu we want to know this how exactly that behaviour is. So, what is Lambda? Lambda the bear parameter is Z Lambda that is the wave function sorry that is the renormalization constant that we had introduced and then to make Lambda R to be dimensionless we had absorbed the dimension of Lambda here. So, we had introduced a parameter mu and raised to the tube to Epsilon now I should be writing it more explicitly.

So, I will write all the arguments of Z Lambda. So, Z of Lambda will be in general a function of Lambda R which in turn is a function of mu should be a will be a function of M R which again is a function of mu the renormalization scale mu can also appear explicitly and of course you will have Epsilon also appearing in the argument that is how we wrote down Lambda earlier all I have done is just made all the arguments of Z of Lambda exclusive.

So, what I have done is I have written all the quantities that can in principle appear without leaving out anything unless I have an argument why a certain parameter will not appear I should write all the things I know when I renormalized I have to subtract Infinities poles will be there. So, there is absolute dependence of course when you calculate farm and diagrams they will depend on Lambda R and M R and you have already seen log mu appearing explicitly.

So, there is an explicit new dependence as well. So, I have listed down everything both the implicit dependence on mu and explicit dependence on here good but see left hand side doesn't know about mu right hand side does. So, if I change mu Lambda should not change but right hand side even though individual pieces Z Lambda mu to the 2 Epsilon and Lambda R individually they will change.

But they all should change together in a manner that the dependence completely gets cancelled because the left hand side is independent and that is all I am going to utilize now no great wisdom just this much.

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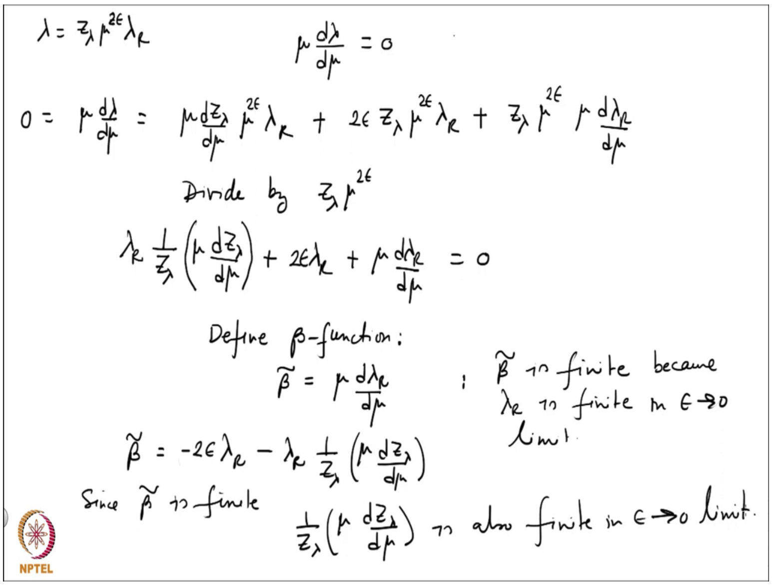

So, because Lambda is a bare object an independent of mu is this recording must be. So, d Lambda over d mu should be 0. That is the simple statement that Lambda does not depend on mu. Now this is zero I can multiply mu here you cannot stop me from doing that. So, I will just multiply that the statement is still true and it is customary to introduce mu here as this Factor now what it means for the individual factors.

So, I will just take the derivative on the right hand side of this equation. So, 0 is Mu d Lambda or d mu that is the expression. So, I am just differentiating that maybe I will write it here again to oh fine. So, the first factor is z you remember that. So, mu dz Lambda over d mu and the other factors were mu to the 2 Epsilon and Lambda R at it here Lambda is Z Lambda u to the 2 Epsilon and Lambda R.

I am differentiating this then the second term is when I differentiate this mu to the 2 Epsilon it gives me a 2 Epsilon times mu to the 2 Epsilon - 1 but then I have a mu multiplying here. So, that makes it again mu to the 2 Epsilon. So, I get and finally a differentiate Lambda R. So, I get Z of Lambda mu to the 2 Epsilon and mu d Lambda R over d B that is what I get now let us divide by Z Lambda mu to the 2 Epsilon if I do.

So, then I get. So, there is a factor of Lambda r. So, this mu 2 the 2 Epsilon goes away from each of the terms here in this one I will get. So, I am writing this Lambda R first Lambda R then Z of Lambda will come in the denominator one of one over Z Lambda mu d z Lambda over do you know Plus 2 Epsilon I have divided by Z Lambda. So, this goes away mu to the 2 Epsilon goes away and I am left with Lambda R and in this term.

Again Z Lambda goes away u to the 2 Epsilon goes away and you are left with mu d Lambda R over d mu is equal to 0 and it is d Lambda R over d mu that is what you want to calculate you want to know how Lambda R changes as you change mu. So, that thing is here which you are searching for. So, let us define this as beta tilde it is called beta function define beta function which is beta is equal to Mu d Lambda R over d.

So, from the definition itself it is clear because Lambda R is a finite object it is a finite number as Epsilon goes to zero and mu is anywhere parameter this beta tilde is a is a finite object beta tilde is finite because Lambda R is funded we are not expecting the derivative to blow up. So, this group is finite. So, substituting this in this equation I will get. So, this is beta. So, beta tilde I take these 2 terms on the other side it is other side it is - 2 Epsilon Lambda R - Lambda R 1 over Z Lambda mu dz Lambda over d.

Now Beta tilde or the beta function is finite in epsilon I am going to zero limit this object is finite in the Epsilon going to zero limit capsule moves to zero. So, that goes to zero Lambda R is anyway finite and that implies that 1 over Z Lambda times mu dz Lambda over d mu this factor is also finite. So, since we total lies finite 1 over Z Lambda mu dz Lambda over d mu is also finite. Now I will look at the let us suppose that we are doing the calculation in the M S or M s bar scheme.

And you remember the advantage of using Ms or Ms bar scheme is that Z Lambda and Z M is the renormalization constant for the mass parameter they do not depend on the scale mu and this is an all order statement and that further implies that Z Lambda and Z M they do not depend on M R also.

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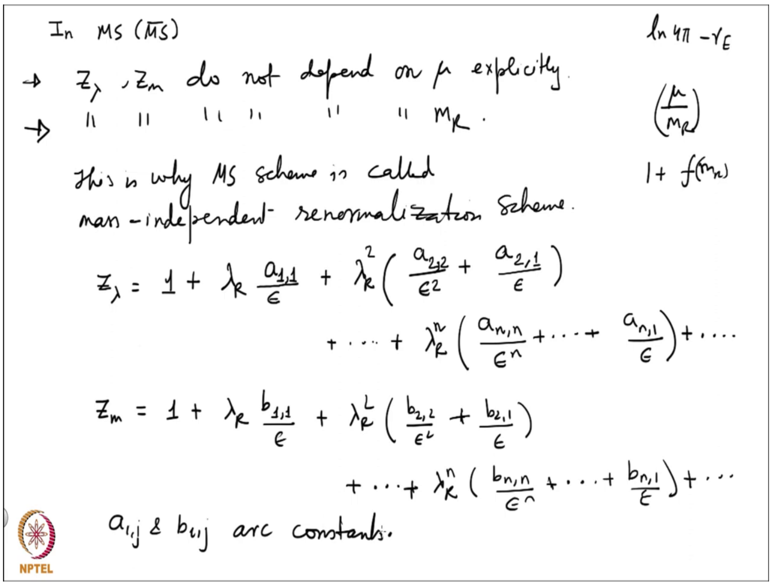

So, let me say this I will be using now M S or M S bar scheme I will call them together as M S minimal subtraction because the difference between M S and M S bar is purely a constant. It does not involve any mu the difference is just log mu - sorry log of 4 Pi - solid gamma you have seen this log of 4 Pi - all of gamma. So, in M S you subtract only the poles and in M S bar you subtract together with the pole this Factor sorry these 2 terms which always accompany the poles that is a statement at one Loop but similar statement is through it hair loops also.

So, I have already told you that Z Lambda Z M do not depend on mu explicitly and because Z is 1 plus order Lambda term and one is dimensionless and because mu does not appear you cannot construct any dimensionless ratios meaning you because you do not have mu there is no way you can construct such an object which is dimensionless. So, there is no way Mr can appear because 1 plus if you were to construct some function which has M R it will be dimension full.

The only way it could have been dimensionless is if you had mu also available to you then you could have divided mu or M R mu by M R and that ratio will be dimensionalized but that is not there. So, it also does not depend on mu and for this reason these are these schemes are called mass independent renormalization schemes because there is no mass scale appearing in these schemes. So, this is why M S schemes scheme is called mass in d and dent renormalization scheme.

So, now let me tell you how Z Lambda would look like and Z M would look like if you were to calculate to had orders in perturbation theory. So, the general form of Z Lambda would be this and you have already seen at one Loop what it looks like. So, it starts with one because you do not require renormalization. If you are at the tree level tree level does not require any renormalization there are no infinities.

So, Z Lambda is one then typically you will get corrections at order Lambda R. So, that is Lambda R some of the coefficients may vanish but in general you will get non-zero coefficients now you have seen that you get a pole simple Pole or one over Epsilon pool. So, you are going to get something here which is proportional to one over Epsilon times some constant what you have in the numerator is a constant it does not depend on mu M R and scales neither the external momenta they are irrelevant.

So, what you get is a constant and let me call that constant as a one, 1 the first one here stands for the order of Lambda and the second one here stands tells you that you have a pole simple Pole right the Epsilon power is one that is what it is telling you then if you go to order Lambda Square meaning you are going higher order in perturbation Theory then you will encounter 2 loops then you will have in addition to a single pole you will have a double pole also.

So, the highest order pole will be one over Epsilon Square it will be more Divergent and again the coefficient will be a constant and I will call it A 2 that 2 first 2 stands for the second order in Lambda R, 2 and this 2 stands. For the second order of the pole so, this is a order 2 pole and that is why this 2 here and you will also have a single pole. So, you will have a 2, 1 over Epsilon this 2 refers to again the fact that you are at Lambda R squared and this one tells you that you are looking at a simple pool Epsilon to 1 over Epsilon to the one.

And the general structure is that it will be Lambda R to the n a nn. So, you will get order n Pole at order Lambda R to the n then you will have lower poles and this n Z finally a simple pool at order n. So, at every order you are going to get poles off. So, at order n you are going to get poles of order n n - 1 n - 2 up to order 1 a simple pole. So, all these will be present. So, that is the structure of Z Lambda.

So let me also write down what Z M would look like it will be identical except for the difference in coefficient. So, let us call let us call them b b 1 b 2 and so on so forth. So, that is the structure and as I said a ij and maybe I should use a ij and b ij are constants. So, that is the general structure of these renormalization constants in MSP or M is working the same thing. Now I can of course rewrite it slightly differently.

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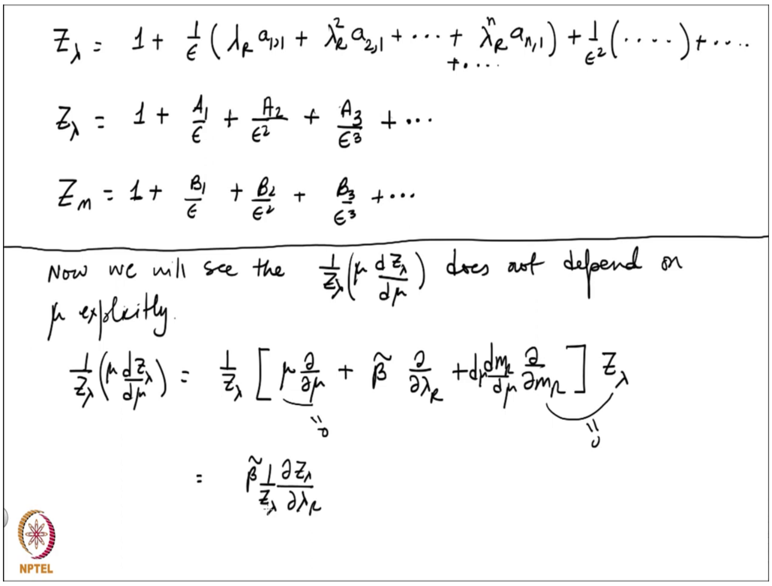

And let us instead of writing it as an expansion in Lambda I will write it as an expansion in the pole the order of the poles and then we are going to see something very interesting. So, I can write Z Lambda as 1 plus so let us collect all the order one over Epsilon terms. So, it will be what it will be Lambda R times a 1,1 that is the coefficient of 1 over Epsilon coming from one Loop the lowest order diagram which is proportional to Lambda R.

Then you will have contributions coming from Lambda R square terms from higher order calculations from order Lambda R square calculation with description and. So, forth yeah I am just regrouping I am not doing anything major and then you will also have other terms like one over Epsilon Square terms and you can similarly write what should be here plus one over Epsilon cube and so forth.

So, the structure will be that Z Lambda I can write as 1 plus A 1 over Epsilon. So, this Epsilon and A 1 is this piece. So, even depends on Lambda R it is a polynomial in Lambda R of I mean in principle you have this goes to this keeps continuing right you I should not have stopped here because you can calculate up to infinite orders. So, this is a polynomial then you will have one over Epsilon Square terms it is called the coefficient to be A 2 and A 2 is again a function of Lambda R.

And the dependence is like this Plus and so forth and the same is true for Z of M where A 1 is for example listing. Now before proceeding further let me first show you that beta tilde or the beta function does not depend on mu explicitly in the MS bar scheme again that is what I am going to utilize later. So, let me do that. So, from here it is not. So, much clear that beta tilde does not depend on mu explicitly right.

So, this term as far as this term is concerned it is clear that dependence on mu is through mu R. So, that is implicit this factor is implicit but then you have a factor of mu here. So, it is not. So, obvious that it does not depend on mu explicitly right the Z's do not depend on you explicitly because you are in M s bar scheme and I should argue that this entire Factor also here at the bottom does not depend on you explicitly do we see that.

So, 1 over Z Lambda times mu dz Lambda over d mu this is 1 over Z Lambda and then let us write the total derivative as the partial derivatives. So, mu Del over Del mu Plus Del over Del Lambda R times Nu Del Lambda R over d Del mu mu d Lambda R over d mu and that is what is called beta tilde Plus Del over Del M R d M R over b mu acting on Z of Lambda and as I have said already that Z Lambda is independent of M R.

So, this derivative acting on this gives you zero similarly this derivative acting on Z Lambda gives you 0 because there is no explicit dependence on mu. So, that also gives you 0. So, what we are left which is only this term. So, you get beta tilde Delta Z Lambda over Delta Lambda R times 1 over Z Lambda. So, that is what you get now let me put that back in this equation this let me box it hmm it is definitely not bad I want also maybe this equation.

So, yeah so I will substitute that thing in here I have just written down one over Z Lambda mu dz Lambda over d mu as as beta tilde 1 over Z Lambda Del Z Lambda Del Lambda R. Now the derivative is with respect to Lambda R instead of mu. So, I will go I am going to put this in here. So, what do I get? So, I will write it again.

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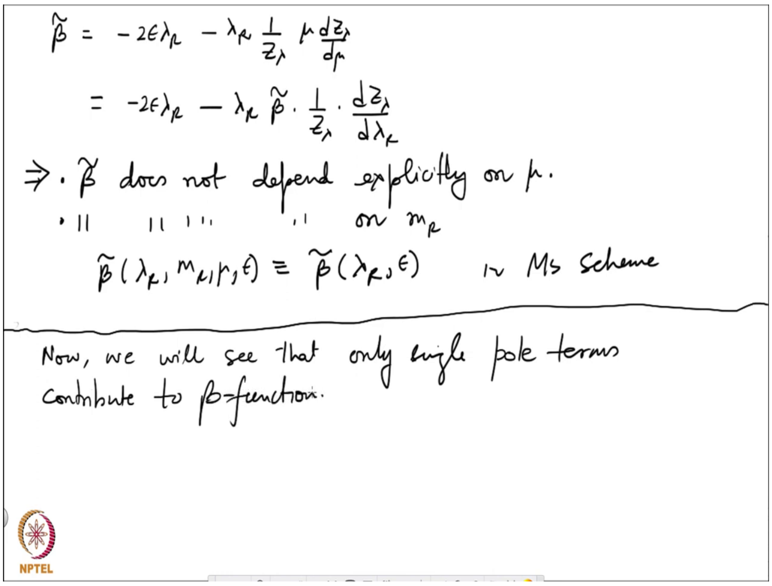

So, we had there we go so, that equation was beta tilde is equal to - 2 Epsilon Lambda R - Lambda R times 1 over Z Lambda mu dz Lambda over d mu 1 over Z Lambda mu dz Lambda over d mu and I have found that 1 over Z Lambda mu dz Lambda or d mu is beta tilde times this Factor. So, it becomes this is beta tilde times 1 over Z Lambda times dz Lambda over d Lambda R not mu.

Now if you look at this equation there is mu dependence which is implicit here through Lambda R again implicit through Lambda R then you have a beta tilde Z Lambda does not depend on mu explicitly. So, the derivative with respect to Lambda R also gives you a function that does not depend explicitly on mu and this of course 1 over Z Lambda does not depend explicitly on mu.

So, when you solve this equation you will get for beta tilde function that does not depend explicitly on mu. So, from here you conclude that beta tilde does not depend x plus settling on Gmail another thing for the same argument from the same argument you also see that beta tiller does not depend does not depend on M R right. Because M R does not appear anywhere here because the only way it could have entered is through Z Lambda but Z Lambda and M S scheme does not depend on M r.

So, beta tilde also does not depend on Mr good. So, yeah so what do I have. So, I started with beta Tila as a function of Lambda R M R and mu and now I have argued that this is and of course would also be a function of Epsilon but now I have concluded that beta tilde is only a function of Lambda R it does not depend on mu and M R is Mu explicitly and there is no depend on M R excellent.

Now yeah now what I will show you is that if you look at the beta function and beta function is important because that is telling you how Lambda R changes as you change mu. So, that you know your a bear coupling constant does not change remember we always we are always trying to ensure that when we change mu. We do not start changing the theory and that is what is ensured by making sure that the bear parameters do not change.

And that has led us to the beta function and beta function is telling you how Lambda R the renormalized coupling constant changes when you change mu. So, as of now I have concluded that beta function is independent of mu and M R and is dependent only on Lambda R and Epsilon and also I can calculate that because here beta tilde can be calculated once you know Z of Lambda because you just take these derivatives.

And Z Lambda how do you get? Z Lambda are the counter terms that you I mean this coupling const sorry renormalization constants that you get by subtracting the Infinities which is arranged by the counter terms. So, when you do a loop calculation to one Loop or 2 Loop or whatever Loop order that will give you the expression of Z Lambda because Z Lambda or Z M these are the things which you adjust to remove the poles.

So, a loop calculation will give you that Lambda take the derivative and that will give you beta Lambda beta tilde that is how you obtain the beta function. So, now I want to show you that the beta function is receives contributions only from the single pole terms. So, what I mean to say is keep this. So, if you look at this expression which is going to give you beta tilde upon taking these derivatives of Z it is it looks like that all the poles will contribute right because Z Lambda has poles simple poles.

Poles of order 2 meaning one over Epsilon Square terms poles of order three and even our Epsilon Cube terms as I wrote somewhere here. So, Z Lambda has all kinds of terms right on our Epsilon 1 over Epsilon square one over Epsilon Q what I am going to show you is that these terms these coefficients A 3 A 2 they do not appear in the beta function what appears is only A 1.

And what is A 1? A 1 is this A 1 gets a contribution from one order Lambda term lowest order calculation will give you a 11 second order calculation will give you a 21 and so forth but the result is going to depend only on one over Epsilon sorry the coefficients of only one over Epsilon will contribute. Meaning if you do a calculation at one Loop 2 loop 3 Loop and so forth what is relevant for beta function is just the coefficient of simple pole terms from one loop from 2 loop from 3 Loop etcetera.

You do not have to include you will not be including this kind of terms that is what I am going to show you now. So, now we will see only single pole terms contribute to Beta function. So, let us see that um.

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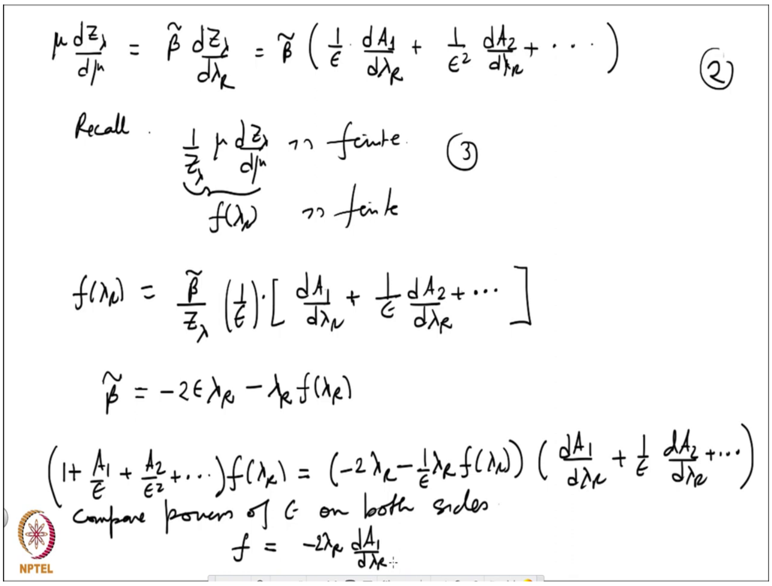

So, let us look at mu dz Lambda over dv this is we have seen just now that there is that this is mu dz Lambda over d mu is beta tilde Z Lambda over Delta Lambda R I'm just dropping out one over Lambda from both the sides. So, it is beta tilde actually I can replace by total derivative it does not matter because there is no other variable in Z Lambda lights just Lambda R. So, let me in fact.

So, what does that give you it gives you beta tilde. Now we are taking the derivatives. So, Z Lambda the first term is one differentiated with Lambda gives you 0. So, that goes away second term is 1 over Epsilon. So, the expression of Z Lambda that I am taking is this one in this form not in this form not this one when they are both the same as there is no difference but this is how I have organized in terms of holes.

So, that gives you one over Epsilon and the derivative of the simple pull term the coefficient of the simple flow plus 1 over Epsilon Square d 2 over d Lambda R + 1 over Epsilon Cube and. So, forth now also recall that we have already argued that 1 over Z mu dz Lambda over d mu is finite and this thing I am going to call as f of Lambda R. So, F of Lambda R is finite that is just a name given to this Factor this object.

So, now you substitute this in one. So, what you get is. So, I am just which one did I call something else one earlier one let me call this 2 let me call this 3. So, I am just putting this here in this equation 2. So, you get F of R. So, I'm dividing the equation 2 by Z of lamb Z of Lambda. So, that gives you f of R sorry F of Lambda R is equal to Beta tilde that is the beta tilde here over Z Lambda.

Now I will pull out one over Epsilon times what you have here is dA 1 over d Lambda R + 1 over Epsilon dA 2 over d Lambda R plus other higher order pole terms. Now we can write beta tilde using F of R. So, beta tilde if you recall it was - 2 Epsilon times Lambda R - Lambda R times F of R sorry F of Lambda R right let me show you if you do not recall here. So, beta tilde is - 2 Epsilon times Lambda R - Lambda R times this is what I have defined as F of R which is finite F of Lambda sorry keep saying F of R.

So, that is good now hmm now I will take this Z of Lambda and multiply on the left hand side and Z of Lambda is 1 plus a 1 over Epsilon plus A2 over Epsilon square and so forth into F of R that is of Lambda R that is equal to. So, this is a Lambda is on the other side Now Beta tilde where beta tilde is - 2 Epsilon times Lambda R and I will divide by this Epsilon. So, it gives you - 2 Lambda R - it is Lambda R times F of Lambda R.

And then because of this division by Epsilon it gives you a factor of 1 over Epsilon times dA 1 over d Lambda R plus 1 over Epsilon dA 2 over d Lambda R and other higher order terms. Now let us compare on both sides powers of Epsilon. So, compare hours of Epsilon on both sides. So, what do you get let us look at the lowest order term which is I mean the finite term of 1 over Epsilon 0 or Epsilon to the zero you get 1 times of f R f of Lambda R.

So, on the left you have f on the right you have - 2 Lambda R times dA 1 over d Lambda R any other term will be either one over Epsilon or 1 over Epsilon square and so forth. So, if I am just comparing Epsilon to the zero term on the left then it is f is equal to - 2 Lambda R times dA 1 over d Lambda R remember what f is f is this piece and F is basically this piece this piece here get that factor here.

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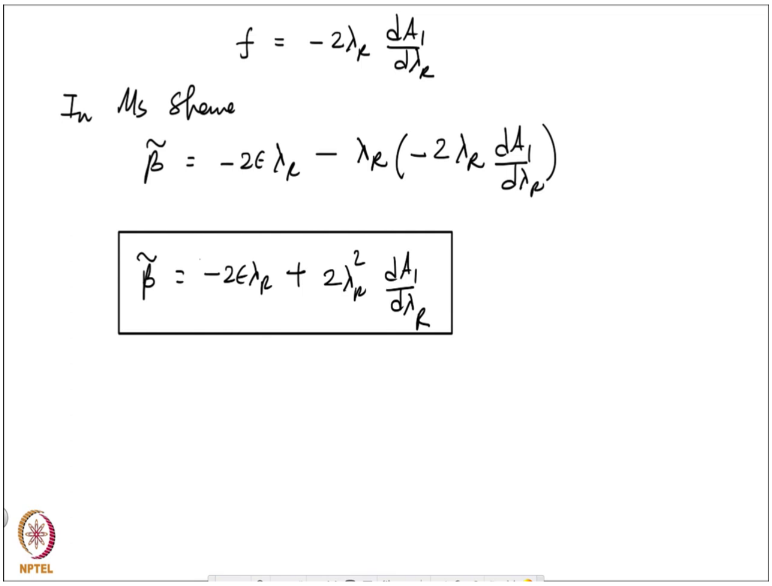

So, what do I get let me write again f is now - 2 Lambda R times dA 1 over d Lambda R. So, in the M S bar in the M S scheme or M S bar also we get beta tilde equal to - 2 Epsilon Lambda R - Lambda R times this Factor right that factor is f - 2 Lambda R check yeah there is a explicit factor of Lambda R here Lambda R times 1 over Z Lambda mu dz Lambda or d mu right that is what we have.

That is what we have calculated 1 over Z Lambda mu d z Lambda over D mu. So, that gives you what was that - 2 Lambda R dA 1 over d Lambda so, beta tilde is - 2 Epsilon Lambda R plus 2 Lambda R Square dA 1 over f Lambda I hope I did not make any mistakes. So, what does that tell it tells that when you are looking at the beta function only A 1 contributes and what was A 1 A 1 was here A 1 was the coefficient of 1 over Epsilon term. So, you see all these other poles of higher order they have not contributed.

So, if you as I was saying earlier if you were to calculate a three loop or for loop whatever for the beta function only the coefficient of simple pole is relevant not the poles of higher orders good. So, we have now some understanding about the renormalization group of group equation for the coupling constant and we also understand what kind of terms contribute to the beta function. We will continue our discussion further in the next video.