Introduction to Quantum Field Theory (Theory of Scalar Fields) - Part 2
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Lecture - 29
Renormalization -Part 1

Okay. So let us begin the process of getting finite answers in quantum field theory for various observables and as I had mentioned earlier this procedure is called renormalization, okay? So let me briefly summarize what we have done so far.

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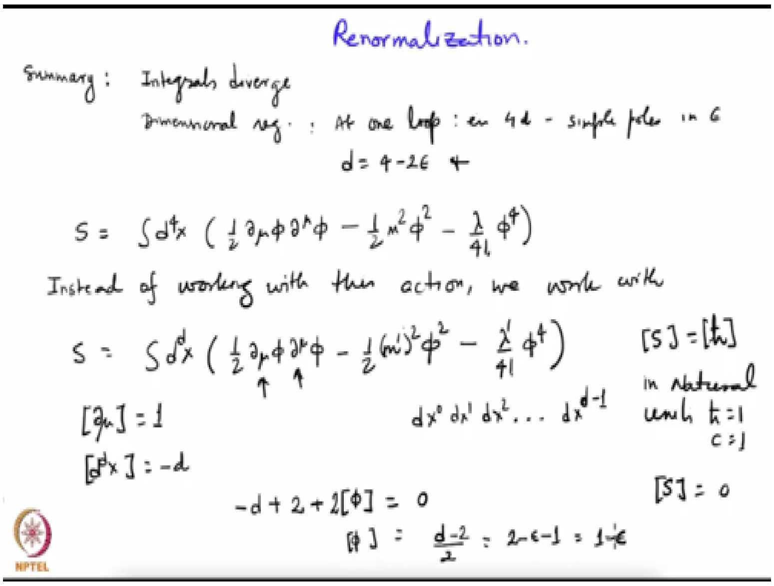

So we have seen that the integrals diverge the moment you go to one loop okay and we have also seen using dimensional regularization how exactly these divergences appear, this ultraviolet divergences appear in the theory, okay? And we have seen that we get simple poles at one loop in 4 dimensions. As you approach 4 dimensions, you get simple poles in epsilon, okay where you take d equal to 4 minus 2 epsilon and let epsilon goes to 0.

So now our next step is to build the dimensional continuation, which we did for the integrals directly into the Feynman rules okay that is what we want to do now. But none of this is going to solve any of our problems as such, but what it will do is it will make it easier to or more systematic to understand the similarities, okay, what kind of similarities will appear in a more systematic manner rather than just looking at the individual Feynman diagrams, okay.

So that is what I want to do. And as I said, this is not going to make anything finite because I have to take epsilon going to 0 eventually, meaning I should work in 4 dimensions and all those integrals will diverge, okay. But nevertheless, this will be the first step towards our goal of getting final answers, okay? So let us begin with that.

So again, my all the work that I will do will be in the phi four theory, whose action is this d 4 x half del mu phi del delta mu phi minus half m square phi square minus lambda over 4 factorial phi to the 4, okay? But now I want to use a different action, not this one, which will be given by this. So instead of working with this section we work with the following.

So I will work instead with integral d dx okay, I am going to d number of space time dimensions, okay? And you have again half del mu phi del mu phi that will not change because after all this is what this is d 0 phi d 0 phi plus d 1 phi d 1 phi plus d 2 phi d 2 phi and so forth. And if you have d number of them let us say d is integer then it is just a more number of terms, right?

So it will have d 0 phi, d 0 phi, d 1 d 1, d 2 d 2 and so forth up to d d – 1 phi, d d -1 phi, okay? But I am going to view it not as a d as an integer but I will let it even take fractional values okay. But we will not worry so much about what exactly it means to have d as fractional okay and worry about those details. Our aim will be just to get Feynman rules which are valid and eventually which give integrals which are of this form.

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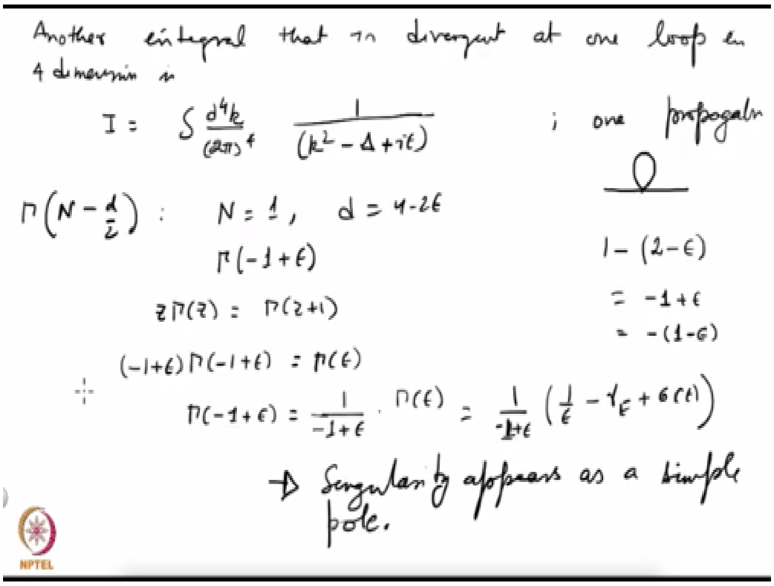

So instead of getting integrals of this form, I will start getting integrals directly of these forms, like these ones, okay.

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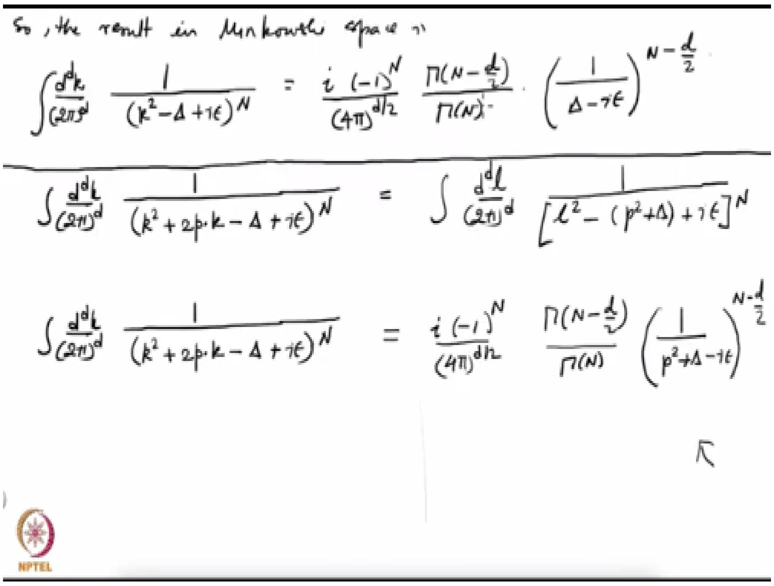

Here, which are already continued. So that is all we are trying right now. Okay, I am not trying to worry about what it means to have an action in fraction number of dimensions, okay or even yeah. So let me write it as minus half, instead of m, I will write m prime. So m prime square phi square minus lambda over 4 factorial, lambda prime over 4 factorial phi to the four, okay?

I have just renamed m to m prime and lambda to lambda prime, okay? Okay, so I want to work with this integral. And now this is in, instead of 4 dimensions it is in d dimensions. So let us do the first simplest exercise of checking the mass dimensions of the fields and these mass parameter and the coupling constant that appear here, okay? And remember to do that, we should remember that S has dimensions of h bar, alright Planck's constant, okay?

They have the same dimension. If you check, look at action okay, and convince yourself that action has mass dimensions of this constant h if you have not already done so. Then because we are in natural units in which h bar is 1, also c is 1, it means that because h bar is 1 a number, it is a dimensionless object okay, in these units, then action is dimensionless, meaning the mass dimension of S is 0 okay, it is m power 0, okay?

So now let us determine the mass dimension of m prime and lambda prime and phi. Now I cannot start that exercise from this term or this term, okay, because here there are two objects whose mass dimensions are not determined i phi, I do not know what is the mass dimension of that okay, because now I am in d dimension.

The mass dimension of phi, what it used to be in 4 dimension is not going to be true. And also in this term, I have another factor m prime square. So I cannot disentangle from here the mass dimension of phi and mass dimension of m separating, okay? So I should go to a term in which I do not have this issue and which is this one, first term, okay?

There is only field phi for which I need to figure out the mass dimension because the dimension of del mu is known. Del mu is just a derivative with respect to a particular coordinate so that its dimension will be 1. So this I know, okay? d dx, the mass dimension of that will be -d right, because you have dx 0, dx 1, dx 2 so and so forth dx and sorry d - 1. So they are each of them has a mass dimension of -1.

So the total mass dimension will be –d, okay? So this has – d, del mu has 1. So -d plus for this it is 1, four that it is 1 so 2 plus must dimension of phi, but there are two such factors. So two times of it. And that all should add up to 0 because action is dimensionless. So each term should be dimensionless. So what does that give? That gives d is equal to sorry mass dimension of phi is equal to d minus 2 over 2.

That is what in general case. But for us, because we are taking d equal to 4 - 2 epsilon, okay. So 4 over 2 is 2 minus yeah, correct. So 4 over 2 is 2 minus epsilon, and then you have -1 coming from here -1. So this is 1 minus epsilon, okay? So that is the mass dimension of phi. Let us now find out the mass dimension of m. So again the same exercise.

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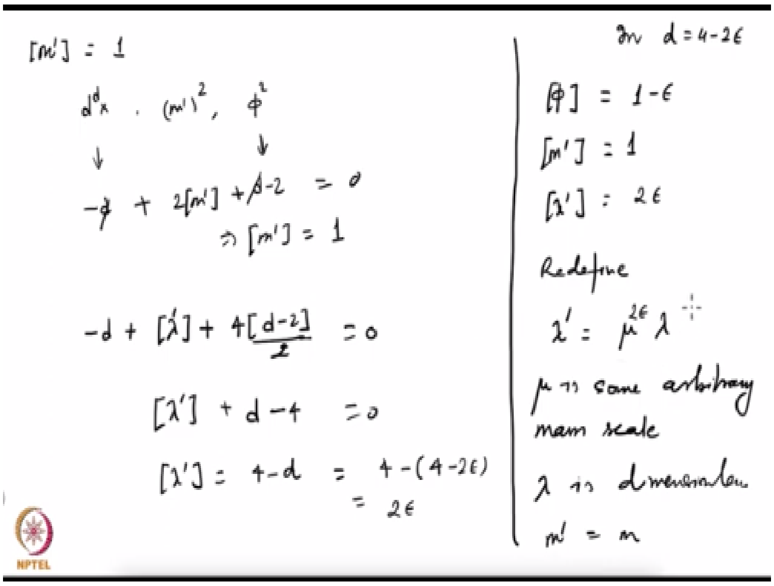

M prime sorry, this you will find that this is 1. Let us check how. So mass dimension of phi, so phi square is here. So it will be d minus 2 over 2 okay and sorry d minus 2 because two factors, okay. So d minus 2 over 2 is for each phi. So two times of it, so it is d – 2. And you have a –d. Okay, so let us write down there. d dx you have and then you have m prime square and then you have phi square okay?

So this is giving you d – 2. That gives you - d and this is 2 times of m prime. And that should add up to 0. And you see that m prime has dimensions of mass so it is 1, okay? From this m prime is 1, mass dimension of m prime is 1, okay? And let us now look at lambda prime. Remember lambda prime or equivalently lambda in 4 dimension it used to be dimensionless, okay?

Let us see what happens when you go to d dimensions whether lambda prime is still dimensionless, okay? So again d dx so it gives you - d plus lambda prime, dimension of lambda prime plus phi to the 4, so 4 times d – 2 over 2 okay because each of phi has dimension d – 2 over 2 from here, okay. So what does that give? This gives lambda prime 4 d over 2 is 2d. 2d - d is d.

And this is -4 or lambda prime is equal to 4 - d minus, okay? So as you see this is correct because if d is 4 then you get 4 – 4 zero. So lambda prime becomes dimensionless in 4 dimension which is something you already knew, okay? And this n 4 - 2 epsilon dimensions becomes 2 epsilon, okay? So lambda prime has dimensions 2 epsilon. Let me record all this here.

Phi has dimension 1 minus epsilon, m prime has dimension 1. That did not change from 4 dimension, it is still the same. Lambda prime has dimension 2 epsilon, okay? So what I will do now is I will redefine. I will just a second. I will redefine this lambda prime such that in this way. Lambda prime I will call it mu to the 2 epsilon times lambda because that is how I want to write lambda prime.

So this is basically defining lambda. mu is some arbitrary mass scale, Okay? So what is the benefit? By doing this lambda becomes dimensionless right, because if you look at the mass dimension of lambda prime this says mass to the power 2 epsilon. So when you take it on the right hand side the mass dimension of this, mass dimension of mu will be, mu to the 2 epsilon will be 2 epsilon.

And because now the mass dimension of left hand side is already taken care of by the factor mu to the 2 epsilon, lambda is dimensionless, okay? And also note that mu is completely arbitrary. This has no relation with any of the physical scales in any problem okay, which you are doing. Because here the purpose of mu is just to absorb the mass dimension of lambda prime okay, so that I can have lambda to be dimensionless.

Also I will instead was calling m prime I will call it m, okay? The reason I put m prime in the beginning because I wanted to check what is the mass dimension and if it also had some, it has changed its mass dimension then I would have similarly introduced some scale okay, mass scale. But because it turned out to be again having the same mass dimension as before I am not changing it, okay.

So that is the notation I want to use now. So I will write that action using these lambda and m.

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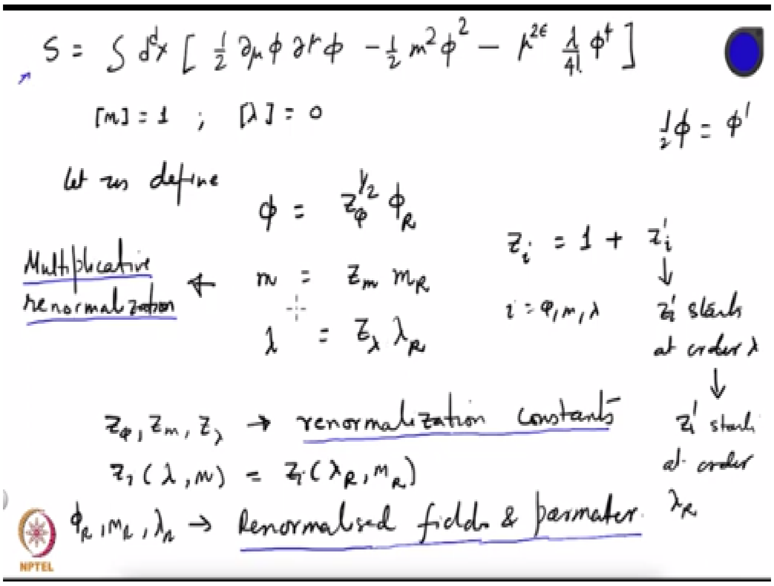

So my action becomes integral d dx half del mu phi delta mu phi minus half m square phi square minus mu to the 2 epsilon lambda over 4 factorial phi to the 4, okay? That is the action with which I want to work, okay? And remember lambda is dimensionless now, okay. So I will just write this here, okay.

Now some time back in I think in the previous course probably I had told you that we can change the, we can redesign the fields, okay? There is nothing secret about half delta mu phi del mu phi. Even the factor half here, this factor half, you could absorb into phi.

You could say I do not want to work with this phi, but I will work, I will define half phi to be let us say phi prime and I will write this action as del mu phi prime del mu phi prime and this half vector will start appearing in these pieces, okay? That is something you could do and we also had introduced some factor of z and redefined. So what I am going to do now is I am going to redefine the fields and these parameters again, okay?

And you will see later why that will be useful. But for now I will do this, okay? So let us define phi to be z phi to the half okay times phi R. And m to be z m times m R and lambda to be z lambda times lambda R okay? Where phi R m Rand lambda R are the fields which are finite and these z's they are so z i where i is phi and m 1 lambda is of this form 1 plus let us call it z i prime okay, where z i prime starts at order lambda, okay?

So I am saying that instead of working with phi, I want to work with phi R, okay. And here is some change in the normalization of the field. Okay, I am just changing the normalizations of the field and of these m's and lambda's. And I have introduced these factors, okay. And I say that these factors are of this form 1 plus that z i prime, okay? Like z m is 1 plus z m prime.

And where the property z m prime has is that it starts at order lambda, okay? It does not have a order one term, it starts at order lambda, okay? It will be lambda times something plus lambda square times something and so forth, okay? So that is the redefinition I do. And these constants z phi z m and z lambda, sorry, these are called renormalization constants, okay.

And these z's are functions of lambda and m okay, and because lambda is a function of lambda R okay, m is a function of lambda because there is a factor of z m; z m depends on lambda and m, but lambda is a function of lambda R, m is a function of m R. So all these z i's are basically functions of lambda R and m R right, because of these relations, okay? An important point is that z i is 1 plus order lambda R terms, okay?

So here I said z i prime starts at order lambda but now because lambda itself starts at order lambda R okay because lambda R times z lambda and z lambda is 1 plus something. So this statement then becomes z i prime starts at order lambda R okay because of this, this relation. So all these renormalization constants are actually functions of lambda R and m R, the way I have defined all this.

So also another thing I should tell you that this, the way I have redefined these fields, you will see soon the nature of the z's that they are going to absorb infinities, okay? And we will see what all this means after we have done the calculation, okay? It is easier to appreciate once the calculation has been done when you have seen everything explicitly.

Then one understands why this has worked or what is the meaning of doing all this, okay? Beforehand it is a bit difficult to appreciate. So yeah, one thing I wanted to say here is that this is called multiplicative renormalization okay, when you have this renormalization constants multiplying the renormalized fields, okay? When I am using subscript R okay to a field or a parameter, I call it as a renormalized field.

So phi R, m R and lambda R these are renormalized fields and parameters, whether it is a mass parameter or a coupling constant, okay? And this because this is multiplicative, the z factors are multiplying the phi R's, this is called multiplicative renormalization. Okay, I should have used some color. Okay, so this is multiplicatively renormalization. These are the renormalization constants.

And these fields and mass parameters these are called renormalized fields and the renormalized mass parameter and renormalized coupling constant, okay? So good I have said a lot of things here but mostly it is just redefining something, okay? Now whether all this will be useful or not we have to see later. But no one can stop me from doing what I have done just now, okay.

There is no way you can stop me from saying that this is not allowed. You cannot say that I cannot write phi as z phi half as phi R, okay because this is allowed. I can redefine the field. Similarly mass parameter I can write it as a product of these two okay where z m is some constant okay, and z lambda is some constant. I can write it this way, okay.

So you cannot stop, not because you are not physically present here to stop me but because there is nothing inconsistent that I am doing. So I am allowed to do this. Okay, so let me put this back into the action and see how the action looks like, okay? But it is still the same action okay, this action. I am not going to change anything. It is the same action I rewrite differently.

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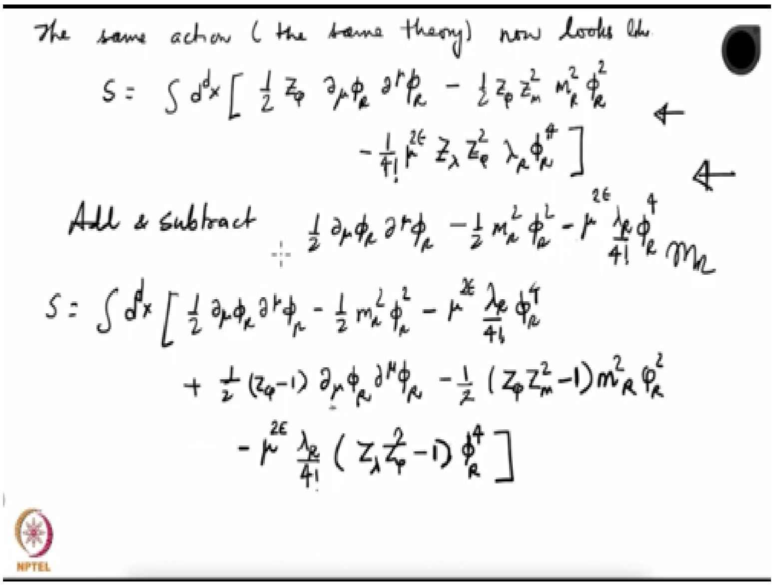

So S the action becomes the same action, the same theory, now looks like the following. Half z phi to the half times phi R, del mu phi R okay, again there is a del mu phi R and that also brings a factor root z, square root of z and those two square roots of z make z phi, okay. So that is first term. Then you have the mass term and that becomes minus half. We had m square so and also phi R square, sorry phi square.

So phi R, phi when I write in terms of phi R, it becomes phi R square and it will bring a factor of z phi, okay? So that is what I am writing here. Then you have m square and when you write it in terms of m R square, it will bring a factor of z m square, okay because z m times m R is what is m. So the square of it gives you z m square m R square, okay.

Then you have minus 1 over 4 factorial, let us stop writing now, mu to the 2 epsilon, okay? Then you can check that you get z lambda times z phi square times lambda R times phi R to the 4, okay? Let us check whether this is correct. Because you had phi to the 4 okay, it will bring 4 powers of square root of z phi that means z square.

And then we had lambda here and that brings z lambda and this is mu to the 2 epsilon, which was present and minus 1 over 4 factorial, okay. So that is how the action looks like now for the same theory. The physical content has not been changed, okay? It is all the same. I have just redefined fields. But that does not change the content of the theory.

Meaning, if you were calculating some observable earlier using S and you do it again, you will get the same result. The only thing is that earlier, you are getting nothing because you are getting infinities and now also it is the same. So you still do not get anything sensible but as far as the theory is concerned it is the same theory, okay? So now this does not look like what we are used to.

So I will now add and subtract the following to this section, okay? Add and subtract the same thing so that I do not alter the action. So add and subtract half del mu phi R del mu phi R minus half m R square phi R square, okay. So this is something in terms of the renormalized fields and renormalized mass parameter, okay? So I will just add and subtract this to the Lagrangian density and my action will become the following.

So S is integral d dx. So I am adding now first half del mu phi R del mu phi R minus half m R square phi R square, okay?

Now I should subtract this from this section right, so that it means that I have not done anything, I have not altered anything in the action. So when I do so I will get the following terms. So I will write this half z phi del mu phi R del mu phi R and subtract from it half del mu phi R del mu phi R, which will give me half z phi, so that is the term which is present.

And from, when I am subtracting this, this gives a -1, del mu phi, del mu phi, the laptop has slowed down suddenly, okay anyway. Okay, let us check whether this is correct. So you have here, if you take this first term and this term and add them up, what do you get?

Because of this -1 this term cancels okay and you are left with only half z phi del mu phi R del mu phi R, which is what you had originally okay, which means that I have not changed the first term as I have written here, okay? Now let us look at minus half m R square phi R square term that I have added. So I should subtract it and I will subtract it from here.

So minus half, it looks like I cannot write at all, minus half, let me try again. z phi z m square minus 1 okay, m R square, I just want to finish this action, not letting me, m R square phi R square, okay. So let us check whether this is fine. Again minus half times minus 1 is half, so half m R square phi R square okay with the plus sign that cancels against this one, okay?

So that I get rid of and what is left with is, what is left is half z phi z m square half z phi z m square with the minus sign m R square phi R square. So that is also correct. Now let me write down the last term. Minus mu to the 2 epsilon okay? Then you have lambda R over 4 factorial, and then z of lambda, z of phi square minus 1.

You can check that this is the right factor and phi R to the 4. This is difficult, okay. So finally I managed to write this. Okay, so that is how the same action which I wrote earlier in terms of phi's looks like when I write in terms of phi R, okay? Now let me tell you why I have done this before I proceed further and anyway I cannot proceed further in this video because I am unable to write at all.

So see the way it was written here, in this on the top here. Okay, now it works. Okay this is working now. Okay, so the way it was written here in, the way the action was written on the top had del mu phi R del mu phi R, that is the usual kinetic term but then you have a z phi. But z phi is a function of lambda R and m R which are your constants, okay?

So I do not want to put it this way but I want to put it in the canonical form the way usually we write the kinetic term without coupling constants, okay. And that is why I have written it this way because once I write it this way, the propagator in the theory will be exactly the same which I had earlier, okay. Instead of m I will have m R but other than that everything looks the same.

So that is why I have added and subtracted that term so that I in the action I have the parts, the parts which were present earlier, okay. I think I missed this one. So not only add and subtract this, I also wanted to add this one, this one I forgo, okay. So you see that I have added and subtracted the entire action that we used to have.

The only difference is that instead of phi I have phi R but looks exactly the same, okay? So here also okay, which is good, because when I am writing the Feynman rules, I will have exactly the same Feynman rules with the only difference being that the names would have changed for the parameters and mu to the 2 epsilon will appear in the vertex. But other than that, everything is the same.

And there is another difference that will come here is, you have more terms, so you have more vertices, okay? Let us look at this term okay? This has z phi; z phi starts with 1. So that 1 cancels 1 and it is left with z phi prime, which I had earlier defined okay, which starts at lambda or lambda R. So this term would contain a factor of lambda. So that is a coupling term, right?

This is a kinetic term. Here, it does not contain any factors of coupling, but here it does. So that is a coupling term. That is a interaction term, which has derivatives here, but it is still an interaction term. Similarly this one, okay? This is also quadratic like this one this term.

But again, it contains factors of coupling okay, coupling constant lambda R and that is why this is also an interaction term and the same is true for this one also okay, because they are factors of lambda and any way this is not even quadratic, that is quartic. So that is also a coupling term, okay.

So compared to the theory written in terms of phi and m and lambda, this has more number of vertices in the theory okay, more number of interaction terms in the theory. So now I will give you the Feynman rules and you can immediately see that these are the correct ones. So let us, first let us write down, let us write down the propagator.

Maybe I should, okay let me take it to the next page. Okay, here and then here I will just paste it. I hope that this will be also easily visible, okay.

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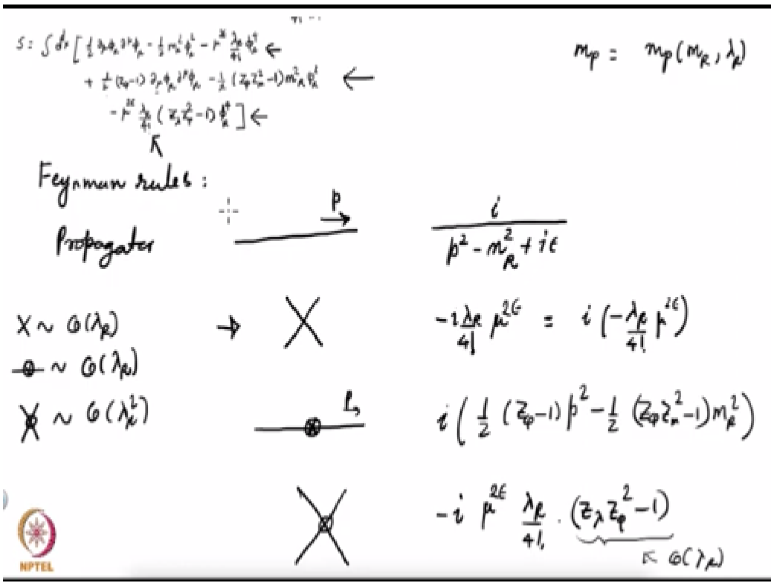

So that is how the action looks like now, okay. So what are the Feynman rules then? Feynman rules. Or maybe I will, okay now I cannot move it. So Feynman rules. Remember it is still the same theory, I have not changed really anything okay, physical content everything is the same. Redefining certain things cannot change the physics content. So how about the propagator?

Propagator is given by these first two quadratic terms and this is of exactly the same form as before so I also get the same thing as before. But now it will be m R square. And remember m R is still a mass parameter, okay? It is not a physical mass of anything, because you remember how we find the physical mass of particles in the theory.

You can look at the poles of Green's functions or let us say 2 point function, okay. And look at the pole and that pole will give you the fiscal mass. The physical mass will be a function of lambda and R, sorry, lambda and m which in our case now has become functions of m R in lambda R. So physical mass will be a function of m R and lambda R, okay.

So m p would be, m p's physical mass will be some function of m R and lambda R, okay? That is what we can say but m R is not itself the physical mass, okay. So that is one of the Feynman rules. Then as before we have this vertex coming from minus lambda R over 4 factorial phi to the R, but you have a factor of mu to the 2 epsilon also. And remember, recall that the vertex here was minus i lambda over 4 factorial earlier, that is what we had minus i lambda over 4 factorial.

So instead of lambda, I will have lambda R, okay. And I also have factor m mu to the 2 epsilon okay, which is basically i times minus lambda R over 4 factorial mu to the 2 epsilon, okay. So you see that minus lambda R over 4 factorial is this minus lambda R over 4 factorial and this is a factor of phi, okay.

If you go back to the first course you will see how this factor of i came out okay, in the Green’s functions because we had written everything in terms of how to say vacuum expectation values of fields and then there was an exponential and e to the i of that interaction term and that is from where the i was coming okay, when you expand the exponential.

So it is that i but now unlike the fact that you had only this term in the expression of Green’s function okay, you also have these terms, okay. So which means that these will also give you vertices. So let us look at this, these two terms now. There are two terms in this line, so let us take both of them and del mu phi R del mu phi R that will give you a factor of p square, okay?

When you take the derivatives it pulls out factors of i p okay? So you will get the following. So this vertex I am drawing it this way because it has only two fields phi’s that appear phi R and phi R okay and so it has two external lines. This one also has phi square, so two external lines, okay? And you get R times half z phi minus 1 p square minus half z phi, sorry, z m square minus 1 m R square.

Let me check. Okay, looks fine. Z m square minus 1 m R square, okay? So you see you have in this theory, I mean it is the same theory, but the way we have rewritten things, you also have a vertex which is a two point vertex, okay? We never had such a vertex before, we always had a 4 point or a 3 point vertex, but it is the first time we are seeing a 2 point vertex, and these are usually denoted by putting a cross, okay?

So all those vertices in this action which carry effectors of z okay, or z m or whatever, they will be denoted by this cross, by putting a cross on the vertex. And there is one more that you have which is a 4 point vertex coming from this last line, okay. And I have put a cross because this also has factors of z's. And it is a 4 point vertex because you have 4 fields, okay.

So 4 lines you can connect that is why it has 4 legs. So this is e to the sorry i minus R times 2 epsilon lambda R over 4 factorial times z lambda z phi square minus 1, okay. So these are our Feynman rules with which we should work. And remember we have a, we always have momentum conservation at each vertex when we are drawing Feynman diagrams and that momentum conservation is now d dimensional, okay?

So it is really 2 pi to the d and delta d and the sum of all momenta that are entering into that vertex, okay? So that is the only other thing that you have to keep in mind. Now given that we have these Feynman rules let us first figure out whether these three new vertices or sorry these two new vertices they are of the same order as the 4 point vertex here or are they different. So this one this vertex is of order lambda, lambda R okay.

So this vertex is of order lambda R, okay. Now let us look at this one. This vertex has z phi and this term. Let us look at z phi. So z phi is 1 plus z prime phi okay or whatever 1 plus some function that starts at order lambda R. So that one cancels and whatever you have here starts at order lambda R, okay? So this two point function clearly starts at order lambda R, sorry this 2 point vertex not 2 point function.

So this is also vertex that is of order lambda R. Now let us look at this 4 point vertex which you have in this way of writing. Here it already has an explicit factor of lambda R, right because you have a factor of lambda R here. Then z lambda and z phi they start at order 1. So z lambda z phi square, the lowest order term in this will be 1. So that 1 will cancel this 1, okay.

And the second term, the next term in these will be of order lambda, lambda R, okay? So you see that this factor contribute something of order lambda R, okay? That order lambda R term times this lambda R makes this vertex order lambda square, order lambda R square, okay? So this is a vertex that is of higher order compared to these two vertices.

So if you are calculating something only up to order lambda R, then you do not need to use this last vertex because that contributes at order lambda R square, okay? So one has to be also careful in using this because these are of different orders, okay. Now we have all these Feynman rules and we have modified or redefined the fields in terms of renormalized fields and renormalized mass parameter and renormalized coupling constant, okay.

But nothing has really changed. You still calculate whatever you want, you will get infinity okay, because just redefining things cannot help. So we will see how to get finite answers in the next video.

But here I want to also bring your attention to the fact that even though you have several interaction terms in this section like this phi to the 4 term, and this interaction term which has del mu phi R del mu phi R coupling and this phi R square and phi R 4, these coupling constants are not independent, okay? They are not all different, they are not all independent.

They all are completely determined in terms of lambda R and m R okay. So because you know a long back I was telling that whenever you write a interaction term in the Lagrangian you have to include a coupling constant and these coupling constants are independent of each other in general unless certain symmetries force them to be related, they are independent.

And here you see that it is just because we have rewritten things in certain way that is why these couplings are still know are given by these two coupling constants, lambda R and m R. The number of coupling constants has not changed. It is the same theory, okay. So all this algebra will be very useful, all these calculations will be very useful now in setting up the procedure to get finite answers okay, and that is what we will take up next.