Introduction to Quantum Field Theory (Theory of Scalar Fields) - Part 2

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Lecture - 28
UV Singularity Structure in Dimensional Regularization

Okay let us continue from where we left last time.

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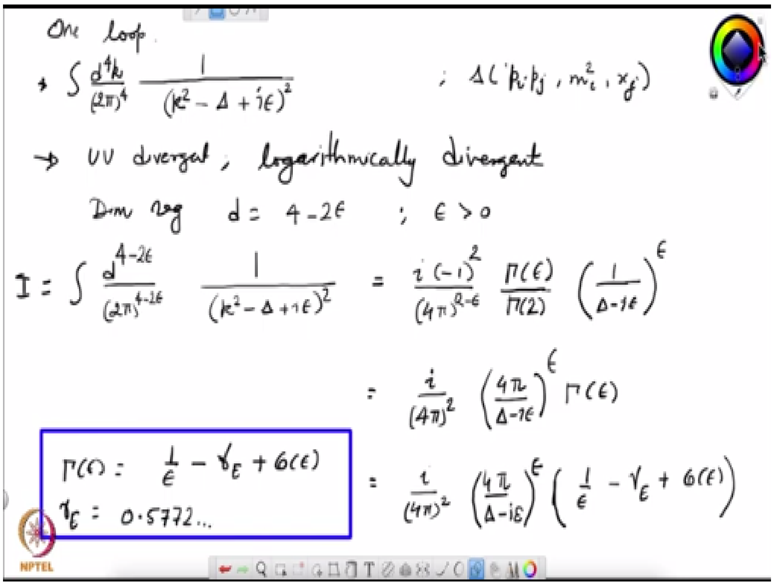

Okay, so we will work at one loop. And suppose we get the following integral from some Feynman diagram that you are calculating at one loop which is integral d 4k over 2 pi to the 4, 1 over k square minus delta plus i epsilon square whereas I have repeatedly said delta will be a function of the external momenta, actually the dot products, okay these are all in cascade of the masses that appear okay and the x i's the Feynman parameters, because these could be different let us put j, okay?

So you do, suppose you have to integrate this one you see that this is divergent in ultraviolet region because you have four powers of k in the denominator, k square is squared and four powers k in the numerator, okay. So this will diverge logarithmically if you were to put a cut off lambda. Okay, let us see how things work out when you are doing dimensional regularization, okay?

So this is UV divergent and we call it logarithmically divergent. Okay when one says something is logarithmically divergent or quadratically divergent, one is imagining a cut off okay. Because if you use dimensional regularization, you are not going to get a logarithmic dependence on the cut off. We will see what you get okay, but it will not be logarithmic.

So the moment you say logarithmic or a quadratic, these things are these statements assume that you are working with a cut off regulator. Okay, so what we will do is to evaluate this because this is divergent, it is infinite. We see no matter what I do to this integral it remains infinite, okay? So I do a regularization and go to d equal to 4 minus 2 epsilon okay where epsilon is positive, okay?

So when epsilon goes to 0, you recover this integral okay, in that limit. So epsilon is some positive number so that the number of dimensions are lower than 4 because 4 minus 2 epsilon is something small, okay? There is no, it is not necessary that you put this 2 okay, you can work with 4 minus epsilon also. It is just a choice that I have made, okay? But if you do not like you do not have to.

Okay so with this your integral becomes the modified integral or the regulated integral becomes d 4 minus 2 epsilon over 2 pi to the 4 minus 2 epsilon 1 over k square minus delta plus i epsilon square, okay? And now this integral is convergent because number of dimensions is less than 4, okay? Okay, so let us evaluate this to see explicit form of the singularity in epsilon going to 0 limit okay, or d going to 4 limit.

So we have evaluated this integral already okay, and I am just going to put d equal to 4 minus 2 epsilon in the result. So let us go back and see somewhere here, I should have that result.

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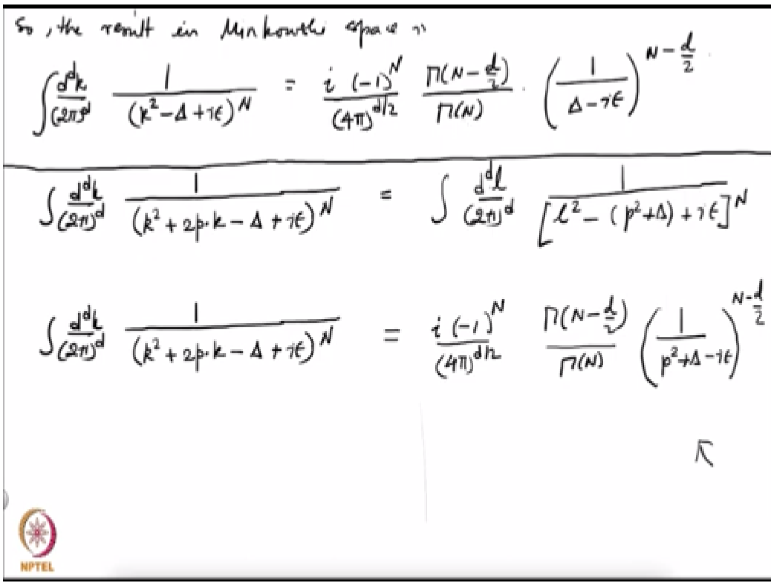

This one okay? d d k over 2 pi d 1 over k square minus delta plus i epsilon to the N. So capital N here in the present integral that I am evaluating n is 2, which means you have 2 Feynman propagators in the diagram, and here I have put 4 minus 2 epsilon. So I will take this result and put d equal to 4 minus 2 epsilon okay, and capital N equal to 2 that is what I am doing.

Let us, okay I will not copy and paste it there, you can verify that what I am doing is correct. Okay, so in that formula that I showed to you, I will put d equal to 4 minus 2 epsilon. So i times it has a minus 1 power capital N, capital N is 2, this 2 okay, this one. Then you have 4 pi power d, which is 2 minus epsilon, sorry d over 2 which becomes 2 minus epsilon okay? Then you have gamma of, we show you again.

So in the numerator you have gamma of capital N, which is 2 minus d over 2. d over 2 is 2 minus epsilon, okay? So 2 - 2 that cancels and it leaves only epsilon, okay? Because this minus cancels the minus epsilon you get. So that makes it gamma of this known factor, I want to write that one before that. So this and then you have now everything was fine, everything was good.

What did I remove? Yeah, gamma of epsilon and then in the denominator you have gamma of capital N, which is gamma of 2 okay? And then we have 1 over delta minus i epsilon and you will see that it is useful to keep this i epsilon okay? And this exponent becomes epsilon okay? It had the same argument, the power here was the same as the argument of gamma function and that is why they both are epsilon.

Okay, now gamma 2 is one factorial, which is 1, so that is what I will put. And this time I can write as i over 4 pi square, okay. Now I am still left with 4 pi to the minus epsilon okay, which goes in the numerator and becomes 4 pi to the epsilon. But I also have this 1 over delta 1, delta minus epsilon, so I will include it here, okay? And then you have gamma of epsilon.

All the kinematic dependence is contained in delta, okay. And this what you remaining things depend on epsilon only. Okay, so where is the singularity in all of this? When I put epsilon to 0, of course this factor is nonsingular, something to the power 0 is 1, so that is not going to give you any divergence. The singularity is in here, in gamma of epsilon. So we should know then how gamma function behaves near 0, okay?

So how gammas z behaves when z approaches 0, because that is the limit we are interested in and here is the result. Maybe I should, yeah here itself. So gamma of epsilon behaves as 1 over epsilon minus Euler gamma. This is again a gamma. It is called Euler gamma plus terms that are of order epsilon, okay? So you see that it is singular, as you already know.

And the behavior is like this, okay? Gamma of epsilon is 1 over epsilon minus Euler gamma where Euler gamma is a number which is 0.5772 okay and of course you have more terms, more digits in this. And then you have terms which are of order epsilon okay?

So you see that this integral, this integral at one loop with two propagators, the Feynman diagram having two propagators okay, it is going to give you a pole a simple pole because gamma of epsilon is going to give you 1 over epsilon. So it provides you a simple pole, okay? So what looks like as logarithmic divergent thing when you use a cut off appears as a simple pole when you use dimensional regularization okay?

So let us put this here, i over 4 pi square okay, 4 pi over delta minus i epsilon. This epsilon is different okay and this epsilon is different. Maybe I should use, let us this is this epsilon, which is coming from Feynman prescription. I will write as this one and this one, this dimensional regulator as epsilon. Gamma, so what is gamma of epsilon? It is 1 over epsilon minus Euler gamma plus order epsilon, okay?

Now I should work with this expression. I will write it again. So did I give a name to this integral; no. So let us call this I.

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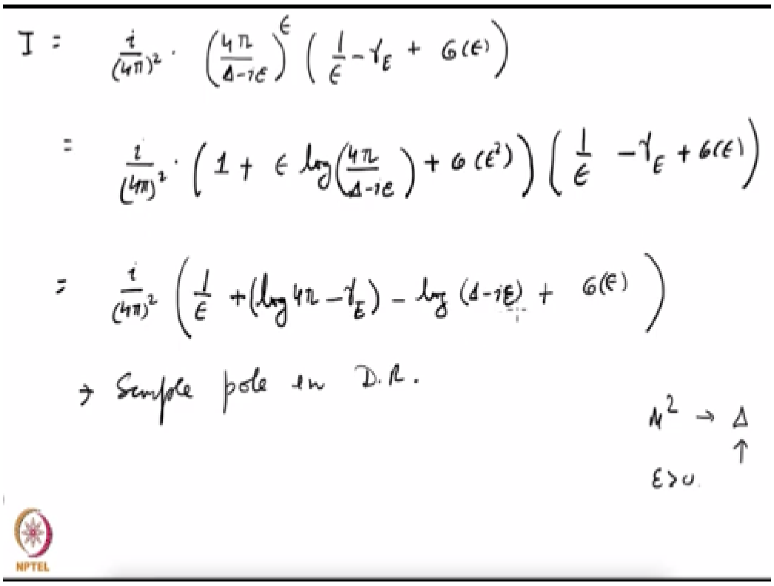

I is equal to, I have found that it is i over 4 pi square times 4 pi over delta minus i epsilon, already, 1 over epsilon minus Euler gamma plus Euler epsilon okay? So this is i over 4 pi square times this one let us expand in epsilon okay, in powers of epsilon. So the lowest order term is 1 plus epsilon times log of 4 pi over delta minus i epsilon okay.

And then you have order epsilon square term okay, this expansion I can write easily. And then you have 1 over epsilon minus Euler gamma plus order epsilon terms. Okay, now let us carefully look at all the terms. Okay, let us look at first the order, let us look at first the most singular piece, okay. You have 1 over epsilon.

So the most singular term is generated when you multiply this with the lowest order term in epsilon which is 1 here, okay? And there is one way, only one way. So that gives you 1 over epsilon. Now let us look at terms of order epsilon to the 0 okay because the next thing in powers of epsilon is epsilon to the 0, meaning a constant okay? So how do you generate that?

When you multiply this constant here with that constant there, that gives you one such term. So let us write it here, minus gamma E. And the other way you generate it is when you multiply this one over epsilon term with the order epsilon term, okay? So that will give you log 4 pi over delta 1 minus delta minus i epsilon. So it gives you log of 4 pi minus log of delta minus epsilon.

So I will write it separately, okay? And now any other term that you generate okay, so sorry this is all order epsilon term, order epsilon to the 0 terms, constant terms, okay? Constant meaning not constant, but order epsilon to the 0 terms. So this is what you have got at this rate, and then you whatever is left is order epsilon terms here, okay? There are many ways in which you can generate order epsilon terms.

For example, by multiplying this epsilon log 4 pi with minus gamma. That will be an order epsilon term. Okay, so that shows you the explicit form of singularity. So as I said, you get a singular or simple pole okay, in dimensional regularization for the integral that we have looked at.

And note that no matter what kinematic factors appear here okay, in the delta, you are going to get, in addition to the singularity 1 over epsilon, log of 4 pi minus Euler gamma. So that constant is going to appear, okay? This will be of, we will have to, there will be things which I will be saying later about this constant.

So let us keep it in mind that when you are working at one loop, these constants are going to appear in precisely this combination, okay? And the kinematic dependencies contained in this logarithm of delta, okay? And also note that you have delta of minus i epsilon, okay? So it is useful to keep this minus epsilon. I did not drop it here, okay? I have not dropped the i epsilon's.

I am being careful in keeping it. And now you see why it is useful. Because see, the delta could be a negative quantity, right? Because you know that I have instead of using M square, when I was in Euclidean space, I switched to delta okay, because delta is not necessarily positive definite. So log of, you could have log of a negative number here, negative quantity here okay, depending on the P i's that you have, okay?

And in that case, it is important to know what is the value of log and that you can know only by knowing whether you are above the cut or below the cut, the branch cut of the log, okay? And for that you need to know the sign of i epsilon, okay? You need to know on which side you are of the branch cut. And that is why you need this i epsilon correctly. And then this is whether minus i epsilon or plus i epsilon becomes crucial.

And since we have carefully carried all this I know that it is minus i epsilon okay where epsilon is positive, this epsilon, okay? So one has to be careful with the arguments of log. So you better keep track of the i epsilon's, okay? So as far as this integral is concerned, this one okay, when I go to, when I use dimensional regularization and go to epsilon going to 0 limit, this is the form of the integral I get, okay?

Now let us look at another integral which you will get and you know that it is divergent and has only one propagator. So here I took a Feynman diagram which had two propagators so that I get to this form, okay? And we already talked that there is one more integral that will be divergent at one loop okay, which you get from tadpole. So let me draw that one or self-energy.

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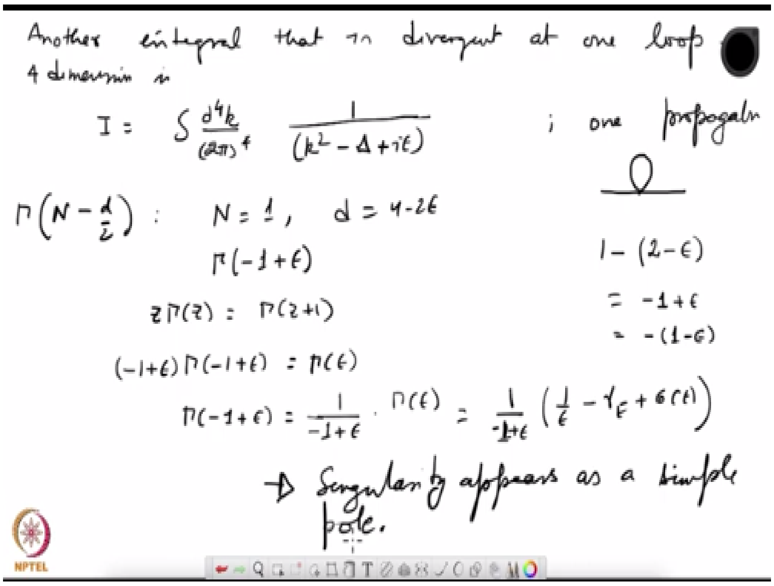

So second integral. Let us look at another integral that is divergent at one loop in 4 dimensions is this one. Here the power is 1. Okay, so only 1, one propagator and previous one was with two propagators. And you know that if you have more than two propagators, let us say you have three or four or whatever, then this integral will be convergent because if you have three here, then four powers in the numerator and six powers in the denominator, so it is convergent.

And that is why I am interested in looking at only these two integrals here, okay? Good. So this is what kind of diagram is this? This comes from these kind of diagrams okay, only one propagator here. So this is a self-energy diagram. So let us look at what kind of singularity this will give. So going back to the integral here. Okay in this I will put capital N equal to 1 and d equal to 4 minus 2 epsilon okay?

And I will, this time I will not worry about this factor because I have already shown you how to use how to work with this. So let me just show you the kind of singularity that you get which comes from gamma N minus d over 2, okay? So N minus d over 2. So capital N minus d over 2. That is the gamma function that you get where capital N is 1, d is equal to 4 minus 2 epsilon.

So what do you get? You get N is 1 minus d over 2 which is 2 minus epsilon which is minus 1 plus epsilon which is minus of 1 minus epsilon, okay? So we have gamma of minus 1 plus epsilon. That is what appears in this integral which will give you a singularity, okay? Now you might already be aware of that the gamma function though it is analytical it is analytic everywhere in the complex plane except for the negative integer values in the origin okay?

So you see when epsilon goes to 0 this goes to gamma of minus 1 which is singular, okay? And I can easily find out the nature of singularity using the result that I have already given to you. Where was that, this one, okay? Let us try to box this one. No, not sure whether it works. Okay, I have forgotten how to do it. Okay, good. So this is what I am going to use now.

And another very useful result that you should always remember is this, is this that z gamma z is gamma 1 plus z or z plus 1, let us call it this way, okay? And I will use this now. So I need gamma of minus 1 plus epsilon okay. So if I use minus 1 plus epsilon times gamma of minus 1 plus epsilon, then it is gamma of minus 1 plus epsilon plus 1. So that is gamma of epsilon.

And this one I know. So from this I can easily determine minus 1 gamma of minus 1 plus epsilon as 1 over minus of minus 1 plus epsilon times gamma of epsilon okay, which is simply minus 1 plus epsilon times 1 over epsilon minus Euler gamma plus order epsilon square, order epsilon terms. Now this denominator okay, it will start with 1 or minus 1 and then it will have order epsilon terms, okay?

So expanding this 1 over minus 1 plus epsilon does not change the order of the pole, okay. So it gives, because it starts with 1 and when you multiply with 1 over epsilon, it gives you 1 over epsilon. So you see again that this integral also is giving you simple poles. But if you had used a cut off regulator you will not get a logarithmic divergent, right because this is not log divergent.

But when you are using these dimensional regulator you are still getting a simple pole, okay? So singularity appears here also as a simple pole. Okay, and this is because we are working in 4 dimensions and at 1 loop. If you were to go to higher loops then this will change okay, these will not be simple poles, okay? And as I said, if N is greater than 2, capital N is greater than 2, more number of denominators then the integral is convergent.

So I have then exhausted all the divergent integrals that are going to appear at one loop in 4 dimensions, okay? So I have completed, I have completely exhausted that set. And now I will start looking at dimensional regularization and how to proceed with that, okay? And I think I should do it in the next video, not in this one. Okay, fine. So let us meet in the next video then.