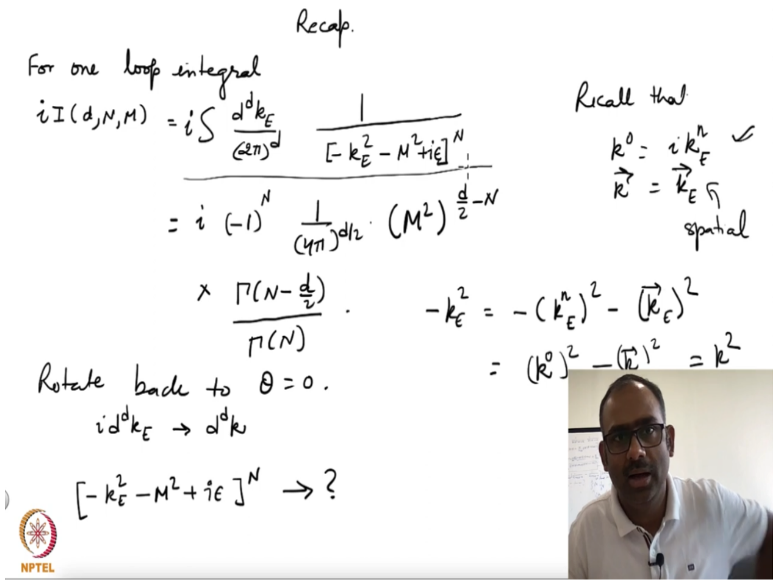
Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
Prof. Anurag Tripathi
Department of Physics
Indian Institute of Technology - Hyderabad

Module - 9
Lecture - 27
Few More Feynman Integrals

(Refer Slide Time: 00:22)



So, last time we wrote down the following integral which was at one loop. So, for one loop integral, we had to find this integral I d N M. So, d is the number of dimensions; N is the number of propagators; and I will show you what M is; and this was in Euclidean space after the Wick rotation. And if you remember, we had dropped the i epsilon because it is not needed as we are working in the Euclidean space, but if you go back and search, it would be a +i epsilon.

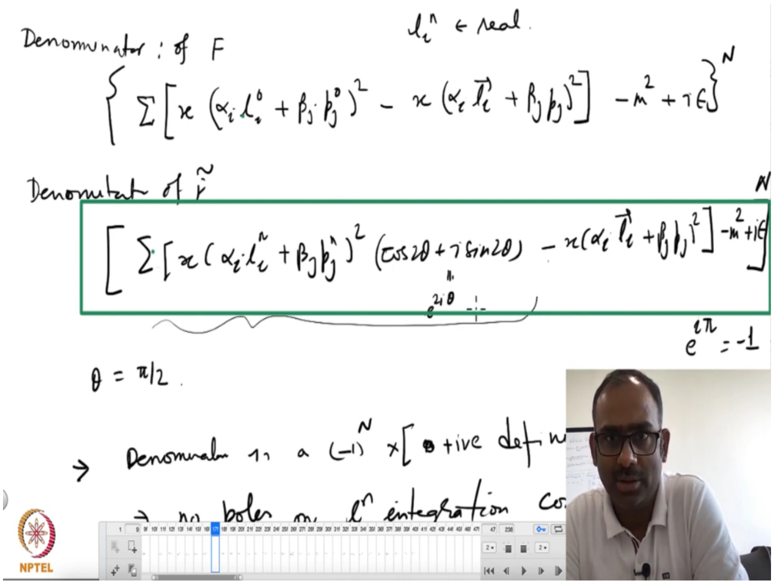
If you keep it, then it will be +i epsilon. It is not really needed here but it will be useful when we continue back to the Minkowski space. That is why I have kept it back; but as far as this integral, it is not needed. Now, also recall that l nought was i times l Euclidean. In fact, it was n that is the nth component. So, zeroth component in the Minkowski space, we call it the nth component of the vector l Euclidean and the remaining special components here, they are identical.

So, l E, the special components of Euclidean vector l, they are same on both sides. I should be writing k, not l. These are the special components. Also because of this factor of I, when I am doing one loop integrals, I am always going to get a factor of I. So, the original integral which is in Minkowski space, that will lead to this integral together with a factor of I, so, let me include that here; and we have seen that the result is this.

So, this I is here and we evaluated this last time and we found that it is -1 to the N, where N is the number of propagators; M square depends on the external momenta. Let me just write M square; I will then explicitly write the arguments, times; that was the result. Now, when you are doing these Feynman integrals, you would want the result in the Minkowski space directly.

So, what I am going to do now is continue this result to the Minkowski space by rotating back to theta equal to 0. You remember what that theta is; where did that go? So, somewhere here; I am not finding it; let us check. We had written an expression with e to the 2 i theta. We had multiplied these, the time components with e to the i theta and; let us see if I can find; here.

(Refer Slide Time: 05:20)

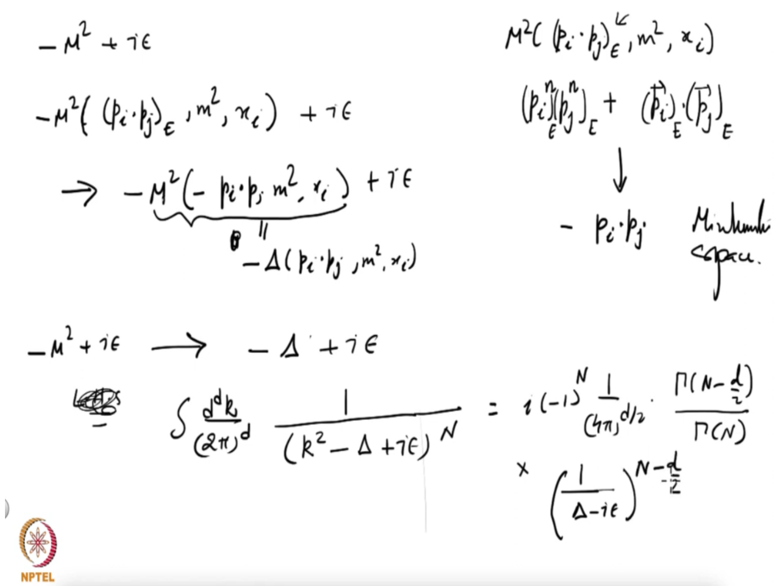


This is the denominator that we had at that time, after Wick rotation, and this is for arbitrary theta but if you rotate by 90 degrees, you get the Wick rotated integral. So, I am going to put theta equal to 0 again. That is what I am doing. So, I am going back to the original integral. So, rotate back to theta equal to 0. So, what will happen? i d d k Euclidean; there is only one factor of i because only one of the components was a time component of the d components.

So, that goes back to d d k. This is Minkowskian on the right-hand side. What happens to the denominator here in the integrand? So, minus k E square minus M square plus i epsilon power n, that goes to what? That is what we should determine, and that is easy; minus k E square is minus k Euclidean n square that is the nth component, and then you have minus k Euclidean square that is the vector.

So, when you use this, you get that minus k E square. This becomes k E n that is of i k 0, i square is -1, so, that gives this -1 and it gives you k 0 square minus k vector square, and this is k square. And so, as far as this is concerned, I know that minus k E square should be replaced by k square when I rotate back. How about -M square?

(Refer Slide Time: 08:02)



So, let us look at -M square plus i epsilon, and remember what is M square; M square is a function of p i dot p j; these are Euclidean and m squares and also depend on x i's, these 5 main parameters. So, let me write down -M square p i dot p j; M square depends on these objects plus i epsilon. That will go to what? It will go to -M square. So, p i dot p j will be p i n p j n; this is still Euclidean, so, I will put Euclidean here.

These are the nth components and then the other components. Now, when you rotate, as we saw just now, this will become - p i 0 p j 0. So, this upon rotation will give you - p i dot p j. I pulled out a minus sign. Now this is a Minkowski space. So, -M square of - p i dot p j m square x i plus i epsilon. Earlier I was writing M square because M square was positive definite, so, it made sense to write M square using the square as a symbol to remind that, that is a positive object, but now once you have gone to Minkowski space, this M square with these arguments is not necessarily positive definite.

So, I will change, I will redefine it to make a better symbol. So, this I will define as minus Delta. So, what is Delta? Delta is the continued version of M square to theta equal to 0. Whatever the functional form of M square is, that is what I am calling Delta. So, -M square plus i epsilon after Wick rotating back, it gives you minus Delta plus i epsilon. And what is Delta? Delta is same function M square continued to Minkowski space.

So, what do we get then? The integral becomes or better still here, this one. So, when I look at this, after continuation, this has become the following; and there is a factor of i that will come from here. So, this i will be gone because i d d k E goes to d d k. So, you get the integral on the left-hand side of; so, this is the left-hand side and this result I am calling right-hand side.

So, you get integral d d k over 2 pi to the d 1 over k square minus Delta plus i epsilon to the N. This is after continuing that integral. So, now I can take the result that I had, the special result here, this one, and also continue it and see what I get. So, all I have to do is take this M square and continue it and I already know what M square becomes; M square is what I call Delta.

Just replace wherever in M square you have these things and continue it to complex, continue it to the Minkowski space. So, this will be the replacement. And here I will get the same; everything will be same except M square will be replaced by Delta. So, you get i times -1 power N 1 over 4 pi d over 2 gamma N minus d over 2 over gamma N times 1 over Delta minus i epsilon.

You will see that you get a -i epsilon because I am pulling out that minus sign. So, here, I will just explain why there is minus i epsilon. So, here you see, this is -M square. So, whatever is in front of this minus is what enters here. If you keep that i epsilon also, then it will be minus of M square minus i epsilon. So, here, M square minus i epsilon and it is this M square minus i epsilon which has become Delta minus i epsilon. So, this is the result. I wish I had written it more nicely which I think I should do. Let me write it again.

(Refer Slide Time: 15:02)

So, the result in Minkowski space is

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-\Delta+i\epsilon)^N} = \frac{i(-1)^N}{(4\pi)^{d/2}} \frac{\Gamma(N-\frac{d}{2})}{\Gamma(N)} \cdot \left(\frac{1}{\Delta-i\epsilon}\right)^{N-\frac{d}{2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+2p\cdot k-\Delta+i\epsilon)^N} = \int \frac{d^dl}{(2\pi)^d} \frac{1}{[l^2-(p^2+\Delta)+i\epsilon]^N}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+2p\cdot k-\Delta+i\epsilon)^N} = \frac{i(-1)^N}{(4\pi)^{d/2}} \frac{\Gamma(N-\frac{d}{2})}{\Gamma(N)} \left(\frac{1}{p^2+\Delta-i\epsilon}\right)^{N-\frac{d}{2}}$$

So, we have shown that. So, the result in Minkowski space is d d k over 2 pi to the d. I am going to use this result again and that is why I want to write it nicely, k square minus Delta plus i epsilon power N, where Delta function depends on all those variables which I listed; i -1 to the N over 4 pi d over 2 gamma N minus d over 2 over gamma N times i epsilon N minus d over 2.

So, good, you can directly use these results when you are doing Feynman integral instead of first going to Euclidean space and then doing continuation because that is exactly what is done here. Now I will also give you another result which will be useful which is the following. So, I want to evaluate now a different integral 1 over k square plus 2p dot k minus Delta plus i epsilon to the N. What is the result for this?

And it is very easy to do; but before I show you what the result is; actually I should not even show you, you should do it yourself; but look here, the Delta contains the dot products, Delta is a scalar, so, it contains the dot products of all the external momenta. Now, here, p is some momentum in the problem and external momentum in the problem or even a linear combination of them and the p is sitting here.

Now, this integral if you see, this is a Lorentz invariant integral. Now, the result can only depend on p because k is anyway dummy, this is integrated over, so, result can only depend on p and the only function that you can construct given only one momentum which is a scalar is p square. So, whatever I do, the result of this will be a function of p square. So, I leave it as an exercise to just do the change of variables by first you complete the square and then do a change of variable and write it as the following.

So, this will become d d l over 2 pi to the d. I am just giving you the result of what will happen after completing the square and changing variables. This you should check. Now, you already see that after doing those steps, p square has appeared in here. Now, instead of having 2p dot k, you have p square plus Delta. So, clearly when I integrate, the result will depend on p square and Delta, and this is equal to; let me try doing it, pulling it out; so, this is a result you already know; we had on the previous page.

Think of this as Delta prime and then use that result and you get i -1 to the N over 4 pi d over 2 gamma N minus d over 2 over gamma N and you have 1 over p square plus Delta minus i epsilon N minus d over 2. That is the result. So, you have these two important results. Now, when you are doing quantum electrodynamics or theories which have spin, then you can get powers of k in the numerator also. Even though we are not doing it in this course but let me still recall that result.

(Refer Slide Time: 21:37)

$$I^\mu : \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{[k^2+2p\cdot k-\Delta+i\epsilon]^N} \propto p^\mu \qquad \frac{1}{\not{k}-m}$$

Evaluate

$$\frac{\partial}{\partial p_\mu} \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2+2p\cdot k-\Delta+i\epsilon]^N} \qquad z\Gamma(z)=\Gamma(z+1) \qquad \not{k}=k^\mu\gamma_\mu$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{-N}{(k^2+2p\cdot k-\Delta+i\epsilon)^{N+1}} \cdot (2k^\mu)$$

$N \to N-1$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2+2p\cdot k-\Delta+i\epsilon)^N} = \frac{1}{2}\cdot\frac{1}{(-N+1)} \frac{\partial}{\partial p_\mu}\left[\frac{i(-1)^{N-1}}{(4\pi)^{d/2}} \frac{\Gamma(N-1-\frac{d}{2})}{\Gamma(N-1)} \times \left(\frac{1}{p^2+\Delta-i\epsilon}\right)^{N-1-\frac{d}{2}}\right]$$

$$= -p^\mu \left[\frac{i(-1)^N}{(4\pi)^{d/2}} \frac{\Gamma(N-\frac{d}{2})}{\Gamma(N)} \left(\frac{1}{p^2+\Delta-i\epsilon}\right)^{N-\frac{d}{2}}\right]$$

So, for those of you who have already seen; some are familiar to some extent with quantum electrodynamics will know that when you are looking in the propagator, you get 1 over k slash -m where k slash is just k mu gamma mu; and when you write the propagator with k square minus m square in the denominator, then you get things like k slash in the numerator which is just k mu gamma mu.

So, you will have integrals which will involve k mu in the numerator. So, let us look at one of those. So, I want to integrate d d k over 2 pi to the d k mu over k square plus 2 p dot k minus Delta plus i epsilon. What will be the result? So, before I give you the result, let us see what it would look like, at least the p dependence. So, again using the same argument, the result will depend on Delta.

Delta is a scalar which is made out of all the invariants and m squares and whatever and x i's. Now, this integral carries the index mu that is a Lorentz index mu. So, whatever you get after the integral should also carry that index mu because it is a Lorentz index. This object transforms like a 4 vector or like a d vector because we are in d dimensions, so, it should carry that index mu. Now, there is only one object that can carry that index mu.

Since k is dummy, it will not appear and you will have p. So, result should be proportional to p mu, p mu times something. So, it should be proportional to p mu. That much we can say. And now let us figure out what it is. So, the trick is, let us call it I; let me put an index mu. Let me take a derivative of this integral with respect to p mu. So, I evaluate del over del p mu where mu is the lower index now of this integral; of not this integral but without k mu.

So, let us evaluate. So, this integral, what I have here in this line, this we already know. That we have seen here. We have already evaluated this one. So, that is known. I can take a derivative with respect to p mu. So, I take the expression and take the derivative, so, I can do that easily; but now let us see what it generates when I do this integral. So, this will give you integral d d k over 2 pi to the d.

So, this is basically, this will give you -N over k square plus 2 p dot k minus Delta plus i epsilon power N + 1 into 2 k mu. Differentiating this will give you 2 k mu. So, that is what you get. Now, I will just, because I want to evaluate this integral with N and what I have gotten is almost the same integral other than some constants but with the power N + 1, so, I will just change N to N - 1 in this expression.

So, what I get is, so, change, let N go to N - 1. So, I get integral d d k over 2 pi to the d k mu over k square plus 2p dot k minus Delta plus i epsilon power N. This is what we want to evaluate, this object. And this is, we divide it by 2, 1 over 2. This N, I am changing into minus N - 1, so, this will become 1 over -N + 1. Let us find. Then you have a derivative of with respect to p mu of the result of after integration which is i times -1 power N - 1.

So, what I am doing is I am taking this result and putting N = N - 1. That is what I am doing. N - 1 1 over 4 pi d over 2 gamma N - 1 minus d over 2 over gamma N - 1 times 1 over p square plus Delta minus i epsilon power N - 1 minus d over 2. And if you use Z gamma Z equal to gamma Z + 1, so, here you have gamma N - 1, this is minus of N - 1. So, N - 1 times gamma N - 1 will give you gamma N.

So, use that and this will give you finally p mu. Taking a derivative from this thing, it will give you; so, the final result I am writing, you can check; - p mu; so, as I said, the result will be proportional to p mu, so, that is what we have got, times i -1 power N over 4 pi d over 2 gamma N minus d over 2 over gamma N 1 over p square plus Delta minus i epsilon N minus d over 2. That is what you get.

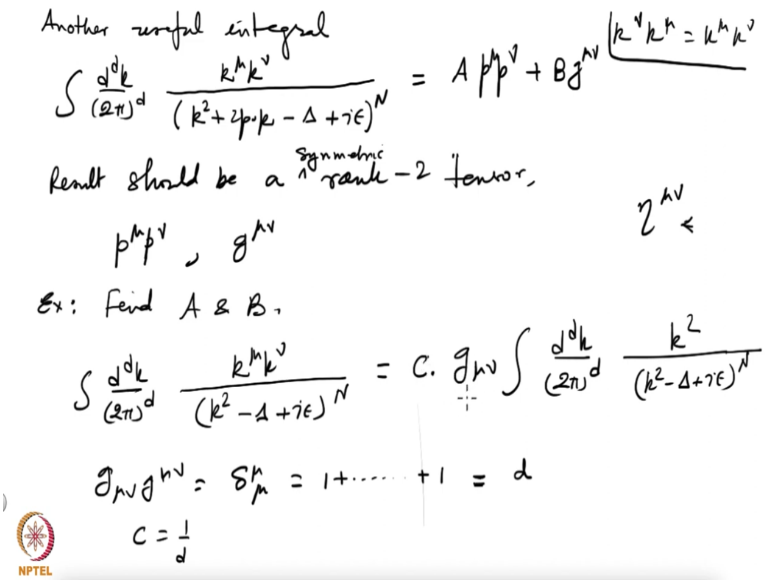
So, what you have here in the square brackets is exactly what we had here; it is identical. So, the result is, this is the explicit result but if I were to relate the 2 results, then it is the following.

(Refer Slide Time: 30:06)

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2+2p\cdot k-\Delta+i\epsilon)^N} = -p^\mu \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+2p\cdot k-\Delta+i\epsilon)^N}$$

So, integral d d k over 2 pi d k mu over k square plus 2 p dot k minus Delta plus i epsilon power N is equal to p minus p mu times the integral without k mu, the scalar integral, d d k over 2 pi d; these are very useful results, we should keep them safe. So, this is nice. We have several; what happened? There is an extra page here; I will delete it later. Now, that we are doing these integrals, I will also give another result to you. So, we will also encounter integrals of this kind when you are doing loop integrals.

(Refer Slide Time: 31:13)



d d k over 2 pi d k mu k nu; so, you may have 2 factors of k in the numerator and then the same denominator. So, let us see even without, even before doing any explicit calculation, what the result would look like as far as its mu independence is concerned. So, this integral is a rank 2 integral; the result should be a rank 2 tensor because you have 2 indices mu and nu but those indices cannot depend on; k is dummy, so, those indices can only appear on p; that is the only vector you have here; there is nothing else that you have.

And the dependence should be such that the integral that you get, the tensor that you get is symmetric. See, if you interchange mu with nu and nu with mu, integral does not change because k nu k mu is same as k mu k nu. These are numbers; k 0 k 1 some number, k 1 k 0 same number. So, you can interchange. So, the result should be a rank 2, a symmetric rank 2 tensor. That is one thing I know from here.

Now, what are all possible rank 2 tensors available to us when we are doing this integral? So, given that the result can depend on p, we have this rank 2 tensor available to us which is symmetric under interchange of mu and nu. There is another rank 2 tensor which is always available, which is g mu nu or better I should call it eta mu nu, but let me call it g mu nu. So, that is your metric tensor which is always available.

So, whatever this result is, it has to be some constant which I will call A times p mu p nu plus some constant B, and that constant B will depend on Delta and up all those other variables n, d, all those things; but the general form has to be this. So, I will leave it as an exercise to find out the explicit answer using these kinds of tricks. So, please do this; you know the trick of differentiating and you can try to figure this out.

I will instead of giving the answer for this one, I will do a simpler version. This you figure out. Exercise: Find A and B. So, now what I will do is, I will put p = 0 so that I can evaluate this integral in a slightly simpler setting. So, p is gone. So, of course the result will not depend on this factor because p is 0 now. So, let us look at this integral d d k over 2 pi d k mu k nu over k square minus Delta plus i epsilon power N.

So, I already know that this result should be proportional to g mu nu because that is the only thing that will be left now; but there is another thing that you see is that if I contract this with g mu nu, this g mu nu times k mu k nu is k square. So, this integral becomes d d k over 2 pi d k square over this factor. Now, let me write it this way. So, it is g mu nu; I claim that this can be written as the following; let us see whether the claim is correct.

g mu nu is fine; that I have already convinced you, but this integral is some function of, it is some constant; it involves all these Deltas and d's and N's but some constant. So, it is consistent with whatever I said earlier that the result should be proportional to g mu nu, but whether it should be k square here and whether the integrand is correct or not, let us check. So, if I contract on both sides with g mu nu, left-hand side will give you k square over this.

If I contact with g mu nu, this g mu nu times the other g mu nu will give you some constant and then exactly here k square as you had here. So, it is clear that the form of the integral is correct, that I should get g mu nu times this kind of an integral but what is not necessarily correct is some constant which could appear here, some number which could appear here. So, let us find out that.

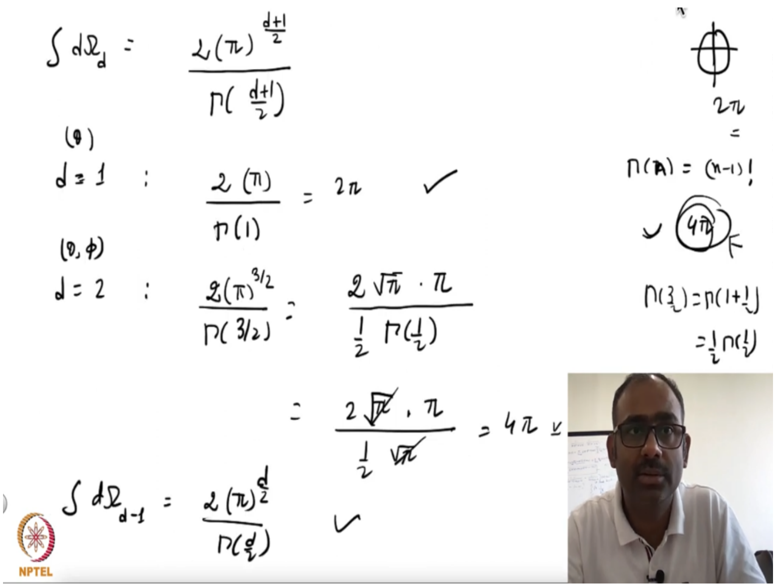
Contracting on both sides with g mu nu, it will produce k square here, and here you already have k square, but g mu nu g mu nu will be what? Is Delta mu mu which is, if you are in integral number of dimensions 4, 5, 6, 10, whatever, 20, 100, then this will be just 1 + 1, those many times. So, if d is integer, then this is just d; and even when d is not integral, it is consistent to define g mu nu g mu nu to be equal to d.

So, I will not go in more details about these issues of dimensional regularisation but it is consistent to define g mu nu g mu nu to be d and the other things related to when you have spinner fields, but that is not what is bothering us in this lecture; but if you are interested, you should look at the book by John Collins on renormalisation. So, anyway, now let us fix the factor of C. So, when I multiply g mu nu both sides, it will give you d.

So, clearly, C should be 1 over d; that is what I should use. So, I will remove that C and put a 1 over d. Let me write that result also. This is; so, now here, so, what I have shown you is integral d d k over 2 pi d k mu k nu over k square minus Delta plus i epsilon to the N is equal to 1 over d g mu nu; indices should be up because on the left-hand side you have the indices up; d d k over 2 pi d k square over k square minus Delta plus i epsilon power N.

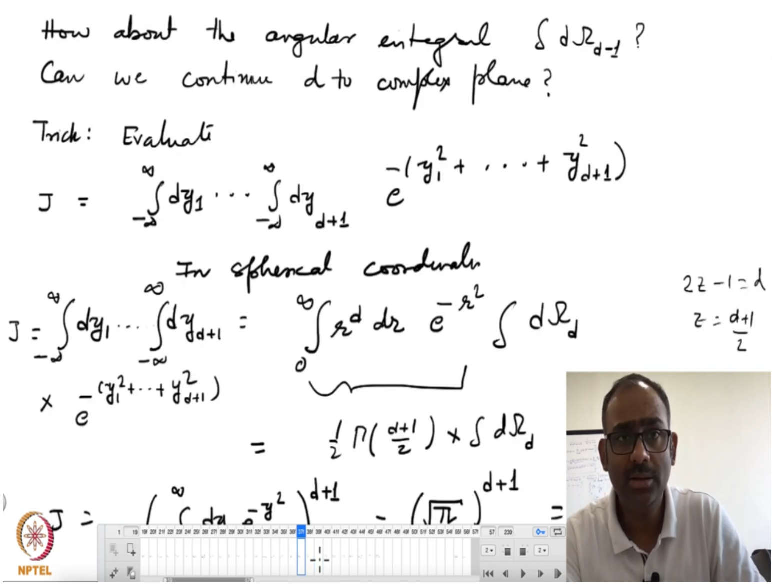
So, these are some useful integrals that you will encounter when you are doing loop integrals and I will stop here. There is one more thing I wanted to tell, which I forgot. This is something I should have told you much earlier, here, after this.

(Refer Slide Time: 41:27)



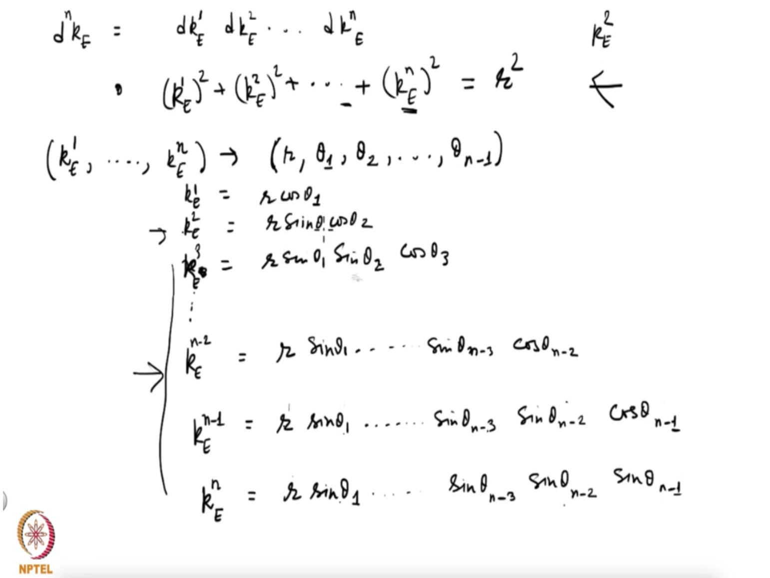
I will add a slide here.

(Refer Slide Time: 41:44)



So, here we had looked at the volume integral and we found explicit expression, but I will also tell you how to parameterise.

(Refer Slide Time: 41:55)



So, we had d d; or I think at that time I was using n; d n k E, n dimensions; this is d k Euclidean 1 d k Euclidean 2 and so forth d k Euclidean n. So, this is n-dimensional. So, you have n variables, k E 1 k E 2 so forth up to k E n and we want to use spherical coordinates because most of these integrals have no dependence on angle; they only care about the magnitude of k E. They involve things like this.

So, if you look at k E 1 square, k E 2 square up to k E n square, then that is a constant, so, r I will, or maybe small r, r square. So, if you have spherical symmetry, then we will use this radial coordinate and the sum of these squares will be equal to the radius square. Now, there are total of n variables. One of the variables I can choose as r. So, I am going to do a change of variables from this set to another set of which one of them is r and I claim that I can introduce n - 1 angular variables.

That is something I can do. You can parameterise differently also but one of the parameterisation is that you have one length in the problem, one coordinate which has dimensions of length in the problem and all others are dimensionless but this is not the only way, you can do different parameterisations where maybe 2 of them has dimensions of lengths and the others are dimensionless like angles, but this is one of them which is allowed, and let me give an explicit expression which tells you that you can make such a choice.

So, here is the explicit result. So, k E 1 r cos theta 1; k E 2 is r sin theta 1 cos theta 2. Let me, even though I am going to write exactly the same thing, I want to write it in a different order, it will be useful. So, k E n, the nth component, I will parameterise as r sin theta 1 sin theta 2 so forth sin theta n - 3 sin theta n - 2 sin theta n - 1. So, if sin theta 1 sin theta 2 and so forth up to here, then k E n - 1 I am parameterising as r sin theta 1 sin theta 2 and so forth sin theta n - 3 sin theta n - 2 cos theta n - 1.

I will soon tell you why I have done it this way. k E n - 2, the n minus 2th component I will parameterise as sin theta 1 sin theta n - 3 cos theta n - 2 and you continue like this. Then you have x 3 is equal to r sin theta 1 and then sin theta 2, then cos theta 3; not x 3, k E 3; and k E 2 is equal to r sin theta 1 cos theta 2 and k E 1 is equal to r cos theta 1; and let us see whether this is a parameterisation that is going to work and I want to have this.

So, if I sum the squares, I should get r square, a constant and that would mean that I am using a spherical coordinate system. So, let us take k E n and square it and take k E n - 1 and square it and add the two. So, I am adding this one and the previous one. So, you will have r square sin square theta 1, squares of all these times cos square theta n - 1. And here k E n when you square, you get exactly the same factors r square you got here, here also you get r square; sin square theta 1, sin square theta 1; up to sin square theta n - 2, sin square theta n - 2; but here you had cos square theta n - 1 and this one has sin square theta n - 1.

So, when you add the two, the squares of these factors in both, they are common. So, you can pull them out and you are left with cos square theta n - 1 sin square theta n - 1. That adds up to 1; cos square theta plus sin square theta is 1. So, what you are left with is k E n - 1 square plus k E n square will be just r square sin square theta 1 up to sin square theta n - 2. Now you add that with k E n - 2 square.

So, when you do that, you see that again the same factors; here is r square, here is r square; here you get sin square theta 1, sin square theta 1 up to sin square theta n - 3, sin square theta n – 3 so that in the sum you can factor this out and you will be left with cos square theta n - 2 sin square theta n - 2, and adding these two makes 1. So, you see, every time you are adding a square of the previous coordinate, the last angle is dropping out.

So, if you continue this way, you will eventually arrive here at, at this stage you will arrive at, when you have added the sum of all these squares up to k E 2 square, you will have r square sin square theta. That cos square theta 2 would have already been taken care of by this sin square theta 2. So, at the end you will be left with this plus all the remaining will give you r square sin square theta and k E 1 square is r square cos square theta 1.

Summing the 2 will give you r square and thus we get this one. So, this is the parameterisation in n dimensions when you are looking at spherical coordinate system. That is something I wanted to tell you which I had missed. Then, so, we have done most of the integrals that are usually required and now I can start discussing more about renormalisation and getting rid of infinities. See you in the next video.