Introduction to Quantum Field Theory(Theory of Scalar Fields-Part 4)
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Lecture - 26
Explicit Evalutation of Feynman Integrals
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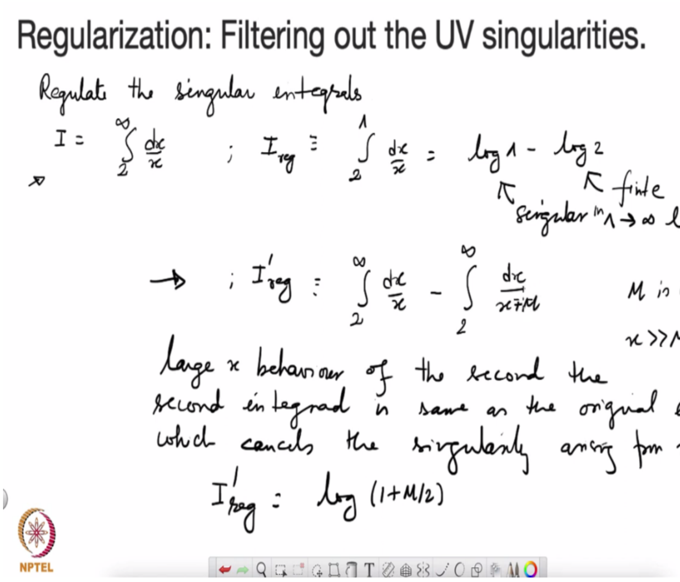

We have seen that the Feynman integrals for loop diagrams are singular and as I said earlier that we will have to use a regularization procedure to filter out the ultraviolet singularities. Now this step is not going to make; I am not going to solve the problem because at the end I should remove the regulator and when I do so the singularities will be present, but nevertheless this is the first step so let me show you how to do this.

And first I will do this in the simplest setting which means I will not be using Feynman integrals, but some ordinary simple integrals and tell you what the idea is. So, we want to regulate the integrals. An integral I which is integral 2 to infinity dx over x. So, this is going to serve as a toy example for Feynman integrals. Now you see that this integral is singular because upper limit infinity gives you divergence.

This integrates to a log of x and when you put the upper limit it gives you log of infinity. So, what I do is I look at a modified integral which I will call I regulated which is defined to be 2 to some capital lambda dx over x. This gives you log of lambda - log of 2. So, here we have regulated the integral as lambda goes to infinity you recover back this original integral I. So I regulated I subscript reg goes over to I when you take lambda to infinity.

But when you put lambda to be a large number but finite this is how this singularity appears. So, there are two terms. One is finite so that is finite contribution and that is the singular contribution a singular going to infinity limit. So, this is an example of a regulator. I will give you another example again let us look at the same integral and I will regulate it differently. So, I can regulate by the following method.

So, I will write it as I prime regulated which is the following. So, I modify the integral to this. This is the original one from this I subtract let us see why this will regulate the integral. Lower limit 2 is not a problem nothing diverges at x equal to 2, but when x goes to infinity this is diverging it has 1 over x and that becomes singular and the integral of it is also singular.

But here what I have done is I have introduced a cut off M; M is a large number. So, when x becomes very large, but M is finite M is large, but finite. Now when x becomes large very, very large compared to M then the contributions from this regions of large x are same in this integral and this integral. So, you can effectively drop M when x is very large and this gives you dx over x.

So, large x behavior of the integrand is as 1 over x and this one also has 1 over x. So, you see that you will not get a singularity because of this upper limit. So, large x behaviour of the second term integrand is same as the original integrand which cancels the singularity arising from is x going to infinity. So, having d x 1 over x - 1 over x + M that gives you a regulated integral.

And you can explicitly check that I prime regulated is log of 1 + M over 2. So, now I should take M to infinity why because when I take M to infinity then this integral vanishes because it is 1 over infinity which is 0. So, this integral just drops out and you are left with only the original integral which we wanted to determine. So, this is a intermediate procedure where I have regulated the integral.

And when I put M going to infinity this gives you a divergent contribution. So, it is a divergent result which you already know it should be divergent, but for any finite M, but large this is a finite thing. So, these are kinds of regulators that we use in the Feynman integrals also. So, these kinds go by the name of Pauli–Villars and these ones by momentum cut off this variety where you put a cut off on the momentum that is called momentum cut off.

And this where you change the propagators it is called Pauli–Villars. So, maybe I should tell you so momentum cutoff you have already seen.

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Pauli-Villars $\quad \dfrac{i}{p^2-m^2+i\epsilon} \rightarrow \dfrac{i}{p^2-m^2+i\epsilon} - \dfrac{i}{p^2-M^2+i\epsilon}$

Dimensional regularization:

We can continue the dimension 'n' to the complex plane.

One loop integral : $\displaystyle \int_0^1 dx_1 \int_0^1 dx_2 \int_{-\infty}^{\infty} \frac{d^n k_E}{(2\pi)^n} \frac{\delta(x_1+x_2-1)}{[-k_E^2 - M^2(x_1,x_2, p_i\cdot p_j, m^2)]^2}$

Multi-loop integrals :

$$\int_0^1 dx_1 \cdots \int_0^1 dx_N \frac{d^{nL} k_E}{(2\pi)^{nL}} \frac{\delta(x_1+x_2+\ldots+x_N-1)}{[-k_E^2 - M^2(x_i, p_i\cdot p_j, m_i^2)]^N}$$

N = # of propagators , L = # loops

n = # of dimensions

So, I will just mention Pauli–Villars. So, here you will replace the propagator where M square is a large number which you have to take to infinity at the end and the argument I gave before applies. So, these are some regulators that you can use, but the best one to use is dimensional regularization and that is what I am going to talk about now dimensional regularization which means that I will take a given Feynman integral.

And reduce the number of dimensions low enough that the integral becomes convergent. You remember that all these divergences for example here in this Euclidean integral the divergence comes because of these powers in the numerator becoming larger than the denominator or equal to the denominator powers of l in the denominator, but if I reduce this dimension sufficiently then I can make the integral convergent or finite.

So, that is what we do. So, we take the number of dimensions to be small enough that the integral is finite, but then I should be able to take the n equal to 4 limit because all the integrals have to be done in 4 dimensions, but thing is that I should be able to continue continuously these numbers, these dimensions I should be able to do to give the expressions of the integrals even for fractional dimensions.

So, what I am going to show you is that this can be done meaning we can continue to I will shall show you that we can continue the dimensions n to the complex plane and this is what I plan to show you explicitly in this lecture. Let us see how it is done. So, we have seen that one loop integral Feynman integrals they are of this form. Earlier I had written only with x not to Feynman parameters x 1 and x 2 because I had used up the delta function already in writing down the expression.

But if you keep the delta function then you get the following. So, here I have instead of putting n equal to 4 I have put n, but then n is discrete here n could be 2 or n could be 3 or n could be 4 or n could be 1 depending on in how many dimensions the given integral converges. So, that is the form of the integral it depends on all these parameters, the M square depends on all these parameters.

And all the momenta that you have the dot products and M square if you have more than one variety of particle then you may have different masses here M 1 square M 2 square and so forth. So, this was for a particular example where you had only two propagators that is why you have a 2 here and we also saw that multi-loop integrals have exactly the same form except for the fact that there are few more Feynman parameters that appear because you have many more propagators in the problem, but the general structure is identical.

So, you get d x 1 dx I am using capital N for the number of propagators. So for each propagator you have a Feynman parameter then we saw that it has to be d n L k Euclidean L is the number of loops and here I have omitted many other constants because I want to only focus on the relevant integral this M square also depends on all these parameters let me write x i p i dot p j m square or let us say I allow for different masses.

So, I will write m i square of course for the theory which we are looking at there is only one mass so do not have a subscript i here, but in general you will have that N is the number of Feynman propagators p i's, are the external momenta mi are the masses L is the number of loops N is the number of dimensions wherein we need results for N equal to 4. So that is the structure maybe I should write here N is number of propagators and L is number of loops, n is number of dimensions.

So, that is the general structure. So, let us look at this k E integral because that is the one which gives ultraviolet divergence. So, its structure is it has some powers of k in the numerator and some different power of k in the denominator. So, I will concentrate on an integral of this form.

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$$I(d,N,M) \equiv \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[-k_E^2 - M^2]^N} \quad ; \ d \text{ is a positive integer.}$$

$$(k_E^1, k_E^2, \ldots, k_E^d) \longrightarrow (\underset{\uparrow \text{ radial coordinate}}{r}, \underbrace{\theta_1, \theta_2, \ldots, \theta_{d-1}}_{d-1 \text{ angular variables}})$$

$$d^d k_E = r^{d-1}\, dr\, d\Omega_{d-1}$$

$$[-k_E^2 - M^2]^N = [-r^2 - M^2]^N = [-M^2\rho - M^2]^N = (-1)^N (M^2)^N [\rho+1]^N$$

Define $r^2 = M^2 \rho$

$$r^{d-1}\, dr = (M^2\rho)^{\frac{d-2}{2}} \frac{M^2\, d\rho}{2}.$$

So, I define I d N M to be the following d d k E over 2 pi d times 1 over – k E square – M square power N. So, here if you are looking at one loop integral then d is equal to n, but if you are looking at a multi loop integral then d is equal to n times L that is the thing here you have to take care of the factors of I when you go from Minkowski to Euclidean by taking k 0 along the imaginary axis.

So, each k 0 will bring a factor of I. Here in this case when you are looking at one loop you get only one factor of I so that is what you have to multiply here to convert to Minkowski space, but here you will have more number of factors of I because you had several loop integrals l number of them so you will have L factors of I the complex I, but anyhow let us look at this Euclidean integral.

And try to see if we can write down the integral in a manner where d is allowed to be complex. See the way it is written right now d is integer because here n is number of dimensions which we are saying it is discrete it could be 1, 2 pr 3. L is some number of loops so n times L is positive integer. So, at the moment here d is a positive integer if you look at the way this integral is written.

So, let us go ahead with this and see if I can write in form of this integral or an expression for this integral which can be analytically continued to complex d dimensions and the simplest way is to do the integral. Let us now evaluate this integral the result will be of course a function of M and d and N and that is why I have put these labels here in the argument of the integral.

So, you see that the integrand does not depend on the different components of the vector k E independently only depends on the k E square. So, it immediately tells us that we should be using spherical coordinates, they are the most suitable coordinates in this case. So, let me introduce that k E1 first component of the Euclidean k and second component and so forth. There will be total d components.

So, instead of using these as the integration variables I do a change of variables and go to the polar coordinate the radial coordinate I will call r and then we will have how many angular variables d - 1 angular variables because we have total d variables here I should have exactly d variables on the right hand side after the transformation I have chosen one to be r then others have to be d – 1.

So, these are the angular variables that we should get. This is our radial coordinate and these are our d - 1 angular variables. So, let us see how about the measure d d k E that becomes what r power d - 1 d r. So, you see that you have to have a d r because you are going to these variables, but then you have to have a factor r power d - 1 because now this balances the dimensions on the both sides.

If k is of dimension momentum then r also has to have dimension momentum and on the left you have d powers of momentum. So, it is clear that you should have r power d - 1 so that together with d r it makes d powers of momentum and remaining volumes are angular variables and the measure I will write as d omega d – 1. So, this is the volume element in the angular space and this d - 1 tells you that there are d - 1 angular variables.

The subscript reminds you that you have d - 1 angular variables. So, the denominator of the integrand which is - k Euclidean square - M square power N this becomes - r square - M square power N. So, there is no angular dependence here that is why we have used it. Now define r square to be M square rho all I am doing is trying to just filter out the M dependence. So, when I do that this becomes - M square rho - M square power N which is - 1 power N M square power N rho + 1 power N that is good that is about the integrand.

Now let us look at the measure you can easily see that r power d - 1 d r is equal to M square rho d – 2 over 2 M square d rho over 2, you can check by finding what how d r and d rho are related. This is easy to do. So, now when you plug in this integrand and r d - 1 d r in terms of rho you will get the following again simple algebra you get the following.

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$$I(d,N,M) = \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[-k_E^2 - M^2]^N} \qquad ; \ d \,\&\, N \text{ are +ive integer}$$

$$= \frac{1}{(2\pi)^d} (-1)^N \times \frac{1}{2} (M^2)^{\frac{d}{2}-N}$$

$$\times \int_0^\infty d\rho \, \rho^{\frac{d-2}{2}} (1+\rho)^{-N}$$

$$\times \int d\Omega_{(d-1)}$$

$$\beta(z_1,z_2) = \int_0^1 dt \, t^{z_1-1} (1-t)^{z_2-1}$$

$$\beta(z_1,z_2) = \int_0^\infty dt \, t^{z_1-1} (1+t)^{-z_1-z_2}$$

$$\beta(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

$$\Gamma(z) = \int_0^\infty dt \, e^{-t} t^{z-1}$$

$$\Gamma(z) = \int_0^\infty dt \, e^{-t^2} t^{2z-1}$$

Let me write the integral also it will be useful easier to see. So, substituting whatever we have written on the previous page we get the following 1 over 2 pi to the d - 1 power N into half M square d over 2 - N times the rho integral which is d rho power d - 2 over 2 1 + rho power - N times del d omega d - 1. Let us look at this factor this line here you can take N to be complex nothing stops you from saying that N is complex in this.

How about here we will see and how about so not just N also about the d. So, d and N you can take complex in this because you have complex functions are defined if you have M square power some complex number here you have this expression and then I have d omega d - 1 where it looks like that d has to be integral because that is the number of variables you cannot have fractional number of variables.

We are looking at a angular volume integral where d is the number of angular variables. So, it looks like there is no way we could get an expression in which d could be a complex number, but let us anyway push further and let us see what we get. So, first we will evaluate this integral and it will be useful to have with us some special functions I will write down here. So, you have already studied beta function so beta of z 1, z 2 where z 1 and z 2 are complex.

There will be some restrictions on what values are allowed, but this is the general expression t power z 1 - 1 - t z 2 - 1 that is the definition of beta function. These are another expression, another integral representation of the beta function which is 0 to infinity. Here in the, integral is from 0 to 1, but you can also write it as an integral where t runs from 0 to infinity and the expression is this.

And this is useful for us because our integral here in the second line is of this form. Let me give you few more results which we will use beta of z 1 z 2 is gamma of z 1 where gamma is gamma function gamma z 1 gamma z 2 divided by gamma of z 1 + z 2 and also gamma function is defined as follows and there is an another form there are several forms, but this is the one we will use t goes from 0 to infinity e to the - t square t to the 2 z - 1. So, these results will be useful.

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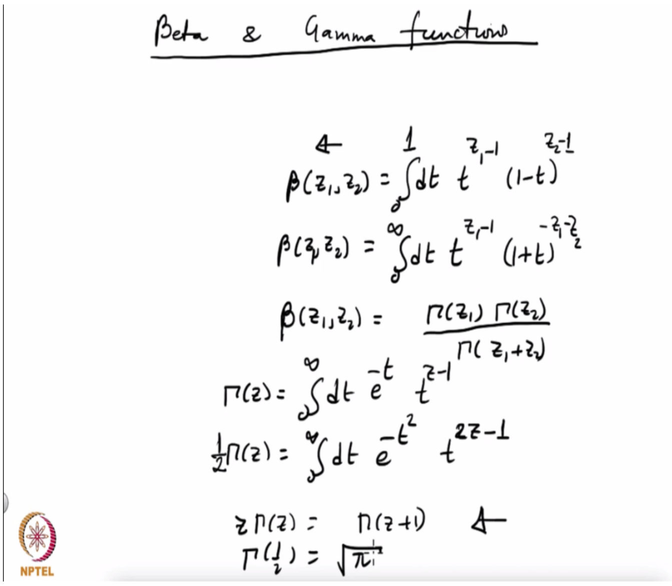

I think it will be a good idea to keep them safely at one place. It did not work why let me try again hopefully it will work this and I should remove this okay good. Let me check if I made any mistake in writing z 2 - 1 the first one is correct, second one is also correct, third one is correct 0 to infinity d t e to the - t power z - 1 that is right and this one is wrong I have missed a factor of half for some reason it is not right and why some strange behaviour.

So, a factor of half was missed here also. So, now you see that this line this is just beta of d over 2 N - d over 2. You can check it from here so you say that z 1 - 1 is d - 2 over 2 and - N is – z 1 – z 2 then you figure out what z 1 and z 2 should be and put it here. So, that is beta of d over 2 and N - d over 2. Now you see that even in this second line now where you have this rho integral even though we had d and N to be integers originally.

I can continue to complex plane because beta is defined in the complex plane for its arguments. So, these capital N and d they can also be continued to complex plane. Now the only thing that is left is the angular integral. So, let us look at that first and then I will combine all the results.

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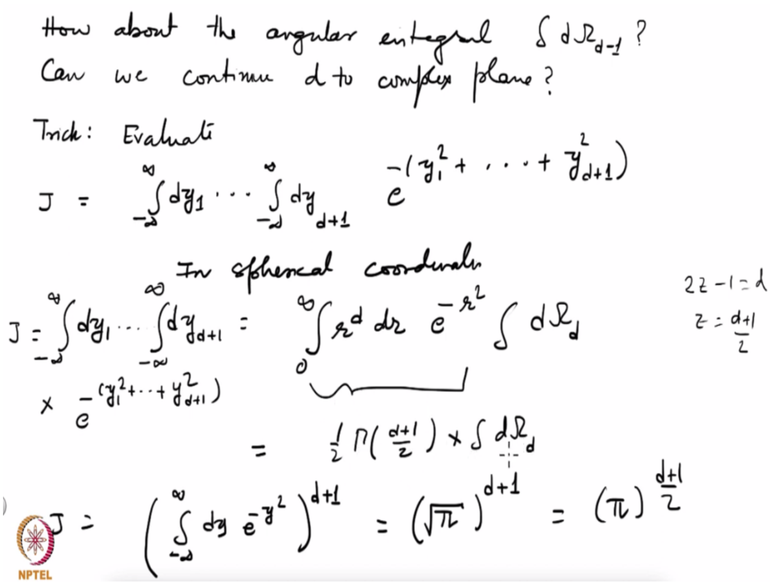

Okay let us see whether that can be done and here I am going to involve this trick. So, I evaluate the following integral. I evaluate the integral let us call it small i that is not good what should I call let us call it J okay integral J which is the following - infinity to infinity d y 1 to d y d + 1 e to the - y 1 square + y 2 square so and so forth to y d + 1 square that is the integral I am interested in evaluating because it will be useful in determining omega d okay.

I am calculating integral d omega d and later I will put replace d by d - 1. So, because again you see that the integrand this exponential involves only the sum of the squares so it tells us that I should use angular integrals and that will be useful I mean spherical coordinates in spherical coordinates I will have d y 1 to d y d + 1 is equal to r power d d r you have seen already this kind of thing before 0 to infinity.

See if you think of y as having dimensions of length or momentum or whatever let us say dimensions of length then these are d + 1 then the left hand side has dimensions length power d + 1 and because I am going to involve all other integrals to be angular the total which are dimensionless. So, the entire dimension of length on the left hand side has to be matched by this. So, r power d r that gives you d + 1 powers of length.

So, that is why you have r d and then you have e to the - r square that is coming from here where r is the radial coordinate in d + 1 dimensions and you have integral d omega d and we are interested in calculating integral d omega d and that is the trick we are utilizing. So, what is J then I have also so here times e to the - y1 square to y d + 1 square now this is correct. So, this is J; J is equal to this which I have written in this form.

Now this is equal to what is this I can evaluate by using this gamma function. You see I have an integral 0 to infinity d t e to the - t square 2 z - 1 that is half gamma. So, this gives you half of gamma d + 1 over 2. Let us check 2 z - 1 is equal to d. So, z is equal to d + 1 over 2 that is why I have put gamma of d + 1 over 2 because that is what you have here gamma z good. So, this times integral d omega d nice.

But now I should evaluate J in a another way so that I can equate these two results and get the expression for d omega d and the idea is that you look at this integral as d + 1 integrals I mean the same integral repeated d + 1 times. So, what you have to do is you write J is equal to integral d y so you can think of this as d y 1 e to the – y 1 square, but then d by 2 e to the - y 2 square that is exactly the same integral as this one.

So, this is repeated d + 1 times and what is the value of this integral? The value of this integral here is root pi that is the Gaussian integral that gives you root pi power d + 1 which is same as pi power d + 1 over 2. So, J is pi power d + 1 over 2 and J is also half gamma d + 1 over 2 times integral d omega d.

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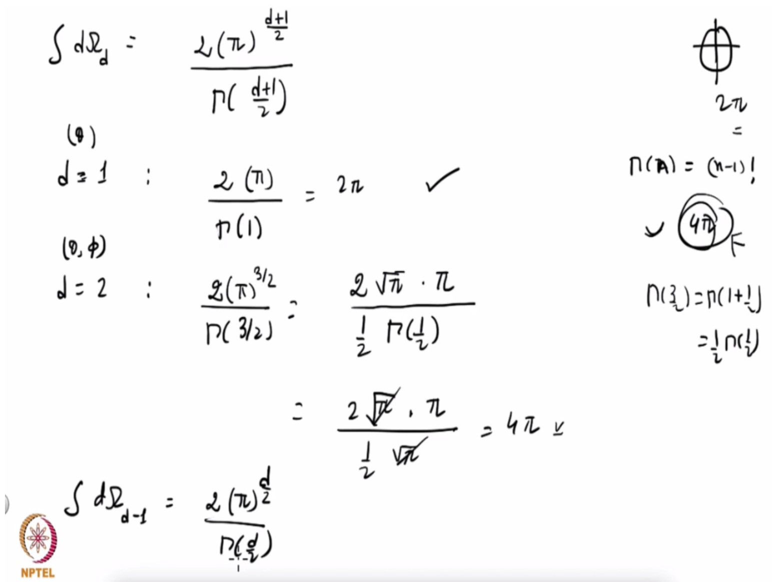

So, from this I can just extract integral d omega d the volume integral which is 2 pi power d + 1 half sorry d + 1 over 2 over gamma d + 1 over 2. Now let us check whether this is correct result, but before that you notice that the moment you have this right hand side no one can stop you from putting d to be a complex number because gamma function is defined for a complex number and pi you can raise to some complex value.

So, the right hand side can be continued to complex values of d even though the left hand side looks like even though the left hand side can be evaluated only for d integral values, but right hand side is also we can continue to a complex plane. So, with this we have now written I in terms of these functions which all can be continued to complex d and complex N plane okay, but before that let us test whether this result is correct.

So, I put d = 1 and d = 1 means I have only one angular variable and when I integrate over this I know what I should get I should get 2 pi and theta goes from 0 to 2 pi that is the contribution you get. So, let us test whether this is correct so 2 pi power d is 1, 1 + 1 2 over 2 is 1 so that is pi over gamma 1 gamma 1 so gamma n where n is an integer is n - 1 factorial so that is 0 factorial which is 1 so it is 2 pi.

So, that is a correct result also let us check for d = 2 so you have now two variables theta and phi. So, here you had only 1, here you have theta and phi and you already know what the result is it is 4 pi that is the angular integral gives and that is/has why you have 4 pi r square as the volume of a sphere of radius r it is this 4. So, you get 2 pi d = 2 so 2 + 1 3 over 2 over gamma 3 over 2.

I will use another useful result which I can write here gamma of let me write this way z gamma z is gamma z + 1 if you want to increase the argument of gamma by 1 you multiply it with z with the same argument that is what gives you gamma z + 1. So, let us use that so this is 2 root pi into pi that is pi 3 halves over gamma 3 half is gamma 1 + half this is half gamma half because of this property. So, you have half gamma half now gamma half is root of pi.

So, also let me keep it here so what do we get? We get here 2 root pi into pi over half gamma half is root pi that cancels and this is 4 pi which is what we expected. So, it means that the formula that we have found is correct and integral d omega d - 1 is equal to 2 pi d I should replace by d – 1. So, that kills that 1 and I have d over 2 over gamma d over 2. So, now I will substitute all these expressions so integral d omega d - 1 from here.

And where is it and this expression of this rho integral which is here given in this form and these factors. So, this one using this formula gives you gamma of N - d over 2 over gamma N. Now I will substitute all these things here and get the following.

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$$I(d, N, M) = \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[-k_E^2 - M^2]^N}$$

$$= (-1)^N \frac{1}{(4\pi)^{d/2}} \cdot (M^2)^{\frac{d}{2}-N} \cdot \frac{\Gamma(N-\frac{d}{2})}{\Gamma(N)}$$

Conclusion: Feynman integrals can be analytically continued to complex d.

I d N, M let me record the entire thing here for later use 2 pi power d 1 over - k E square - M square power N and this is equal to - 1 power N did I change the result – 1 times 1 over 4 pi power d over 2 times M square power d over 2 - capital N then you have gamma of N - d over 2 divided by gamma N. So, that is the result, but this is in Euclidean space if you are looking at one loop then you have one power of d, you have one time component d k 0.

And that will bring a factor of I so you will have a factor of 5 this expression was written in the Euclidean space, in the Minkowski space, but let us keep the Euclidean. So, now you see that the conclusion is that Feynman integrals can be continued in two complex d complex dimensions or complex and we used to call it small n earlier, but now I am calling it d. So, this is good so two things achieved.

One that I can write it as a expression in d and now you see that the singularities are contained in these gamma functions, all the singularities are contained in here d is acting as a regulator so that you do not have our naked divergence sitting here, it is not just I have infinity, but it is regulated and that singularity I can recover by going close to d equal to 4. So, this is a good situation we are in at least we can regulate the integral.

But this does not solve any of our problems because at the end I should take d equal to 4 and everything is divergent and next thing is to learn how to remove the infinities. We will see that in the next video.