Introduction to Quantum Field Theory(Theory of Scalar Fields-Part 4)
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Lecture - 25
UV Divergences: Part 3

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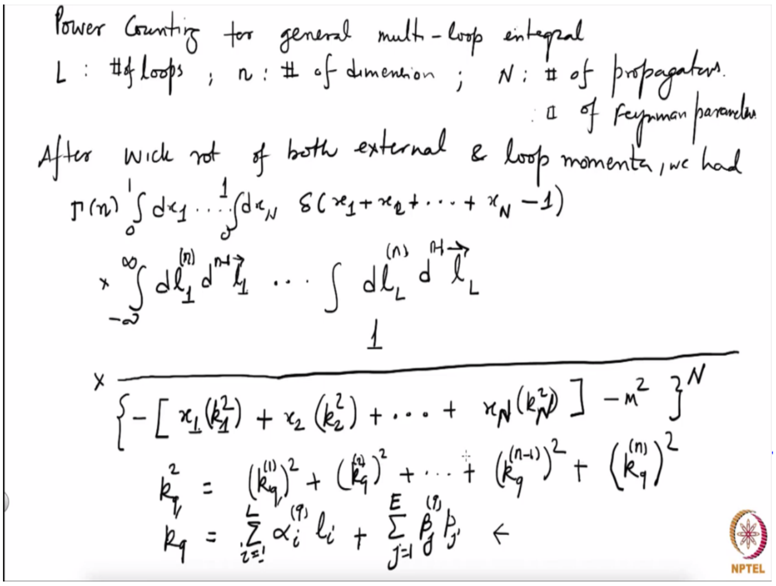

So, power counting for general multi-loop integral or Feynman diagram. So, let me remind you about the notation l stands for the number of loops. We still want to work in n dimensions. We will see later why, but for now it does not matter. So, instead of 4 dimensions I am working in n dimensions. So, for now you can imagine that the universe you live in has more than 4 dimensions which is of course not true, but we can imagine.

And N is the number of propagators which is same as the number of Feynman parameters and after doing Wick rotation of both the physical and loop momenta sorry not physical let us call it external. You can call it physical also and loop momenta; loop momenta we had found the following. I will drop these factors of i pi n - 1 factorial or gamma n what we had earlier and just concentrate on the important part.

Should I go to the next page or maybe not. So, we had let me keep gamma in which is n – 1 factorial. So, we had this following we had d x 1 up to d x N then you have a delta function which constraints all these Feynman parameters such that the sum is 1 then after doing Wick rotation our loop integral momenta or loop moment are like this so l 1 for loop number 1 this n.

And you remember what this is right this is l 0 when you take the zeroth component of l 1 and you do Wick rotation that is what you are calling l n and then these are the remaining spatial components which is n - 1 dimensional this is not a good notation and the limits are from - infinity to + infinity. We have l loop momenta so l n d n - 1 L then it stopped working then here is the denominator and the denominator is this - x 1 k 1 square.

I will tell you what I am doing. First let me write down x 2 k 2 square x n k n square - m square and I do not put any i epsilon because I am already in the Euclidean space. You do not need i epsilon there because there is a definite sign here and what is x 1 x 2 so forth sorry k 1 k 2 so k q; q runs from 1 to k n (()) (06:08) the number of propagators and the number of propagators I am saying capital N.

So, this should be capital N that you have for each propagator you have one x so we should have x 1 up to x capital N. Let us say q is that index which labels this 1, 2, 3 so k q square is this k q 1 square that is the first special component of k q 2 square or maybe I will put this otherwise it is confusing k q. We are working in n dimensions n - 1 square and then the zeroth component after Wick rotation we were writing it as k q n that was not (()) (07:32) that was for the loop momentum.

And I am using the same notation for q because k and l are related. So, this is in Euclidean space you see all the signs are plus there is no minus sign entering anywhere and k q is the following. The vector k q is l, but that is the vector k q I will just remove this arrow is alpha i q l i summation from i equal to 1 to i runs from 1 to what? It should run from 1 to the number of loop momenta so that is what you are adding so 1 to L + summation over j beta j q beta q j.

And then you have p j and this is what is p j; j runs from 1 to the number of external momenta, external lines E. So, that is the definition. All I have done is define let us go back here. So, you see here indices on x is suppressed, but this is giving you a factor of - 1 so that comes out here - n sits here. So, you minus of this minus also let us pull out both minus signs and just keep them aside for a moment then you have only plus signs relative to them.

So, you have x times this square - x times this square and this momentum this entire thing alpha l i + beta j p j. The sum of all this momenta I am calling k. So, this becomes x times k square this plus this becomes x times k square and there is an overall minus sign and that is what I have written here. So, x times k square or for x 1 we have k 1 square x 2 k 2 square and what is x 1 and x 2.

These are just the sorry what are k 1 and k2 these are just the momenta that are flowing in these propagators. So, propagator number one has this momentum flowing in it. I hope you remember see whatever momentum flows let me show you one of the diagrams it will be easier to let us see if I can find here for example if you look at the propagators okay this will have l 1 + p 1 whole square.

So, it is both linear in the loop momentum and the external momentum the k the corresponding k here this is what I am calling k 1 or k 2 or whatever. These are the k's. So, this is both linear in loop and external momenta if you go to any of these you will have loop momenta summing up adding up linearly and also the external momenta will be adding up linearly science may be plus or minus.

But they will be all linear and this is exactly what I have written in the general case and this will be true for whatever diagram you draw. This is the sum of all else with either coefficient + 1 - 1 or 0 so l 1 + l 2 - l 3 - l 4 whatever and similarly here pj's with all of the pj's with some of the betas could be 0, some of the betas could be 1 or – 1. So, that is a plus not a minus.

So, now we have this integral we are almost there and we can just to remind you what we are trying to do. We are trying to achieve equivalent of this so that I can do the power counting. Good then now because each of the case is linear in l the loop momenta and because we have a square it becomes quadratic in a l i's. So, this thing here in the curly brackets is quadratic in l i's. So, I will just write down the most general form of quadratic function can take.

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Denominator : $\left[\text{Quadratic } f^n \text{ of } l_i\right]^N$

Quadratic f^n of l^i

Degree 2 term : $a_{ij} l_i l_j$ $\quad a_{13} l_1 l_3 + a_{11} l_1^2 + \ldots$

Degree 1 term : $b_i l_i$ $\quad b_1 l_1 + b_2 l_2$

Degree 0 term : c $\quad b_i = b_i(x_j, p_j)$

Denominator : $\left[-(\underbrace{a_{ij} l_i l_j} + b_i l_i + c) - m^2\right]^N$

After diagonalization : $\sum a_i' l_i'^2 + \sum b_i' l_i' + c$

Ex: Complete the square, shift the loop momenta.

So, denominator is quadratic function of l i power capital N and we are looking at this. So, as I said the degree of this thing is 2 sorry it is a quadratic. I will write down what it looks like then so it will have a degree 2 term and what will be the degree 2 term? It will be l i l j it will be multiplying n 1 l 2 l 1 square l 2 square l 1 l 3 l 4 l 3 like this and there will be some coefficients there is Einstein summation convention which I am using so it is, for example, a 1 3 l 1 l 3 this is one of the terms plus a 1 1 l 1 square and so forth.

Then let us look at the degree 2 term sorry degree 1 term it will have the form l i some constant b I. So, this is b 1 l 1 + b 2 l 2 and so forth and these bi’s are functions of stopped writing I am unable to write now. Let us call it j here all the external momenta and all the Feynman parameters. So b's will depend on that you can see from here k q has b and there are also x is multiplying here.

So, when you are looking at the sum of these k squares multiplying these axis the linear terms will also have this these p is entering them and multiplying them and also the axis that is why I am writing b is a function of x and p. If you are not fully confident then you can write down a diagram with three propagators and a few propagators and two loops and row explicitly and then you will see or just do this from here you will see.

And then degree 0 term which is the constant term I will call it c this will also be a function of these m square and external momentum. So, there is nothing else we have exhausted all the possibilities so quadratic function can only be a function of we will have a degree 2 term, degree 1 term, degree 0 term there is nothing else. So, the denominator is of the following form.

Now you see a i j is a real symmetric matrix it is multiplying l i l j which is symmetric so a i j is also symmetric. It is a real symmetric matrix which can be diagonalized and furthermore none of the eigenvalues will be 0. So, it is a positive definite matrix and you know because you have loop momentum is l it is not going to be something which involves lower number of loop momenta after diagonalization.

So, this is a positive definite matrix and all the eigenvalues will be non-zero and then after diagonalization I can write it as a i prime l i prime square b i prime l i prime + constant when I look at this thing. So, basically I am doing an orthogonal transformation. So, l i go to l i prime and these coefficients will change of course so this is the linear term. This term because we are trying to diagonalize this one by an orthogonal transformation.

This is the diagonal form and these are the coefficients multiplying each of the l i and then this is a constant term. Now, you can do another thing you can see going from l i to l i prime the measure will not pick up any Jacobian because this is orthogonal and determinant of this matrices will be 1 the transformation matrices, but then you can do one more step you can absorb these a i into this l i.

So, you define a new l i which absorbs that. So, these terms these quadratic terms come with power 1 each term and this will of course then change and the constant will also change and then you complete the square that is exactly the same step as we had done long back somewhere here, how did we reach here? We reached here by completing a square here this step and then we had to shift the momentum.

So, after you have completed the square so this is a loop momentum and that is what you get after completing the square so you do this similar step and then you shift the momentum and that is what you should do here as well. So, I leave it as an exercise for you to do. So, exercise complete the square after you have gotten rid of these coefficients by redefining and then you shift the momentum, shift the loop momentum and it will give you the following. So, that is an exercise you should check.

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$$\text{factors} \times \int_0^1 dx_1 \cdots \int_0^1 dx_N \; \delta(x_1 + x_2 + \cdots + x_N - 1)$$

$$\times \int \frac{\boxed{dl_1^{(n)} \, d^{n-1}\vec{l}_1}}{(2\pi)^n} \cdots \int \frac{dl_L^{(n)} \, d^{n-1}\vec{l}_L}{(2\pi)^n}$$

$$\times \frac{1}{\left[-(l_1^2 + \cdots + l_L^2) - M^2 \right]^N}$$

$$M^2 = M^2(x_i, p_i \cdot p_j, m^2) \qquad M^2 > 0$$

$$l_1^2 + \cdots + l_L^2 = (l_1^{(1)})^2 + \cdots + (l_1^{(n)})^2 + \vdots + (l_L^{(1)})^2 + \cdots + (l_L^{(n)})^2$$

$$\text{Define } l_E \equiv \left(l_1^{(1)}, \ldots, l_1^{(n)}, l_2^{(1)}, \ldots, l_2^{(n)}, l_L^{(1)}, \ldots, l_L^{(n)} \right)$$

$$\text{Denominator} = \left[- l_E^2 - M^2 \right]^N$$

Apart from some other factors which I am not writing. You will get the following 0 to 1 d x 1, 0 to 1 d x N number of propagators delta of x 1 + x 2 + x N - 1. times integral d l 1 n I hope I am not changing the notation then d l 1 d n - 1 l 1 over 2 pi to the n did I write 2 pi to the n earlier I forgot. So, here I should have written 2 pi to the n and for each loop momentum you have this d l L because you have L loop momenta and over 2 pi to the n good.

And then you should check that this is the result that you are going to get times 1 over I have renamed the loop momenta again to l l 1 l 2 so I am writing the result because anyway l is dummy so I am writing it this way square - capital M square i epsilon has been dropped power N. Check this that this is what you get where M square is of course the function so it is not writing now the function of x i.

Then it will be function of p i dot p j and it will be a function of m square. These are the things which will enter m square and now this is looking very good I, will just write one more step. So this l 1 square + l 2 square + l l square let me just write it in a more expanded form. So, that we are 100 percent sure that we understand what we are writing this is the following. What is l 1 square?

L 1 square is l 1 the first component of it squared plus the second component of it squared so and so forth nth component of it squared then what is l 2 square similarly you change the subscript 1 by 2 and then finally you have l L which is l L first component squared. We are in Euclidean space that is why it is all pluses there is no minus l L n square. So, now let us define a new vector which I will call l E which is let me define it first it will be easier which is l these are the components l 1 1 l 1 2 l 1 n then it has l to 1 up to l 2 n then l L l l L n.

So, that is the l E vector in the Euclidean space that is a new vector. So, what is in the denominator this just becomes l E square. So, the denominator is - l E square l Euclidean square - M square also check that M square is positive power N that is your denominator. Let us look at the numerator, numerator is what? It is just this so it is d l 1, d l 1 2 d l 1 3 so and so forth d l1 n which is exactly this one.

Then you have d l 2 1, d l 2 2 and so forth up to d l l n that will be what you have here and this is the last one. So, you see that what you have here in the this integral the differentials that is same as the following.

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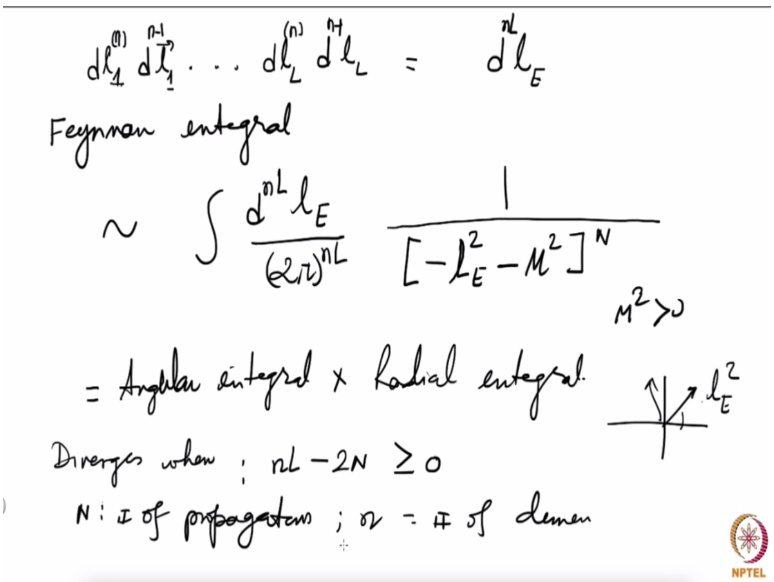

D l n - 1 here I wanted to write n up to d l L n d n – 1 l L this is what the way I have defined the new vector l E this is d l E, but what is the total dimension of this differential? It has total l such and each of them has n. So, this is n L and they are total n L differentials here that is why I have written d n L l E. So, the integral the Feynman integral apart from other factors is this. It is not writing times 1 over - I will use a square bracket - Ll Euclidean square - M square power N where M square is positive good.

So, that is how an arbitrary Feynman integral is going to behave. There are other factors which I have omitted, but they do not carry the loop momenta. Now you can do the same thing you can do the angular integrals and the radial integral. Angular integrals are not going to give you any divergences and here anyway angular integral just separates out easily because what you have here is only radial it depends only on the radial part.

There is no angular part in here because just look at this two dimension this is this is l E square. The square of the magnitude will be l E square and this does not depend on the on the theta even if it is here it is the same l E square. So, the denominator does not depend on the angular variables. So. the angular integral is trivial we will see the angular integral later, but at least it filters out and sits outside of the radial integral.

And then you can do the radial integral. So, this is same as angular part times the radial integral. So, when does this diverge? Now the reason I have written it in this form is because earlier we had problem because of the Minkowski space now I do not have any such problems. So, I can just count the degree of divergence. So, when does this diverge? It diverges when n L - 2 N is greater than or equal to 0 if this condition is satisfied.

Then it is divergent. If n L – 2 N is less than 0 then it does not diverge. Let me remind you again capital N is the number of propagators and small n is the number of dimensions. So, what does this say? Let us take n equal to 4 and l equal to 1.

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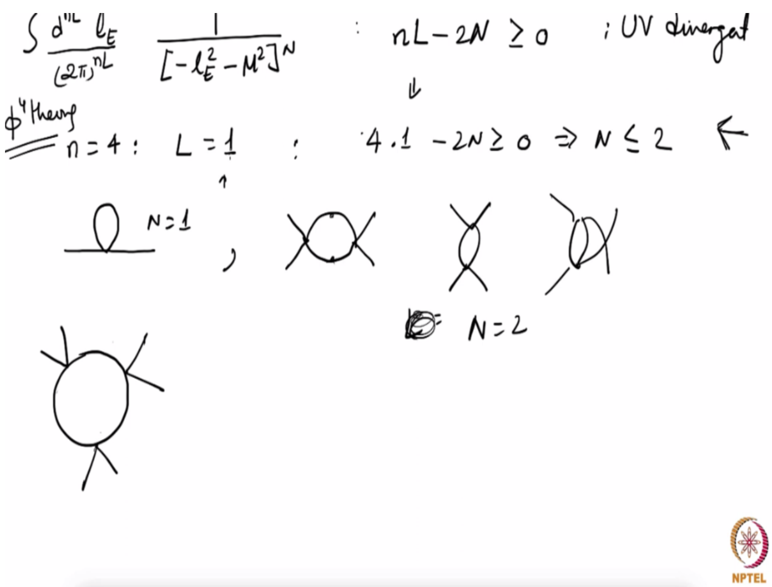

So, let me write down this again I will write down the expression once more. So, here integral dn L l E over 2 pi n L did I write that earlier? Yes then you have 1 over - l Euclidean square - capital M square power N and as I wrote just now if n L - 2 times capital N is greater than equal to 0 greater than or equal to 0 then this is UV divergent and remember angular integrals cannot give any UV divergence they have finite volume.

So, let us look at n = 4 meaning 4 dimensional space which is our space and L is equal to 1 so one propagator. So, this condition says that you have yeah so this is n is 4 L is 1 - 2 N should be greater than equal to 0 which means that N should be less than equal to 2 and what was capital N was the number of propagators. So, it says that if you are looking at one loop in four dimensions.

If you are looking at Feynman diagrams which are defined in four dimensions and it is at one loop then that Feynman diagram is going to diverge if the number of propagators is less than 2 or equal to 2. You remember if you increase the number of propagators the integral starts behaving nicely in the ultraviolet anyway because each time you include have a additional propagator it brings an additional power of 1 over k square in the denominator.

So, this is saying that things are bad till you are having one or two propagators which you have seen already. This is consistent with what we have learned earlier. So. if you are looking at four dimensions and at one loop L is 1 meaning one loop and you have this is divergent, it has one propagator this is the case for N = 1 propagator. Then we have also seen that at one loop these diagrams diverge.

And these L equal to 2 sorry not L N equal to 2 the number of propagators is 2 1 2. So. you see all these diverge precisely for this reason this is the counting which we have found. So, this is what is giving the divergences and if you go to this case where this is still one loop so L = 1 N = 4 and look at this diagram this did not diverge because it has 3 propagators and 3 is bigger than 2.

And when N is bigger than 2 then this is not going to diverse. So, you can see that the calculation that, we have done is right we have not made any mistakes because this math is what we already knew. So, this is in phi to 4 theory whatever examples I have just now give. We can also look in phi cube theory and see whether it is all consistent.

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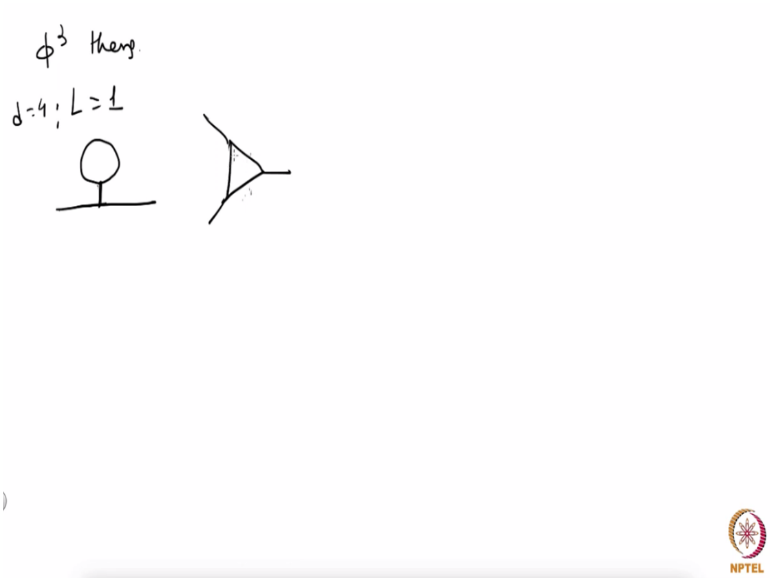

So, in phi cube theory let us again at L = 1 that is one loop and let us look at d = 4 four dimensions. This is 3 point vertex in phi cube theory. So, that is how many propagators you have here which are involved in the loop integral there is one it is only one this one this propagator not this one this one is not involved in the loop integral. So, this one and it is diverges because it is only one propagator n is equal to 1, how about this?

This is 3 propagators. one loop, 4 dimensions so this will be finite ultraviolet finite and which is true because it brings 6 powers in the denominator you have 4 powers in the numerator. So, this is also consistent with what we have found here. So, we have now seen that the source of ultraviolet divergences and how to see them correctly by doing a Wick rotation.

Now our next task will be to look at regularizations different kinds of regularizations I will talk about in the next video. You have seen already one the cutoff regularization I will also talk about (()) (39:45) and dimensional regularization and then once we have done the regularization meaning how to make integrals finite, but dependent on the regulator we will proceed to do the renormalization.

How to remove, how to subtract the infinities to get finite answers. So, see you in the next video.