Introduction to Quantum Field Theory(Theory of Scalar Fields-Part 4)
Prof: Anurag Tripathi
Department of Physics
Indian Institute of Technology – Hyderabad

Lecture - 24
UV Divergences: Part 2

So this program of getting finite results out of a perturbation theory which is seemingly divergent I mean it is each element is diverging in ultraviolet.

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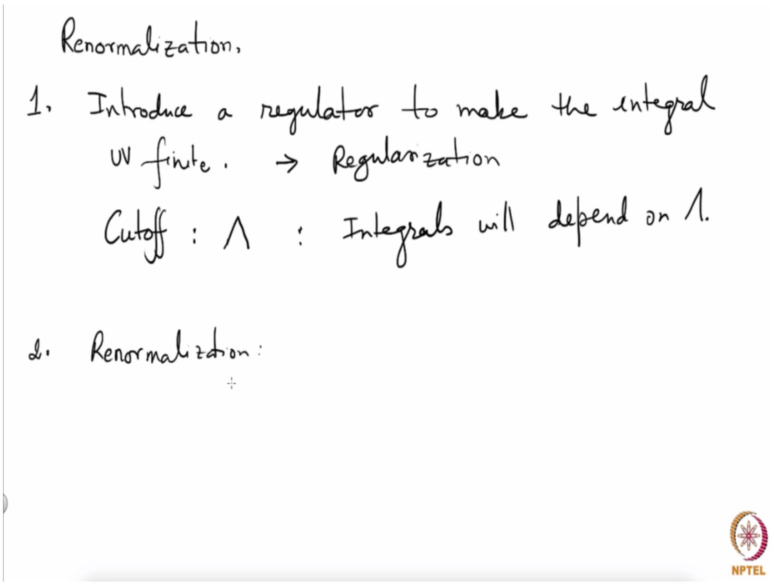

That program is called renormalization. So, there are two steps involved in this. So, step number 1 is to introduce a regulator whose job is to make the integrals finite. So, you introduce a regulator the integral then depends on that regulator, it becomes a function of that regulator and the purpose of that regulator is to make the integral finite in ultraviolet, but of course if you were to remove the regulator it will become divergent again and we need to remove the regulator.

So, introduce a regulator to make the integrals UV finite and this procedure is called regularization, for example, we have already seen one example, too many examples in the same sentence not nice. So, one regularization method we have seen and that is cut off. So, here you introduce a cut off and say that I will cut off the integrals at some momentum value high momentum value lambda.

So, now your integrals will depend on this cutoff because there is lambda, but our real thing is when lambda goes to infinity. So, at the end I have to take lambda going to infinity so it looks like this is not going to help which is true having just a regulator is not going to help something else has to do the magic so that we get finite results and that magic is called renormalization and I will tell you in more detail how that thing works.

So, the idea is you introduce this cutoff and calculate whatever observable you want to calculate with the cutoff in place and your result will depend on the cutoff, but we will see that we can arrange things in a manner that this divergences which will appear as function of lambda because lambda is to filter out the divergences. So, within that observable we will be able to cancel the ultraviolet divergences.

So, all the divergences which are functions of lambda they will cancel and then once the divergences have canceled we are free to take lambda going to infinity in our result will still remain finite and that is what is called renormalization. These procedure of subtracting infinities. So, I will talk about different ways, different methods of regularization first and then we will move on to renormalization.

But here something I want to point out based on what we have already seen here. So, how many parameters do we have in this action? We have two parameters m and lambda and let us look at one loop. We will worry about higher loops later, but if you look at one loop then this two point function is divergent because of this diagram that is one diagram which is divergent or one two point Green function which is divergent at one loop.

Then you have another one where did that go here. This one 4-point function which is also divergent at one loop. So, you have two Green's functions which are divergent at one loop, but when we went to higher order Green's functions like this one this is not divergent at one loop, this one is not divergent and you can imagine drawing other one loop diagrams that will be just introducing more vertices so you get higher point functions.

And they will also not be divergent. So, two Green's functions which are divergent at one loop and you have two parameters m and lambda in the theory which is not physical quantities. So, you never measure m, you never measured lambda these are not physical objects. What you measure is a physical mass you can measure the mass of an electron or you can measure the strength of the interaction of one electron with another electron.

These are the things you can measure you can measure cross sections, but there is nothing you can do to measure m and lambda. So, right now the situation is that you have two parameters which are not measurable and you have two Green's functions at one loop which are divergent and what one does is hides these infinities which you are seeing at one loop in these two parameters.

And then you can get a result that is finite up to one loop and we will see how this can also, why this two similar things will work at two loop also, but roughly that is the thing and I will tell you in detail later how to do these steps to get a finite cross section, but before that I want to look at the ultraviolet divergences in all these finite diagrams a bit more carefully as I have been saying repeatedly that I will do it later.

So, now let us do that part and we can see that ultraviolet divergences become very transparent if we do a wick rotation both in the loop momentum and the external momentum. So, that is what I am going to do and when we go to the physical external momentum. So, we have gone to I am saying these steps let me remind you just a second yeah here for example or even earlier here.

So, when we were doing this integral we had gone to p square which was the external momentum p square that small p square we made it space like which is not the physical region which you can also do by taking the zeroth component of p to be imaginary then p square will become negative and here we had done a wick rotation so that we got this denominator.

This integral was well defined. Here it is once you have done this here it is easy to see the ultraviolet divergence because now everything is adding up it is unlike the previous case where k 0 square - k vector square you had a minus sign and it is not so clear what kind of divergences you are getting because if you take all k 0 and k 1, k 2, k 3 all to be of the same order then you can get cancellations between k 0 square minus these other terms, but here it is all very clean because there is no cancellation between different components.

So here you see that in this case you have two powers in the numerator so both k 1 and k 2 are large let us say we put a cut off lambda. So, for both of them you have a cut off lambda so two powers here and then you have two powers here and get squared. So 4 and 2 here so this is convergent and this is convergent because we had gone to two dimensions. This integral we evaluated in two dimensions.

So, that is what we want to do again and the lesson from here I am going to rewrite this thing in the following way.

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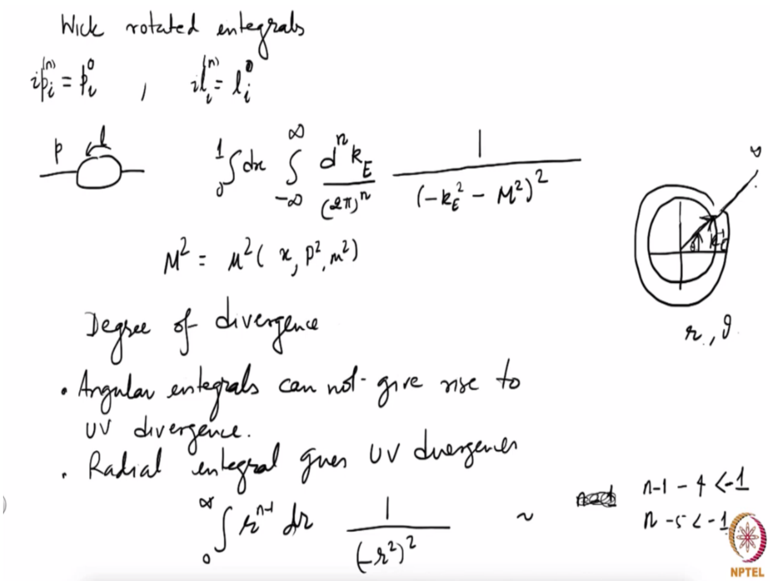

As I said I will take the external momenta which are pi’s and I will rotate them I am thinking of doing integrals in n dimension. At this moment it is not clear why I am doing this, why I want to be in n dimensions, but let us say that for whatever reasons I want to work instead of 4 in n dimensions, but if you do not like it you can put n equal to 4 there is nothing is going to change in the in the reasoning.

So, this and also the loop momenta we had done this. So, that is the wick rotation and we just saw that thing in the red box that if you look at this 2-point function and do the integral then it is of the following form integral 0 to 1 d x integral so that was two dimensions, but I am saying n dimensions now and then that was in the Euclidean space. So, I call the loop momentum vector is k E for the Euclidean and here similarly you will have 2 pi to the n.

And this factor of minus k E square - m square whole square. Let us go back and check whether indeed we have the same form. Here you see this is; so let us bring the minus sign inside I had taken it out, but let us put it back. So, there is square so that minus goes in here and it multiplies everything. So, it is minus of this square minus that square which is minus of k Euclidean square that is a definition.

Then minus this object is positive so minus of a positive object because x is positive and 1 minus x will always be positive because x lies between 0 and 1 and p square was space like because it is wick rotated and that is what we are so this is minus this positive object - m square ok because of that minus sign. So, it becomes negative of a positive quantity and that is what I am writing as capital M square.

So, that is the form and I have dropped i epsilon because there is no need for it when we are in the Euclidean space and here M square is a function of x, p square and m square. Now we can very cleanly see what is the degree of divergence? So, I have this integral see the x integrals are not going to give you any ultraviolet divergence. They run from 0 to 1 they do not go to infinity.

So, that is not going to give ultraviolet divergences, they give rise to something called infrared divergences, but right now we are not concerned with any infrared divergences, infrared our collinear divergences they are given theories where your massless particles, but they cannot give rise to any ultraviolet divergences. So, ultraviolet divergences are completely contained in this and as I said that going to the physical, external momenta.

And going back to the real axis that is not going to generate any ultraviolet divergences any additional things. So, whatever ultraviolet divergence is contained in this wick rotated integrals is the real and traveler divergence and that is why I am showing you these integrals. So, from here now we can estimate what is the degree of divergence. So, I hope it is clear, but if not then you should do a radial integral and an angular integral.

This is a vector k E in n dimensional Euclidean space for example n could be 2. So, it is a vector k E in this two dimensional space, you can do integral by integrating this way. So as you go out you are integrating over the radius or the magnitude of the vector r and the other integral will involve, integrating over angle which is theta. So, these two integrals are in theta and the generalization of this for n dimensions you have to do.

Now angular integrals do not give any divergence any ultraviolet divergence. These angular integrals have finite volume like here it is theta goes from 0 to 2 pi that is something finite if it was sphere then the solid angle will be 4 pi and so forth that does not give you any infinity. So, what gives you infinity is because this radius goes to infinity. So, that is the source so we can just forget about the angular integrals also and just worry about the radial integral.

So, suppressing the angular part let us look at the radial part. So, radial part will be integral 0 to infinity you will have r to the n - 1 d r and that makes n powers of momenta dr has one power of momentum and r power n - 1 has n - 1 powers so together n powers and this is anyway - r square power I have dropped m square because m square is something finite, p square you have chosen that is something given to you by your process m square is finite.

This is finite x ranges from 0 to 1 that is finite. So as r becomes large m square you can drop in comparison with large r and then you see the divergence is like this when n – 1 – 4 is 4 here and this 4 should be less than 1 because n – 1 – 4 let us say if it is equal to 1 not 1. So, if n – 1 – 4 is let us say 0 then it is divergent because then you have just integral d r because there will be no power of r coming from this because n – 1 – 4 is 0 so that will be divergent.

Then even if you had a power of r in the denominator which is r – 1; that would also be divergent and that is why n - 1 - 4 should be less than – 1. This is fine then what next. Then we have also looked at the general multi loop integral earlier which is here that is the let me remind you. So, we were looking at this Feynman integral and you see here we had n propagators that and for each propagator we had an x i so that is why you have n number of Feynman parameters and just a delta function.

I think this is something I inserted today I had forgotten in the previous video and then you had this denominator and these are the loop momenta which are still in the Minkowski space then I did a wick rotation and after doing a wick rotation this is the denominator where l 0 so here l i 0 is l i n times e to the i pi by 2 which is i. So, here that is the denominator where here I should put e to the 2 i pi by 2 which is going to give you – 1.

And I had suppressed the labels on x. So, this is what we had in the denominator. So, let us look at this one and try to carefully find out the degree of divergence for such diagrams and this time I am doing properly in the Euclidean space just like for this diagram.