Introduction to Quantum Field Theory-II
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Lecture – 22
Wick rotated Green's functions

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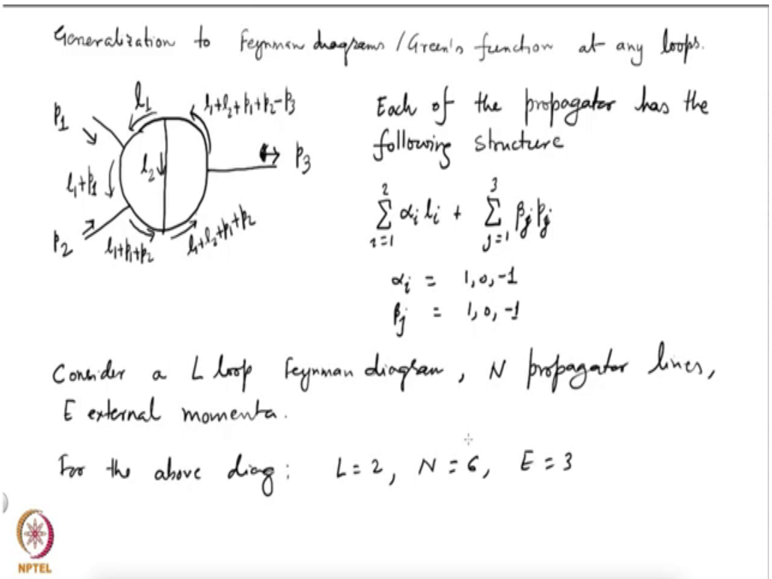

In this video, we will look at the generalization of the results that we proved last time. Let me show you what we did last time.

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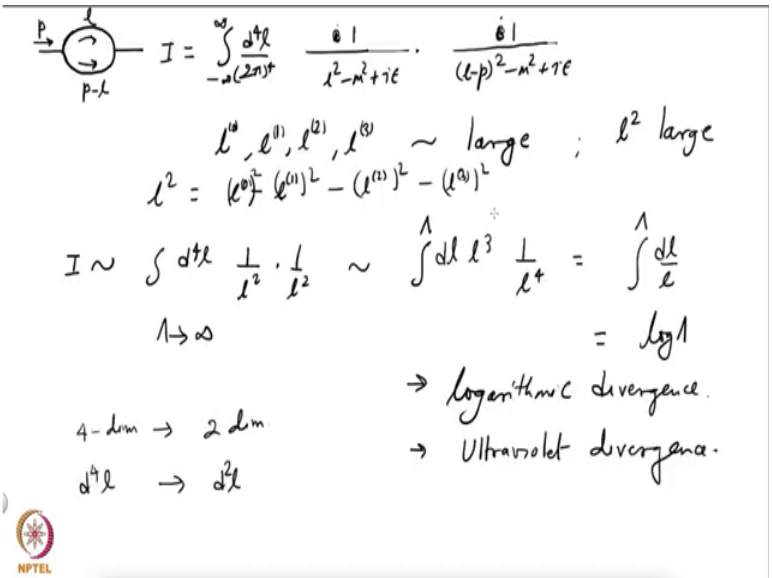

Here, so, we were looking at this Feynman diagram where p is the momentum that enters into this. This is the external momentum. Where p could be time like or space like momentum.

We leave it a general for now. So, this is the integral that we were looking at in 4-dimensions. So, you are still in 4-dimensions but then we saw that this integral is divergence in 4-dimensions. It is ultraviolet divergence and it divergences logarithmically.

And to avoid that complication, we said we will look at this integral in 2-dimensions where this integral will converge. And that we could see by power counting.

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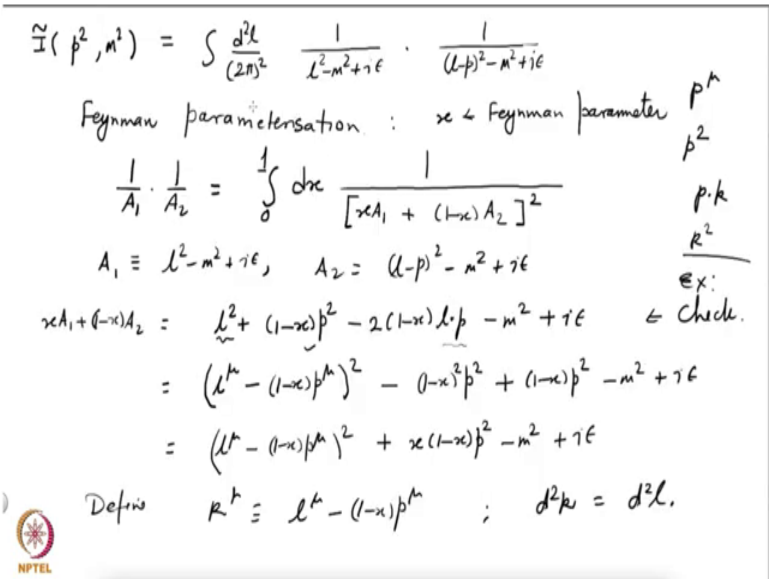

So that is why you have d 2 l here the propagators are the same but we are working in lower dimensions. Then we use Feynman parameterisation and we did some shifting of momenta and then we looked at the external momentum.

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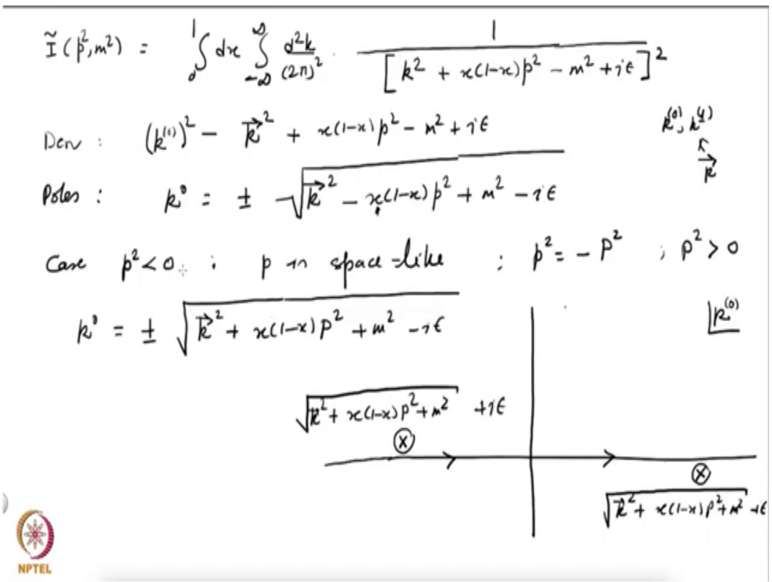

We looked at the case where external momentum is space like. And in that case we saw that we could rotate the contour access, the integration contour to the imaginary axis.

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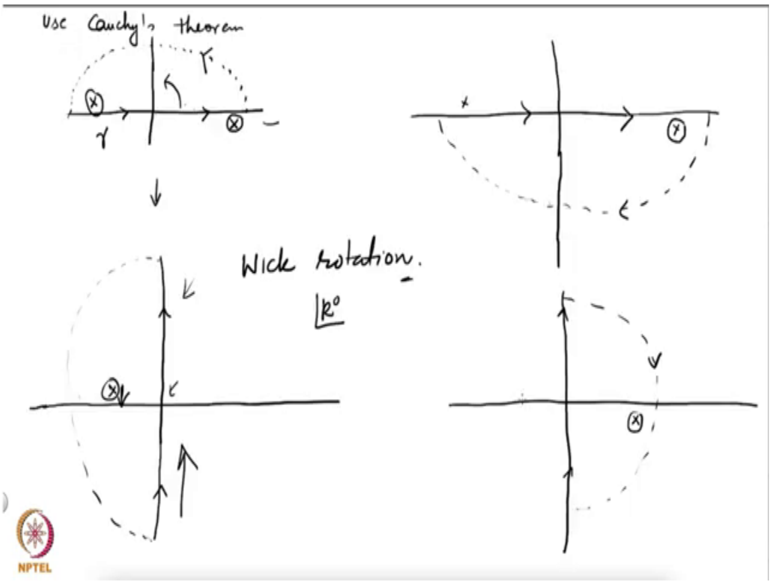

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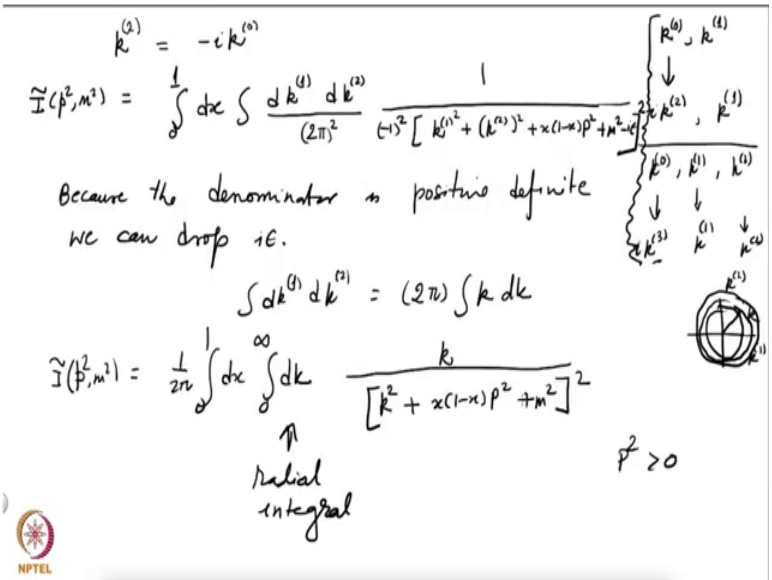

And that is called wick rotation. And then we could find the integral there were no singularities. This is the denominator here in the integral we found that it was positive definite. So, there were no singularities on the integration contour of k 2. If you are doing an integral over k 2. And then we found the results for physical momenta, where p square is time like by analytic continuation.

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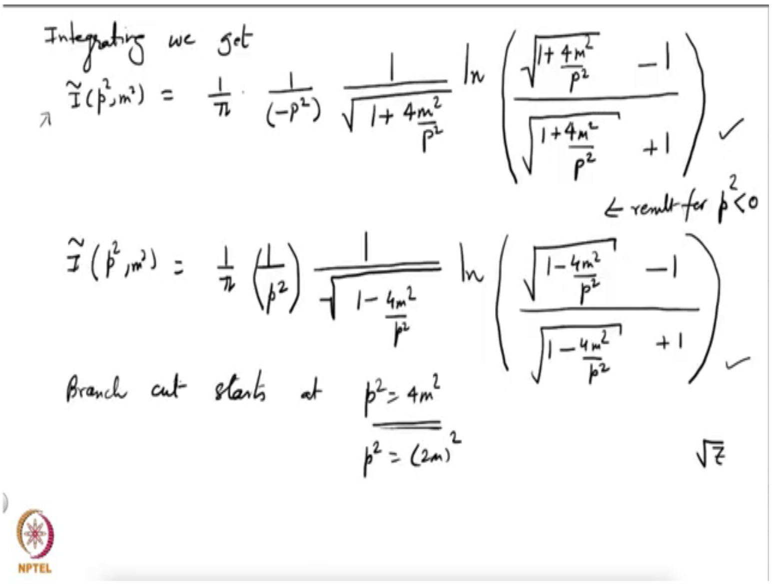

That is what we found here. And now, what we want to say is? That the same procedure goes through even when you are looking at Feynman diagrams which are having more than one loop. So, as you know that greens functions, the analytic structure of green function is going to be given by these integrals. The remaining overall factors and other things that sit in the numerator does not bother us.

So, we will not worry about those things and we will only worry about the Feynman integrals. So, let us proceed with that. I will draw one diagram to give you some idea of kind of denominators that we are going to get in these integrals. So and I am drawing it in 5 cube theory it does not matter because we are not looking at the vertices and other things anyway. So, you could do the same thing in 5 4 theory, similar diagrams and arrive at the same conclusion.

So, let us say that P 1, P 2 and P 3 are the external momenta. So, they let us say P 1 square is m 1 square, P 2 squared is m 2 square, P 3 square is m 3 square or you could take all of them to be equal does not matter. It does not change our argument. So, this is a diagram of how many loops this is a two loop diagram. And let us do a some assignment of momenta, so, I will put a loop momentum l 1 to be flowing in this direction.

You do not have to put it this way. You can put it differently. There are many ways in which you can assign momenta. But this is one particular choice of momentum assignment. And as you are already aware that result does not depend on how you assign these loop momentum.

So that is l 1 at this vertex P 1 and l 1 are entering. So, what should be flowing here is l 1 + P 1? Then at this vertex P 2 is injected.

Then what should be here is l 1 + P 1 + P 2. Now, here the momentum what flows in here is undetermined. Because I do not know what is coming from here? And that I do not know because this is a two loop diagram and you will need two loop momenta, they are undetermined, they have to be integrated over. So, you need one more loop momentum and let me assign l 2 to be flowing in this direction.

Then this one gets fixed it is l 1 + l 2 + P 1 + P 2. I want to change the direction of this one. I will put it this way does not matter. You can take it in also but I am going to take it out. Then you have this momentum l 1 + l 2 + P 1 + P 2 entering here and at this vertex P 3 is taken out. So, what flows in here is l 1 + l 2 + P 1 + P 2 – P 3. You could have drawn this differently and I mean not drawn but momentum assignment could have been different.

So, if you took l 2 to be going upwards instead of coming downwards then this l 2 would have come with a minus sign. So, with this, we see that the propagators any of the propagators in this entire diagram. As the following structure that it is a sum of loop momentum, loop momenta and external momenta they appear with some coefficients. Where, for example, here that coefficient is –1 and all others have appeared as +1.

But as I said, if l 2 was going upwards, if I had done a different assignment l 2 going upwards then here would you would have got –l 2. So, each of the propagator has the following structure. So, it is some alpha i l i so, you have to sum over the number of loops. So, in this case this is alpha 1 l 1 + alpha 2 l 2 maybe I should make it explicit later I will drop this summation but here I will put.

So, i = 1 to 2 for this case plus here you have three external momenta. So, a beta 1 P 1 + beta 2 P 2 + beta 3 P 3 where alpha i takes values 1, 0 or –1. A particular propagator may not have any of the external momenta flowing through it. For example, here in this one there is no P 1 or P 2 and in this one there is no P 1 or P 2. Similarly, it is possible that some line does not have some of the external moment of flowing through it.

So, these are the values that is alpha and beta will take. It can be a little bit more careful and put j here. So that is the structure that is what flows in the propagators. So now, what I will do is I will look at a general l loop greens function and that will have, of course, an l loop, Feynman diagram. And write down what the expression of that integral would be. So, let us hear itself so, consider l loop Feynman diagram.

So and it has N propagated lines. So, let us say it has N propagated lines and there are E external momenta. For the above diagram you have L = 2, N is equal to how much 1, 2, 3, 4, 5, 6 and E = 3.

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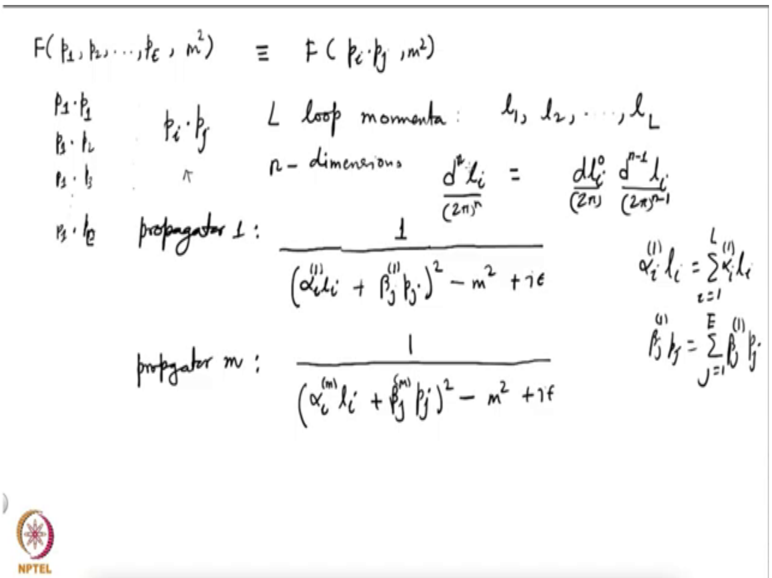

So, let us write down expression of a general Feynman diagram. So, I will denote the Feynman diagram by F. It will depend on this external momenta P 1, P 2 and so and so, forth, to P E. Then it will depend on all the masses. But let us say all the masses are equal. It does not change arguments so, they are all equal to m square. Now, as you know that these Feynman integrals are Lorentz invariant objects.

So, they will not depend on P 1, P 2 separately but only on Lorentz invariant combinations. And there are several Lorentz variant combinations that you can make. So, of course you have given this, you can make P 1 dot P 1 then you can make P 1 dot P 2, P 1 dot P 3 and so, forth P E. So, it will depend on these combination these Lorentz invariant quantities. It will not depend on P 1 mu separately.

But on either on P 1 when it will depend on P 1 square and P 1 dot P 2 and so, forth. And of course, similar other combinations so, in general, it will depend on P i dot P j where i and j both take values from 1 to E and also on m square. And if your momenta are on shell then P 1 square is itself m square. So that is the dependence that we have but remember that not all these will be independent.

Because you have energy momentum conservation constraint. So, not all these invariants are independent of each other. But I will not worry about making them or writing down the arguments with only independent ones. I will just leave it like this without any harm. So, it is really Feynman diagram will have the structure it will this dependence. I am going to denote these l loop momenta by l 1, l 2 l capital L.

So and also I want to work in N dimensions just like I previously worked in 2-dimensions. Because it was the integral was divergent in 4, I want to write expressions in N dimensions. And of course, this integral will be divergent in ultraviolet but I will pretend as if this problem does not bother us. And later, we will see that the reason why we can ignore that problem is because we can do renormalization.

But for now, we just pretend that is a problem which is not going to disturb us. Later, we will see that we can remove these singularities and then this analysis that I am going doing now goes through after you have done those subtraction of singularities. So, I am going to just write down expressions in general in and definitions. So, let us take a particular loop momentum l i so, d and l i that is what will enter into the integral.

And I am going to write it as dl i 0 dl sorry, there is a time component and now I have special components and there are n – 1 special components l i. So, if you are N is 4 then you have d4 l i that is dl 0 d cube l i so, 4 – 1 is 3 and in general n – 1. So, how about the first propagator? Let us call one of the propagators as propagator number 1 then another one is propagated number 2 and so, forth up to propagator number what did I use for and propagator number N.

So, let us write propagator 1 whichever you have assigned as one. So, I am again not keeping whatever comes in the numerators. So, although I am doing scalar theory right now but because I am not bothered about the numerators this analysis will go through even for other theories like QED. And other theories where you have fermions and other gauge fields so, this analysis is not restricted to only scalar theories.

So, propagator number not 1 and let us take it in general, propagator number sorry I want to keep it 1 propagator number 1. So, you have alpha i like I had previously written l i so, these are all the loop moment are flowing through the propagator. Then you have beta j, P j. These are all the loop moment that are flowing through the propagator. But then I should put a label 1 to remind me that this is for first propagator square it – m square.

And this is our i epsilon coming from i epsilon prescription. And I have dropped the summation, so, summation convention is now implied. So, alpha i, l i is basically alpha i, l i where i runs from 1 to the number of **(()) (18:08)** momentum which is l. And similarly, beta j, P j beta 1 j P j is j = 1 to the total number of external moment. Because we are adding this sum here contains a E number of terms.

So, beta j 1 P j that is one of the propagators and similarly you will write for other propagators. So, propagated number m will have same thing. But only this index 1 will get replaced by m. So, alpha i m l i + beta j m P j square – m square + i epsilon. Now, let me write down the expression which will be just writing d and l i for so, d n l 1, d n l 2, so, forth and each time you divide by 2 pi to the n.

So, I will always divide by 2 pi to the n in 4 dimension. That division is 1 over 2 pi to the 4 and here it will be 1 over 2 pi to the n. So, we have to just do that integral help in writing maybe I will write this entire integral again.

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$$F(p_i \cdot p_j, m^2) = \int_{-\infty}^{\infty} \frac{d^n l_1}{(2\pi)^n} \cdots \int_{-\infty}^{\infty} \frac{d^n l_L}{(2\pi)^n}$$

$$\times \frac{1}{(\alpha_i^{(1)} l_i + \beta_j^{(1)} p_j)^2 - m^2 + i\epsilon} \times \cdots \times \frac{1}{(\alpha_i^{(N)} l_i + \beta_j^{(N)} p_j)^2 - m^2 + i\epsilon}$$

$$= \Gamma(N) \int_0^1 dx_1 \cdots \int_0^1 dx_n \; \delta(x_1 + x_2 + \cdots + x_n - 1)$$

$$\times \int_{-\infty}^{\infty} \frac{d^n l_1}{(2\pi)^n} \cdots \int_{-\infty}^{\infty} \frac{d^n l_L}{(2\pi)^n}$$

$$\times \frac{1}{\left[x_1 \{(\alpha_i^{(1)} l_i + \beta_j^{(1)} p_j)^2 - m^2 + i\epsilon\} + \cdots + x_N \{(\alpha_i^{(N)} l_i + \beta_j^{(N)} p_j)^2 - m^2 + i\epsilon\}\right]^N}$$

It is a little wastage of time but I think it will make reading things easier. So, F P i dot P j m square is d and l 1 over 2 pi to the n and this integral limits run from minus infinity to plus infinity and then you have d n l total number l that is the loop number of loops 2 pi to the n. And what you integrate over is? These propagators 1 over alpha i 1 l i + beta j 1 P j square – m square + i epsilon this is the integral that we have.

Total number of propagators is n so that is why this is index n + beta j and P j whole square – m square + epsilon. So that is the integral which we want to study. And these alphas and betas take values 0 + 1 and –1. Now, we will use our old technique of using Feynman parameters to com bine these denominators.

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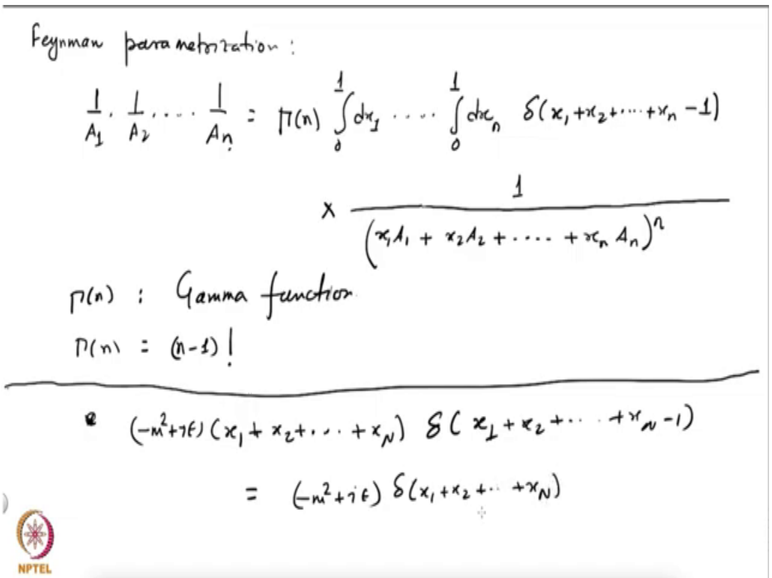

So, let me go to the next page and I will come back to this one again. Here Feynman parameters so, I will need to use a formula that is generalization of the formula that I have shown you earlier. Let me go back and see where it is? Here. So, here we had 1 over A 1, 1 over A 2 as integral over dx with these factors, where 2 is the this two coming because you have two functions A 1 and A 2.

So, I am going to write a generalization of this formula when you have several such functions A 1, A 2 and A n and the result is the following. This is just multiplication 1 over A 1 times 1 over A 2 so and so, forth, times 1 over A n. This is integral 0 to 1 dx 1. So that is x 1 is one of the Feynman parameters and actually there are n of them dx n this n this is not the number of dimensions this is, I am writing for.

If you have n factors 0, 2 and 1 and then you have a delta function that constrains these x 1 x 2 values. So, it says x 1 + x 2 + so and so, forth + x n – 1 meaning this delta function hits only when the sum of all these Feynman parameters is equal to 1 not otherwise. And then the integrand is this so, these all these denominators, A 1, A 2 and A 3 up to A n they nicely combine. And they give you x 1 A 1 + x 2 A 2 + x n A n and the power is n.

And here we have a pre factor which is gamma n so, gamma n is our familiar gamma function. And you know that gamma n = n – 1 factorial, if n is integer but gamma n is also defined for non integral values. You have gamma and defined for complex values as well. So, this is the Feynman parameterization and this is what I am going to use in manipulating the function here, the integral here.

So, using this, I can now write down this expression as gamma n. So, I am including this first time putting the Feynman parameter integrals dx 1, dx n. So that is the delta function. Then you have integrals over loop momenta and this run from minus infinity to plus infinity. From there you have all these denominators have been combined. So, you have x 1 alpha i 1 l i + beta 1 j + P j in that whole square – m square + i epsilon.

So, this is the contribution from the first propagator and this gets multiplied with x 1. then you have the remaining ones and the last one gets multiplied with x n because we have n such denominators, capital N denominator because we have capital N propagators. So, for each Feynman propagator, we have introduced one Feynman parameter. That is what we have done. So, alpha i n l i + beta j and P j square.

This entire thing squared – m square + i epsilon. and then this thing raised to the power n because we have n number of propagators. So, this is what we have to now analyse. So, let me first look at this term we have x 1 multiplying minus m square then you have next term will have x 2 multiplying, x 1 multiplying – m square + i epsilon. Then you have x 2 multiplying – m square + i epsilon and the last one is x n multiplying – m square + i epsilon.

So that is easy to deal with so, what you get is minus m square + i epsilon times x 1 + x 2 so and so, forth plus x n. But then the delta function here that you have says that the sum has to be equal to 1. If it is not equal to 1 then the integral is 0. It does not hit. So that condition has to be satisfied, so, I can use already use that condition here and put the sum to be 1 or let me make it more explicit.

This is equal to – m square + i epsilon times. See this is in the denominator and it does not come like a product like this but it is because of this delta function that I should put this equal to 1. So, you cannot use it directly and put it there but I hope you understand this argument. So then I will just replace as far as this – M square + epsilon from each term is concerned, I will just have an this thing will just sit outside of all these alpha l i + beta j l d terms.

So that one part is done and now, I will write this again. But this time I will also write explicitly the integral over time components of the loop moment. I will make that explicit.

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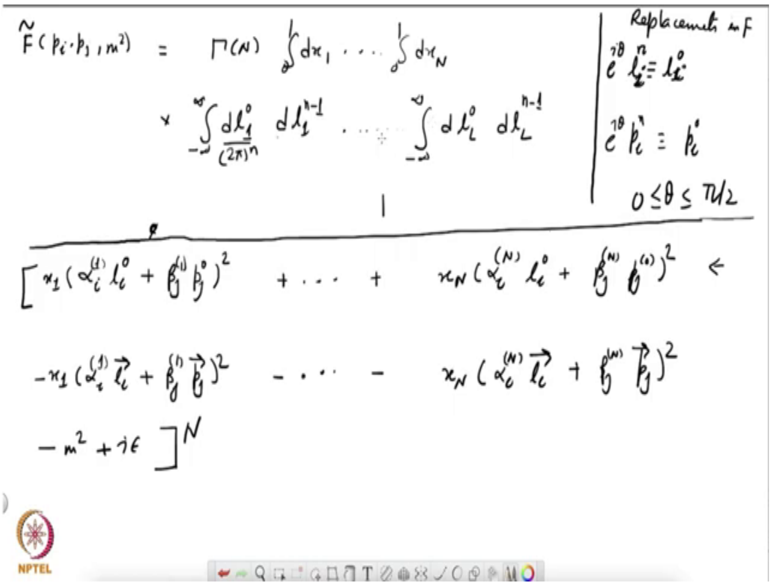

So, I am not doing much I am just carefully writing everything again. So, this Feynman integral is now gamma N then dl 1 0 component of it, dl 1 the remaining and management components I will just here. And then so, this these limits are for all the variables not just for the zeroth component. So, dl 0 L dl L n – 1. It is small n is for the dimensions in which we are working times. Let us write it carefully.

So or may be we can do a little better. Let us see so, we have first I will write down the time components or even better in just a second does not remember how I write it. So, you have x1 alpha i 1 l i 0 beta j 1 P j 0. So that is the time component and that has to be squared. So, this is this one so, I have just written the time component part of this. So, if you take this entire thing as Q squared then I have written Q 0 square and I should now write down – Q vector square.

That is what I am doing so, minus this is in the denominator. So, this one is this square, is what makes up the first term in this in this denominator? And – m square + epsilon I am going to write at the end. Then you have: Let us keep it a little bit space x n alpha i N l i 0 correct + beta N j P j 0 that squared. And I should have similarly, here the square of the space component which is alpha i N l i + beta j N P j squared.

And then you have – m square + i epsilon and this entire thing should be raised to the power. So that is what we have for this integral and I should see whether I can repeat the same line of reasoning which I had earlier for the case of one loop. I will now just copy this entire expression. Let us see if I can. There is another way like this and duplicate. So that is the Feynman integral which I want to evaluate and look at it is analytic structure. I am not going to evaluate it actually.

So now, let us take this integral and modify it slightly. So, I will define a new function, new integral which I call F tilde. So, this is different from F, F tilde, where the difference is the following. So, this F till I defined by replacing the following I make the following replacements in F. So, what do I do? I say l 1 0 this I defined as e to the i theta l 1 n. This n is referring to the number of dimensions.

This is let us go back here. You remember, I had e to the Sorry k to the 0 is i times k 2 and this 2 is for the number of dimensions k 0, k 1. Then k 0 I r label, as k 2 times this vector, so, I go to a 2 dimensional space k 0 and k. Similarly, because I am in N dimensions. So, I have these space components as k 1 k 2 up to k N – 1 the 0 is component. I am redefining using k n so that is the notation.

And not only for loop momentum. I will do the same thing for so, this is for all of all the loop momentas I will put an index. I and the same thing I will do for all the external momenta. So, if I take momentum P i 0, 0th component of P i then I defined P i n to be by this relation. So, if I do that and also before that I should I restrict theta between 0 and pi by 2.

I think now I have said everything that I should good, so, theta from 0 to pi by 2. So, here this is fine this d N, this dl 0, dl 1 0, will become dl 1. There is no gain in copying this I have to unfortunately rewrite it or maybe I will not maybe a lot a little tired of writing this over and over again. So, dl 0 I should replaced by E to the I theta l i N and similarly, here and then what are the other places?

Here l i 0, here l i 0 they get replaced and in this something I have in this P j 0's. So, if you look at the first line here, this entire thing gets modified because it picks up a factor of e to the 2 i theta because each term here is multiplied by e to the i theta square which is e to the i 2 theta. So, everyone is multiplied by e to the i 2 theta. So that is the factor it picks up and nothing happens to these.

And there is some factor of e to the i theta coming from each of these here. So, let us look at the denominator only the denominator.

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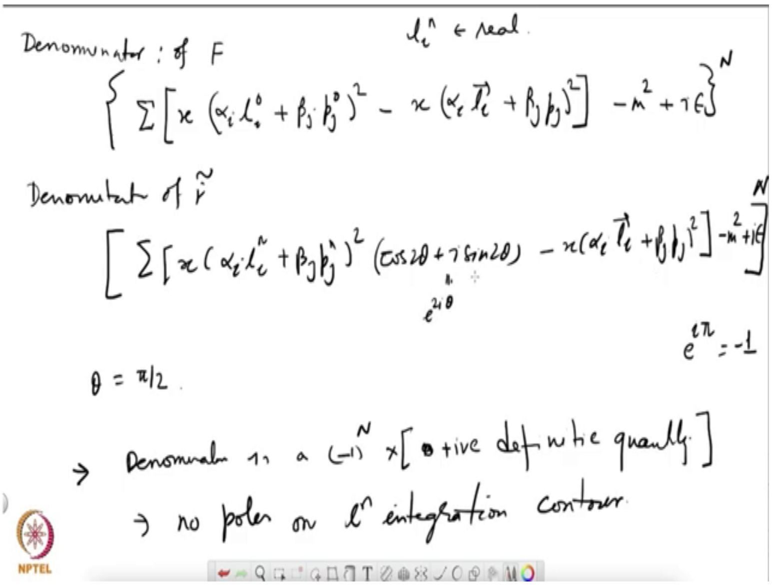

So, denominator what is the structure? Structure is I will drop these labels. I will this 1 1 here and n n here, I will just suppress it because it will be easier to write that way. So, the denominator is summation over all those indices that I am suppressing. I am not showing here and it is x alpha I l i 0 + beta j p j 0 squared – x alpha, i l i + beta j P j square and then also have – m square + i epsilon correct just a denominator.

And you understand the sum is over these suppressed indices. I will good and this is the denominator of f and denominator of F tilde is what is summation x alpha I, now this becomes alpha i l i n + beta j P j n squared – times cos 2 theta + i sin 2 theta This is the factor e to the i 2 theta and then you have minus that is what we have? So, now let us take theta to be pi by 2 and see what happens.

So, if theta is pi by 2 this is e to the i pi and this is so, e to the i pi and e to the i pi is –1. Now, when this factor becomes – 1, here it is anyway square of all real numbers, see l i n they are all real. So, l i n they are real. So, if l i n are real the square is going to be a positive number this gives provides you with a – 1. So, this is minus of some positive number. This is similarly, here minus of some positive number.

Then you have –m square, so that is again a negative number. So, this denominator other than the psi epsilon is a positive it is a negative definite number. Or if you take out minus I overall it is a positive definite quantity. It has a different sign, it does not change somewhere and it because it is negative definite. It never goes to 0. Here I should put a n that is the denominator, so, this denominator never vanishes.

So, it is well defined for theta = pi by 2. This is what is Wick rotation. So, let me write it down so, we have a the denominator is a negative definite quantity is – 1 power n times a positive because I have pulled out – 1, – 1 power n positive definite. And then I do not need i epsilon because if it never ends, if it this denominator, never becomes 0. Then you do not have to have an i epsilon, it does not become 0 when on the l i n axis. So, I will drop the type epsilon.

So, this is well defined because there are no poles on l and integration contour. And I have also included this –m square in the argument because let us just answer.

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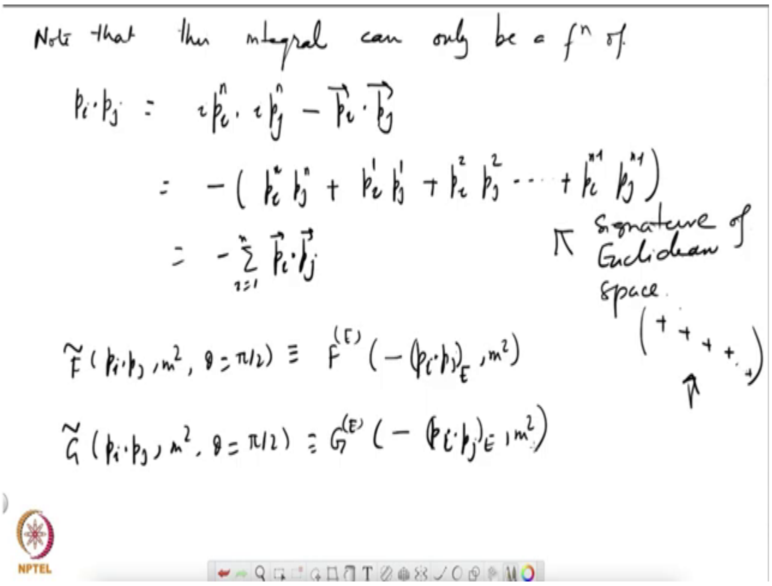

So, this integral now can only be a function of so, let me show note that this integral can only be a function of so, I earlier told you that it is a function of P i dot P j. I am taking theta = pi by 2 now. And arguing about F tilde 4 theta = pi by 2. So, this is, i P i n dot, i P j and this i is complex i – P i dot P j. P i and P j are the external moment is external momentum ith external momenta.

So that is what P i dot P j is and i square is – 1. So, minus summation over whatever let me let me write explicitly so, P i n P j n – P i 1 P j 1 P i 2 P j 2 up to P i n – 1 P j n – 1. Sorry, I made a mistake. I should have a plus because I have put lot of minus sign which is what you write as P i dot P j. So, you see that this this dot product now this that enters into the function F tilde does not have the dot products are not **(()) (49:20)**.

They, the signature is now not – 1 + 1 + 1 so, forth or + – –. So, here the signature is Euclidean. So, you do not have P i 0 P j 0 minus something all the terms are added together. So, this is what you have in Euclidean space, so, the signature is + + + + + to n times. So, this is what is usually defined as the Euclidean greens function. So that is F tilde with this P i dot P j m squared.

And if you put theta = pi by 2 then this is what is called as Euclidean Feynman integral P i dot P j. You have to evaluate it with this signature, so, I will make it explicit by putting Euclidean here and then you have m square. And because these Feynman integrals enter the green functions correspondingly the green function will be given by this. So, this is the expression for the greens function.

This is called Euclidean greens function. But we still have to do a little bit more work to say something about our original function. So now, let us look at what happens when theta is between 0 and pi by 2.

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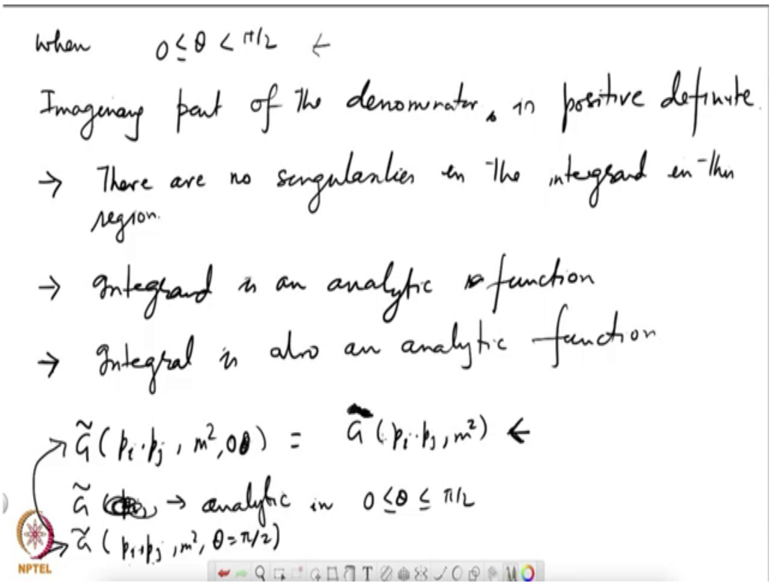

Here, so, when theta is between 0 and pi by 2 I want to look at the imaginary part of the of the denominator and from where do I get those imaginary parts? See these are all real objects I am sorry it is real. There is 1 part coming from i epsilon and another coming from this, i sin 2 theta term. That is what gives you the imaginary part. Now, sin 2 theta will be positive when theta is from ranges in the region 0 to pi by 2.

So that is positive. This overall coefficient which multiplies this, is also positive. So, the imaginary part and here also epsilon is positive. So, here the imaginary part is positive definite. So which means that the there are no singularities in the integrand in this region. Because that is positive definite when this the denominator the many part of the positive of the denominator is positive definite.

So, it means that there are no singularities in the integrand in this region. You are integrating over all the real values your the integral has all these this here. So, when you are integrating this denominator, you are integrating over l i n which takes real values. So because there is a imaginary component which is negative which is positive definite. You are the denominator the it never becomes 0 on the on the axis of on the control on the integration contour.

So, it never becomes 0 so, there are no singularities and and now your integrand is an analytic function. There are no singularities and when you integrate such a function, such an integrand you what you get is again an analytic function. So, what does that mean? How does that help? Before that I will tell you how it helps but before that pay attention to the fact to the i epsilon here.

It if so, happened that the imaginary part coming from here coming from this term or precisely from this multiplying test, came out to be negative. Instead of positive as we have got then it could cancel this i epsilon term. Because you get minus I times some quantity and for some values of these other variables that equals whatever epsilon you have chosen. So, there will be a 0 there will be a 0 sitting.

There will be a pole that you will have in the in the function. So, it will not be an analytic function and integrating it will not give you another analytic function. So, here we have seen that the way we are rotating the contour in this manner from 0 to pi by 2. We do not have any singularities that appear. So, what is the upshot? The upshot is that our G tilde or F tilde is same as what is here m square and then we had a theta is same as G tilde with so, not 0.

If you put 0 then it is same as G P i P j m square so that is the same greens function and for this value of theta and this is what we want to understand whether this is well defined object or not. So, as far as G tilde is concerned G tilde same as G when theta = 0. Then G tilde is an analytic function for in this range. So, G tilde is analytic in this entire range and also G tilde is well defined when we have taken theta = pi by 2.

That is the Wick rotated greens function. So that Wick rotated greens function exists because we saw that there are no poles on the contour. And then G tilde then it says that what it means is that the G tilde provides the analytic continuation of the Wick rotated greens function. And then we can continue this G tilde on the for theta = pi by 2 which is well defined to G tilde.

Here, because this is analytic function, I can get analytic continuation for that. So, with this we have argued that these objects are well defined. We can compute them and also find a analytic structure by by using this continuation from this Wick rotated greens function to the greens function with physical momenta. At here the moment are not physical because we have taken space like momenta and which we saw here.

Where was that here you see that P i dot P j which is what enters in the greens function. This is positive because this is a Euclidean and there is a minus sign. So, you see all these dot products are negative. All the moment are P i square they all become negative. So that is why I am saying that they are space like so you can go from this greens function defined for space like momenta.

And all space like dot products to greens function which is defined for physical momenta in which you are interested in. When you are looking at scattering processes. Because when you look at scattering processes, the external moment are on shell and they are physical momenta. So, this proves that we can finally do anything Continuation and everything is well defined. We will continue further in the next video.