Theory of Scalar Fields
Introduction to Quantum Field Theory - II
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Lecture – 21
Renormalisation
Loop diagrams-1

So, we will start talking about contributions to different observables that come from higher order corrections in Quantum field theory and these will involve fundament diagrams that contain loops. So that is the plan and we should understand that the real content of quantum field theory resides in these loop diagrams.

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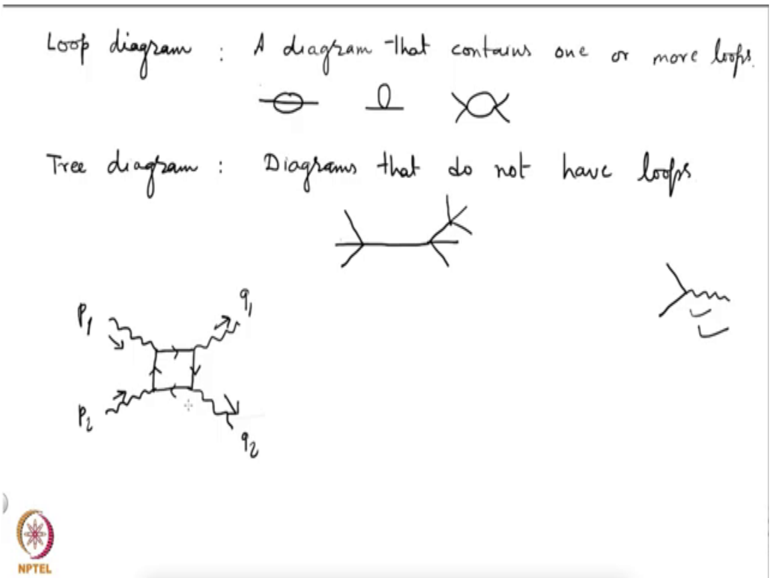

So, let me first define what a loop diagram is. So, a diagram that contains one or more loops. We have seen examples of such diagrams, for example, if you are looking at the two point function in 54 theory. This is a possible diagram, this is a two loop diagram, this is a 1 loop diagram and if you are looking at two point functions, this is one of the contributions which is at one loop and so, forth.

So, this is what we call a loop diagram and then we have something called tree diagram. Tree diagrams are those that do not contain loops. For example, you can look at this. This is in 54 theory again, 1, 2, 3, 4 sorry this is a tree diagram and if you put more lines here. It still remains a tree diagram. So, this is 1, 2, 3, 4, 5, 6, 7, 8 this has 8 external lines, so, it contributes to a greens function with 8 fields and this is a tree diagram because it does not contain any loops.

Now, the reason I am interested in loop diagrams is because if you are interested in some observable. Let us say cross section, and if you calculate cross section using only the lowest order diagrams then the result that you get it is not going to match with experiments. Because there can be a big difference between this experimental outcome and what you are predicting from theory not because your theory is not describing your experimental result correctly, meaning you do have a theory.

But because you have not included contact field, theoretic corrections into your calculation. I will talk more about this later and I will show you also the example of fix production, where it is very much clear that if we had not gone to higher orders in the calculation. Then we would obviously misunderstood the 2 photon peak that was found in that was observed in the experiment.

People would have wrongly concluded that what we are seeing in experiment is some new physics signal and not standard model physics. So, it is important that one gets the contributions of loops into the calculations, otherwise predictions are usually far from reality. There is also another reason that if you do not consider loops then some processes will appear to be impossible which in principle which in fact can occur.

For example, if you look at scattering of 2 photons. We have not done this we have done only scalar theory but I am giving an example. So, if you look at scattering of 2 photons, these are represented by these wavy lines in literature. Now, there is no vertex which which connects 2 photon lines but there is a vertex which is like this. So, this is some **(()) (00:05:37)** line and that is a photon if that vertex exists in QED quantum electrodynamics.

So, if you want to scatter a photon from another photon, it looks like it is impossible at tree level because it is not possible but if you include this vertex and make a loop, this is a fermionic loop using this vertex. I can draw this fundament diagram. Where here you have fermionic propagator, this could be an electron. Then you see that 2 photons can scatter non-trivially to give 2 photons in the final state that are going in different are having momenta which are different from the incoming moment.

Let us say P 1 and P 2 are the incoming momenta and Q 1 and Q 2 are outgoing momenta. Then this scattering can give you Q 1 and Q 2 being different from P 1 and P 2. But this process cannot occur at lower order than this. You have to involve a loop so that is another reason why we need to consider these loop diagrams as well. So, we are going to see that there is lot of trouble the moment one wants to include these loop diagrams.

But fortunately, the machine to deal with these things is well developed and that is what I am going to describe in this I mean I will start describing this lecture today.

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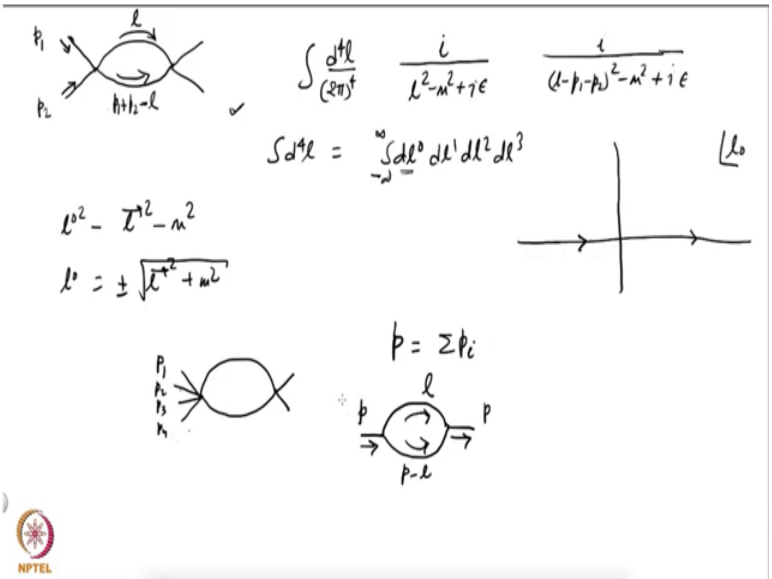

So, let us look at one example this is the 2 point function. Let us represent the loop momentum by L. We have already learned in the first course that when you have loops the number of the number of loop moment are running in the loops is equal to the number of loops. So, l is undetermined it is not fixed by the external momenta. It remains undetermined and we have to integrate over it.

So, if P 1 and P 2 enter they enter it at this vertex as P 1 + P 2. That is what enters here and l goes out in this direction and here you will have P 1 + P 2 – l. So, if you take it backwards, it will be in this direction – P 1 – P 2 + l and this will be – l so that at this vertex, the sum of all moment I entering is equal to 0. So, let me write down the expression it is d 4 l over 2 pi to the 4.

I am just writing the loop integral over l square – m square + i epsilon. And then we have to do this loop integral where d 4 l is dl 0 and dl 1, dl 2, dl 3. And you see that the poles if, if you do not take into account this, i epsilon then you get poles at poles on the real axis. So, you should view this as an integral over l0 in the complex plane. And here this is l 0 square, so, l 0 square – l square – m square.

So, you see that when l 0 is l square + m square +– square root of this then you have a pole on the real axis. And the integration contour also goes from – infinity to + infinity along the real axis. So, this is the integration contour and then you have this poles sitting on the contour. So that the integral is ill-defined it is not defined if you have poles sitting on the contour.

But then you are safe because you have i epsilon that takes the poles off the contour. And we are going to see how exactly there is i epsilon help us to make these help us in having these integrals well defined. So that is what I am going to show you. So, let us take a generic case, not generally not necessarily a 2 point sorry of a 4 point function. But any 1 loop diagram of this kind, so, I will just:

So, here you have, even if you have several lines depending on whatever theory you are looking at. So, here. 1, 2, 3, 4, 5, 6 so, you have a 6 point vertex. So, clearly we are looking at a theory in which you have 5 to the 6 term in the Lagrangian. And here, let us say 5 4 so, you have both such vertices and if you have momentum, P 1, P 2, P 3, P 4. Here then, as I said in this case, the momentum that enters in here is the sum of all these.

So, let me define p to be so, all these sum enters and similarly, here you will have the corresponding sub. But anyway, whatever goes in has to come out. So, I will instead of putting all these external legs. I will draw this simply as but if this does not necessarily mean that I am looking at a phi cube theory. It could be coming from phi cube theory but this line stands for all these lines and carries the momentum P.

Which is sum over p I and if this is l then it says this one is P – l. So, this is the diagram I want to look at. This is particular 1 loop diagram. Now, if you look at this, the corresponding integral is already I have written here.

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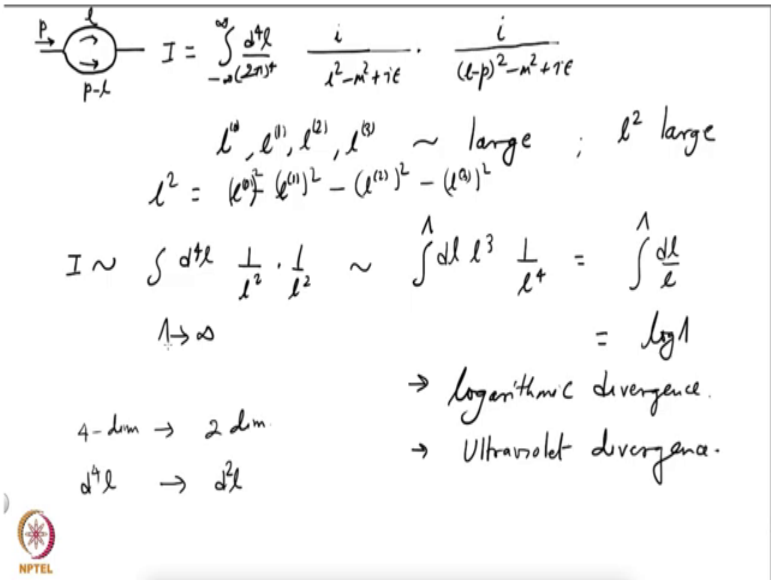

Let me write it on the next page and integral is d 4 l and remember that the limits run from – infinity to + infinity. Now, I want to look at this integral and pay attention to those configurations for which l 0 l 1, l 2 and l 3 they are all large. They take very large values because the integral runs up to infinity. So, this is something these configurations will appear.

And I also want to look at those configurations or I want to add a qualification that l square is also large. You see because we are in **(()) (00:15:28)** l square is l 0 square – l vector square. I will put it like this l 1, so, instead of this I will put here. So, l 1, l 0 square – l 1 square – l 2 square – l 3 square. So, even if all these components are large l 0, l 1, l 2 and l 3 it is possible that this difference is not large.

So, it is possible that l square is not large but I am looking at I want to look at those configurations for which l square is also large. In that case, I want to know the behaviour of this integral. How does it behave when all the components are large? Such that l square is also large. So, in that case, this m square is irrelevant. This i epsilon is also irrelevant, this m square I can drop this P also I can drop in comparison to l.

Because P is some fixed for momentum and I am talking about l being very large. So, P in comparison to a very large momentum can be dropped. So, this integral I behaves as d 4 l 1 over l square times 1 over l square, which is so, you do the angular integrals. I am being very not being very careful right now. But all I am doing is counting powers of l so, this is dl l cube that is d 4 l.

And then you have 1 over l power 4 which is dl over l. And I am interested in only the upper limit infinity so, to see clearly how things behave, let me put a cut off lambda and then I should take lambda to infinity. When I take lambda to infinity I am looking at again this integral in the large momentum limit. So, how does this behave? dl over l is log of lambda log of l and if I put the upper limit, it gives you log of lambda.

I am not worried about what is on the lower limit because I am not evaluating this integral. All I am doing is just trying to figure out the way this integral behaves in large level, large in the limit of large l. So, I am concerned only with the upper limit of l which is lambda now, not the lower limit. So, you see that now this has log of this is log of lambda, meaning as I take lambda to be very large, this diverges logarithmically.

So, we have a logarithmic divergence so, this integral is divergent and such divergence is all are called ultraviolet divergences. I will discuss in more detail about ultraviolet divergences later but now, my interest is not in this ultraviolet divergence but rather looking at this integral that whether this integral is well defined or not. Because you have these poles which are close to the contour.

And these epsilons are helping us but we have to see that indeed, whether they are really making the integrals well defined. So, what I will do is, I will not consider this integral because this is ultraviolet divergent. But I will look at an integral which has the same propagators, so, everything remains the same but I change the d 4 l to d 2 l. So, I go to a lower dimension instead of working in 4 dimensions. I will work in 2 dimensions.

Then this d 4 l gets replaced by d 2 l and this is helpful because as far as this divergence is concerned, this goes away. So, you see here the power counting was that your 4 powers of l in the numerator 4 powers of l in the denominator. And that gives you a logarithmic divergence. So, if I have only d 2 l here, it will be 2 powers in the numerator and four powers in the denominator and that is convergent.

So that integral will converge for large lambda. So, I will get rid of ultraviolet divergences and I can then focus on the issues that I am interested in. So that is what I am going to do.
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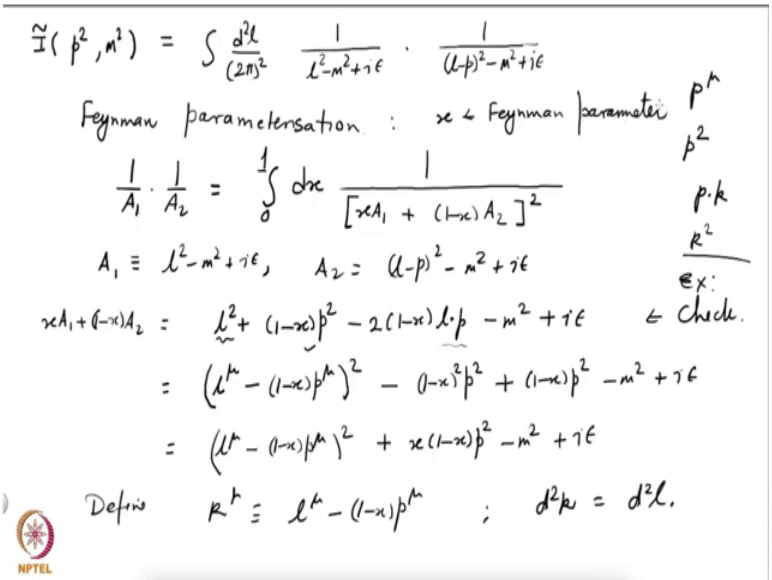

So, I will define a new integral I tilde, which is d 2 l, l is for loop momentum 2 pi square. I have also changed this factor 2 pi to the 4 to 2 pi to the 2 and 1 over l square – m square + i epsilon times I have dropped the factors of i, I should have kept actually. Let us drop it let us drop them from here as well. Now, one thing I should have mentioned here is that this P square is not necessarily equal to m square.

If it was a particle then if P square was on shell momentum then it would have been equal to m square. The P square but here P is sum of all these momenta which let us say they are on shell. So, P 1 square is m square, P 2 square is m square and so, forth. Then P 1 + P 2 + P 3 + P 4. That is some that is a square of the sum will not necessarily be equal to m square. So, you do not have that condition.

So, I will not make use of that relation which is only true for onshore particles where I was? So, I want to look at this integral I tilde. First, let us look at the arguments of this integral meaning of what quantities this integral is a function of so, l is integrated over, so, this quantity, this integral cannot depend on and on l because l is dummy. It can depend on m square because there is this constant that appears then it can also depend on P.

But you see this integral is made up of Lorentz invariant objects, l square is Lorentz invariant m square is Lawrence, invariant l – P square is Lorentz invariant, d 4 l or D 2 L if it was d 4 l that would have been Lorentz invariant. But if it is 2 dimensional space this is also Lorentz variant. So, this integral is only a function of the vector P. P mu but because the integral is Lorentz invariant, it cannot be a function of P mu.

It can only be a function of P square there is no other possibility. So, it has to be a function of P square and m square. See if you had this integral in which you had some other vector also appearing. Let us say you had l – P 1 – P 2 or, let us say, l – P – k, where k is also some external momentum then you will have P dot k also available to you. This is also a Lorentz invariant quantity and k square is also a Lorentz invariant quantity.

So, all these things would all these would also appear in i tilde, if you had k also appearing in this integral. But now because you have only p and the only invariant you can construct using P is P square and that is why I tilde will be a function of P square and m square So now, we want to look at the evaluation of this integral I tilde. So, for that I am going to introduce a technique which is called Feynman parameterization.

So, if you have 1 over A 1 times 1 over A 2, where here for example, in our present case A 1 is l square – m square + i epsilon and A 2 is l – E square – m square + i epsilon. So, the technique is you can combine this A 1 and A 2 like this, so, we introduce a parameter, called fireman, parameter, dx and the integral runs from 0 to 1. And the denominators combined in this fraction xA 1 + 1 – xA 2 and here you will have a square.

You have A 1, A 2, so, 2 denominators, so, there is a square here. And I will encourage you to prove this relation that this is indeed true but I will use. I will just assume that this is to be true and I will work with it. So, let us look at this integral now, so, A 1 are defined to be l square – m square + i epsilon and A 2 to be l – P square – m square + epsilon. So, the denominator is xA 1 + 1 – xA 2 and if you calculate this, you are going to get l square + 1 – x P square – 2 1 – x l dot p – m square + i epsilon.

I am keeping carefully i epsilon check this check that you get this that is an exercise. Now, what I want to do is, I want to complete the square for l. So, here you have a linear term in l that is a quadratic term, so, I want to complete the square. So, let me write down the following. I am looking at this one and this one. Now, so, I write this I write l mu l mu square l mu, l mu will be l square.

So that will generate this term and here you have – 1 – x P mu. So, when I square this it generates l square which is here it generates a square of this which I should subtract so, it is 1

– x square P square. And then it generates the product of these two terms with a factor of two which is what you have here – 2 1 – x l mu P mu which is l dot P. So that is this term and I have removed the constant of now, let us include this one this.

So, I have completed the square for l which is same as + x 1 – x P square – m square + i epsilon. So, this is what now sits in the denominator. So, what I will do now is define k mu to be l mu – 1 – x p mu. So, what do we have? Then we have d 2 k is equal to d 2 l. So, I will now express I tilde in terms of the variable k mu rather than l.

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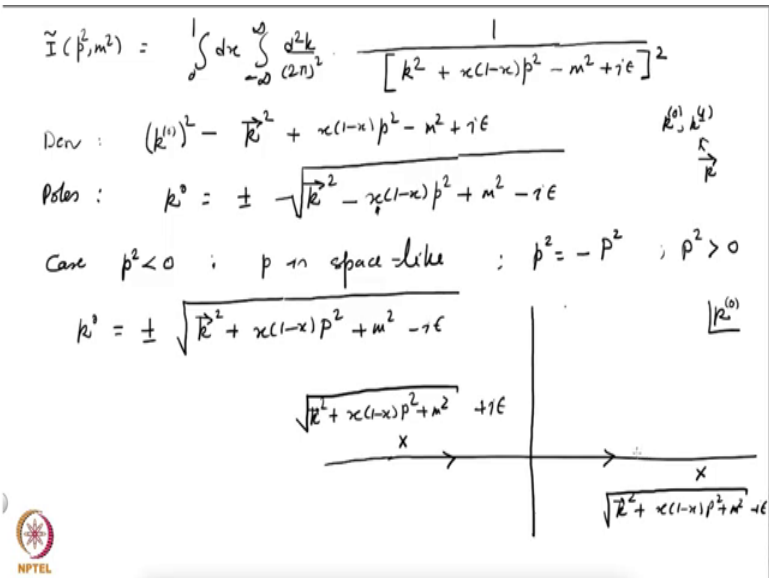

So, I tilde P square m square is equal to integral 0 to 1 dx and I should have mentioned x is called Feynman parameter. It is a Feynman parameter. Now, you have d 2 k over 2 pi square times k, square k square because this has been defined as k so that is k square + x 1 – x P square – m square + epsilon and then you have a square of this. That square comes because you have a square here.

And that square here is really because you have two product of two functions one over A1 and 1 over A 2. That is why you get a square there. So now, we are in this space 2 dimensional space k 0, k 1. But I will keep writing this as vector k because you could have also instead of looking I tilde the other one I am looking at, you could have looked at dQ well for example.

That would also have been convergent because if you have 3 powers here and 4 powers below that converges. So, I am just keeping the notation more general instead. In fact, I could have also looked at instead of d 2 l d n l, where n is anything 2 3 or whatever. So, I will just write k instead of k 1 but you understand that in 2 dimensions the time component is k 0 and then you are left with only one component.

But if you do not like what I am doing you can you write only k 1. So, k square the denominator is k 0 square – k vector square. So, where are the poles? So, this this is an integral and the integral integrand has some holes. It vanishes at some values and we are asking what are those values at which this integrand vanishes and the poles are at and of course I am looking at things in k 0 plane.

So, k 0 is equal to plus minus these are the locations of the poles. Now note that x is running from 0 to 1 so, x is always positive which means sorry x is always positive, of course but also smaller than 1. So, this x is always positive and this factor 1 – x is always positive which means x times 1 – x is always positive. But here the sign is not fixed because it depends on what the value of k square is and what is the value of P square, whether it is positive or negative?

So, what I will do is? I will take the case P square less than 0 meaning p is space like just choose this configuration other P square configurations are also present. You can work for those also but I want to look at the case when p square is less than 0 and see what happens then. Now, if I take P square, it will be less than 0 or negative space like then, this term – x times 1 – x P square that becomes positive.

It becomes positive, so, let me define p square to be minus capital P square where P capital P square is positive. So, this minus sign takes skills care of the fact that P square is baseline. So, k 0 the poles are at + x – x capital P square + m square – i epsilon. So, this is the place where the poles are and let me draw the k 0 plane. So, where are the poles on the plus side you have here because of this minus epsilon.

You have here a pole which is and on this side you have here. Because minus makes that epsilon plus and this is our integration contour because the integral over d k 0 runs from minus infinity to plus infinity. So, poles are poles, are located at these places.

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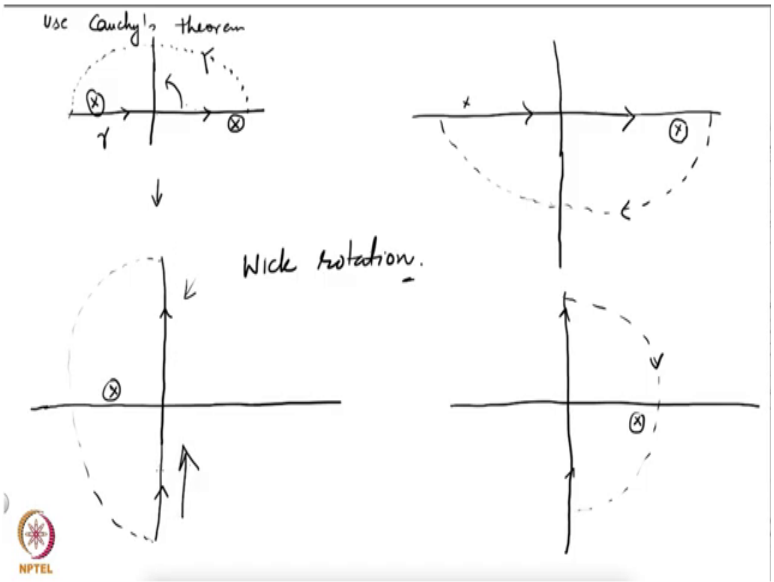

Now, what I can do is I can use Cauchy's theorem and do the following. So, you have this situation where the poles are located like this and the contour integration is this let me call the integration contour as gamma you can. You can put a semicircle which is infinite in length, infinite semicircle here. And doing so then you are picking up this pole or if you were to close it below then you are picking up this board.

Now what I will do is see this contour this encloses only this pole, not the other one. And similarly, here this contour and closes only this pole, not the other one and if the contribution from semicircle goes to 0 then you are allowed to I will close the contour like this. So, let us look at this one here I can do the following what happened? So, here I can just rotate this contour see as long as I am not crossing any poles.

I am allowed to deform the contour. If the poles that are enclosed within the contour, they remain unchanged using Cauchy's theorem I can deform the contour without changing the integral. So, I will just what I will do is I will rotate it and this I will take to here. So then the contour becomes so, take the x axis and rotate by 90 degree clockwise sorry just it is you rotate it 90 degree counter clockwise.

And doing so we are in this situation and which is fine because I still enclose the same poles and no new pole has entered in this region. So, I can deform the contour this way and similarly, here I could have if you close this way then again, deforming is allowed because then it becomes: So, you see you enclose exactly the same poles and no new poles are in this region. So, these deformations allowed using Cauchy's theorem.

So, instead of evaluating along the x-axis along the real k 0 axis, I can evaluate along this imaginary k 0 axis. That is the this is what is called Wick rotation. So, we will do this week, rotation and we should also note that this rotation from real axis to the imaginary axis is possible because of the way these poles are placed here. Because these poles are not hindering the rotation of this axis.

So, we can do this but if and that is also due to the fact that we are looking at a space like P square. So because we chose P square to be space like these, poles were coming out to be located at these places which allows us to do a Wick rotation. But if you take P square to be positive in general, you will not get an arrangement which allows you to do a Wick rotation. So that is why we are going to always look at space like case space like P square.

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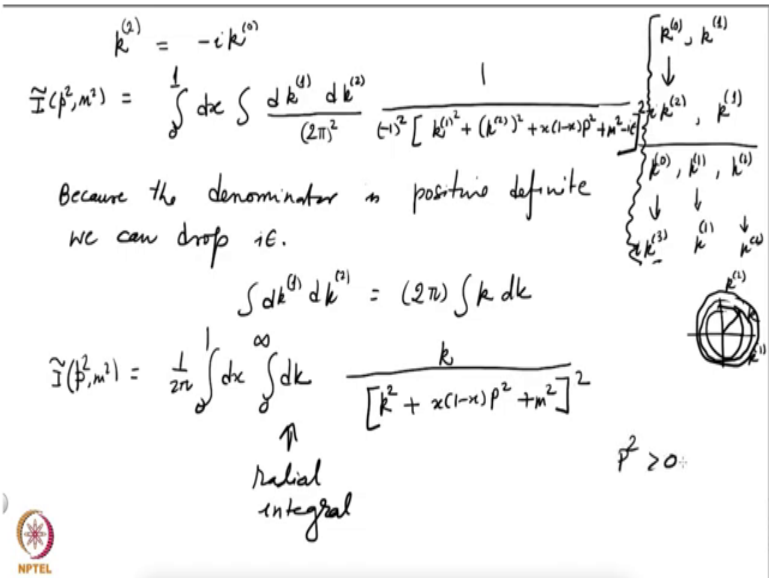

Now, that our integration is along this imaginary axis imaginary k 0 axis, I will define k 2 to be – ik 0. So, let me explain this notation, so, you are working in k 0 k 1 plane but now this k 0 I am writing as ik 2 and this is k 1 So, 0 1 has become 1 and then this has become 2. If I had taken instead of 2 dimensions like here d 2 k if I had taken d 3 k then it would have been k 0, k 1 and k 2 3 dimensional.

And then the notation would happen in this case when I have done the Wick rotation after that I would have called this as I k 3 so that I have k 1, k 2 and k 3 This will be the variables then so that is a nomenclature. So, sorry that is the notation. So, I have define this and with this definition, the integral becomes integral over the family parameter x dk1 and then dk 2.

The 0 thing has become dk 0 has become dk 2 over 2 pi square and then you have 1 over – 1 square times k 1 square + k2 square + x 1 – x P square, where capital P square is positive + m square – i epsilon square. And note that k 2 is real because k 0 is imaginary because we are doing integral over the imaginary axis see here, this is k 0 plane. I have turned the contour integration from this to this.

Now, k 0 takes imaginary values on this 3 on this imaginary axis but because I have multiplied, k 0 with – I this has become k 2 is a real number. So, k 2 takes only real values k 1 also takes only real values. So, this k 1 square, k 2 square these are all positive numbers. P square is positive x times 1 – x is positive, as I argued earlier m square is positive. So, everything in the denominator is positive.

So, it does not become 0 because it is positive. So, there is no need of i epsilon you can drop it. Can then, of course you have this this square also, so you do not need this so I just remove it. So, this is an integral that we need to do. We can do the angular integral first. That is easy because your integrand, it does not depend on angles. It is just depending on k 1 square + k 2 squared. So, it is easy to do the angular integral d k1, d k 2.

This is equal to you see we are doing in this plane k 1 k 2 plane and this angular integral will give you 2 pi times k dk where k is now the magnitude of k 1 square + k 2 square root. That is what is k. So, this is, let us call it k 1 2 and this is k. So, I tilde P square m square is equal to 1 over 2 pi 0 to 1 dx 0 to infinity dk. This is the radial integral and in angular integral I have done I have to now do the radial integral.

So, radial, integral is dk, k dk so that k is here over the denominator that I had written earlier which now becomes k square plus: this is what you have. Now, if you integrate this, you will get the following.

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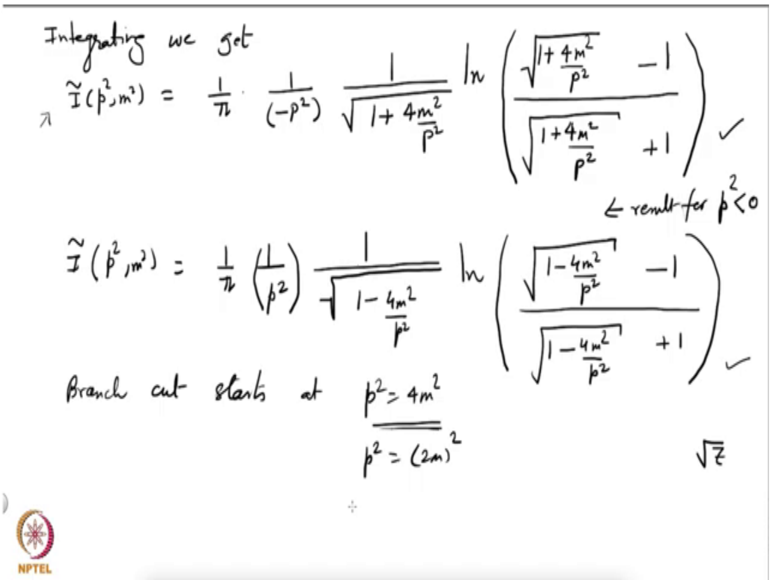

So, this is what we get and we see that this is because you see P square is positive m square is positive, so, what you have in the square root is larger than 1. So, subtraction from 1 will still give you something which is positive, so and denominator is also positive. So, log of a positive number, so that is positive here, is the square root of a positive object which is fine. So, this is an analytic function of capital P square.

Now I can analytically continue this to this is still for but I can continue this to all of complex plane. So, I am creating P square as complex but I have removed this restriction of P square being negative Sorry, this is for any P square. That is the continuation because this expression is, I can put a minus here and make a P square and which is valid for all P square, less than 0.

I can then continue for all P square but then I see that this I tilde which is the continuation of this title I should have used a different symbol but it is this analytically continued function has branch cuts. And where is the branch cut? Branch cut starts at P square equal 4m square. Check that this is indeed the place where the branch cut starts. So, you know the analytic structure of where the branch cut starts for a square root.

So, you can look at square root of Z. Where the branch points are for square root of Z and you also have log. You also know where the logs has branch points and then look at them together and convince yourself that the branch point is at 4m square. So that is the place where branch code starts also, there is a 1 over P square. So, it looks like there is a pole at P square = 0 in I tilde.

But that is another exercise convince yourself or maybe I will show that to you. So, in general you have a 1 over P square at different branches of dysfunction but in the principal branch or in the physical sheet you will not have a P square = 0 pole. And that is easy to check you can and you can also analyse this function and then convince yourselves that P square = 0 is not a call when you are in the physical sheet.

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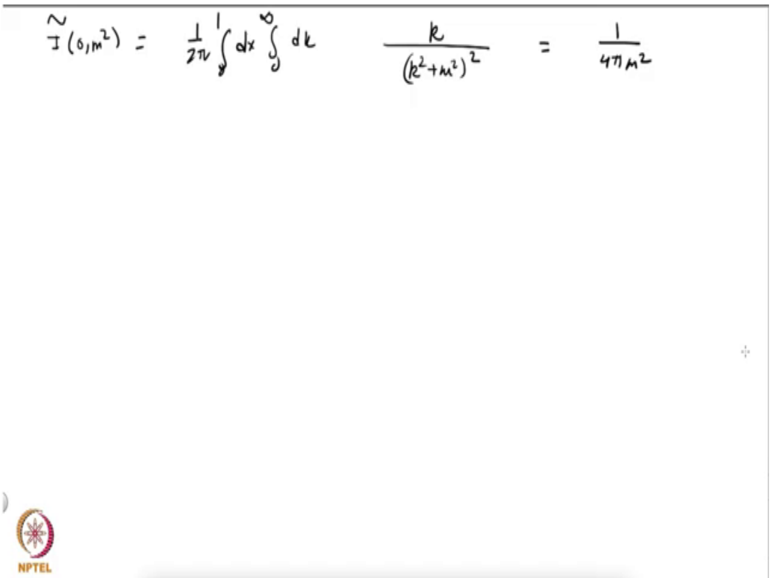

But you can do it differently also, you can just look at I tilde and put P square = 0 and evaluated. So, this becomes 1 over 2 pi 0 to 1 dx dk is 0 to infinity and this is k over k square + m square whole square. And you evaluate this and you will get 4 pi m square. I hope I made no mistakes or this is what you should get and clearly there is no similarity here. So, I tilde of 0 which is p square = 0 is a finite object, not a singular object.

And that is why I made this claim that when P square = 0, there is no similarity. There is no pole at P square = 0 but that that pole can appear can be present in the other sheets. You see this is a function which is which has a branch cut. So, when you go to other branches that pole may still be present. So that you have to carefully analyse and understand. So, what we have seen is that because if I take the external momentum of this Feynman diagram to be space like then I can do a Wick rotation.

And the integral is clearly well defined because there are no poles on the integration contour. Because this pole, when you take epsilon, goes to 0, it goes here. It does not go here, so, there is no issue of the integral getting being ill defined because whole is not migrating to the contour. Here it was other way around. The poll was actually, going to the contour. So, you see that taking space like external momenta allows you a Wick rotation.

And that gives you an integral that is well defined. and then you can do a analytic continuation to other values of P square, like here. And get the get the result for physical momentum because your physical momenta, if you are looking at a scattering process, then these physical momenta will be time line. They will not be space-like, so, you can do a continuation to those momenta after Wick, rotation.

So, I will next analyse the Feynman integrals without restricting to 1 loop and I will arrive at the same conclusion that this, whatever procedure I have outlined can be repeated there which means that these Feynman integrals or these greens functions are well defined. For space like external momenta and then we can. We can do analytic continuation I will, make a side remark here. This p square = 4m square.

You have seen earlier at some point when we were talking about the spectral density and I had given a remark that spectral density has a branch cut starting at 4m square which is the threshold of creating two particle states. So, you see, p square is 2m whole square. If the external energy is such that if the external energy is such that it produces barely two particles at rest then p square is to m square.

And this is the threshold of producing two particles and this is what you are seeing here that this two point function has a branch cut starting at 2m square. And it comes because of this integral here. So, you see when this p is such that the energy which you pump in is just sufficient to produce two particles at rest. Each having mass m then you have 2m, 2m square is 4m square and that is why you have a branch cut starting at 4m square.

This is what we had said earlier and here you now see after an explicit integration. So, we will continue this discussion in the next video