Theory of Scalar Fields
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Lecture – 20
Introduction to Quantum Field Theory-II
2-2 Scattering Cross-Section

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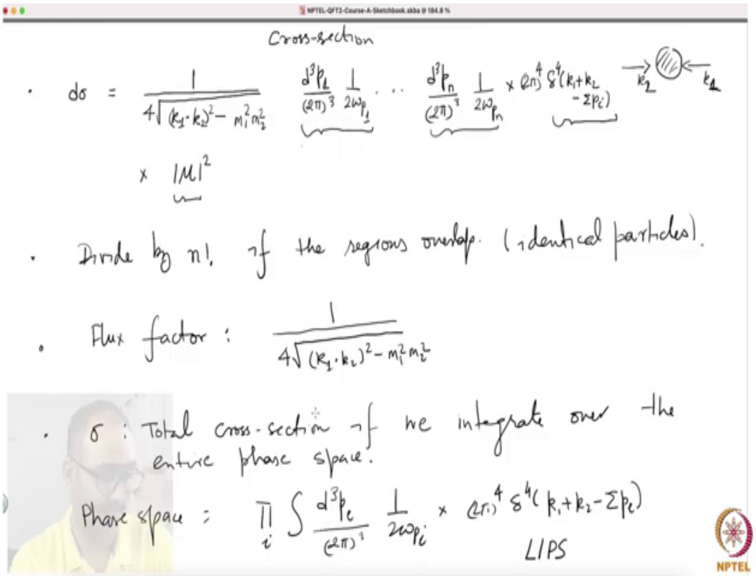

Let us continue our discussion of differential cross sections or cross sections in general. So, let us recall what we wrote down for the cross section? It was this expression here I am just rewriting it, it will be useful. So, d sigma is 1 over 4 I hope you have already done this exercise I gave it last time –2 square. So, we have some scattering happening and these are the incoming particles of momentum, k 1 and k 2.

This beam had momentum I think I called it k 1. So, I should label it like this. So then you have all these differential elements. These correspond to the final state particles. Then you have matrix element squared. I will suppress the arguments and then you have I should also this k 1 + k 2 – summation over P i this delta function also times matrix element square. And right now, all the differential elements, P 1, P 2, P n they are in different regions they do not overlap.

So, one thing I forgot to tell you last time I realized later that I did not mention. Here till this stage we had this probability given by this expression, where we had f 1 tilde and f 2 tilde still present here. And after that I we use the normalization of f 1 tilde and f 2 tilde which I have shown you earlier. And I was assuming that I mean I was working in the limit in which these folding functions are smearing functions are sharply peaked around values k 1 and k 2.

So, that only those values are picked up by these factors. And this is no problem because even though the width of these Gaussians which we are taking as smearing functions, they are finite they are not 0, something small. But that does not cause any trouble because if you look at detectors when they detect, they detect a particle hits somewhere and they can detect the momentum.

But they detect the momentum with some finite resolution. They cannot distinguish between a particle which is having a momentum sharply peak that k 1 or if it has a small spread around k 1. So, when you prepare these states, our measurements are not of that quality that they can really resolve the difference between a very sharp peak and a slightly broad peak. So, we are not making any error in that sense.

So, these are all consistent with the measurement requirements that we have. So, I can replace these by their just normalizations. And I am assuming that I have a very sharply peaked Gaussian. Also, some names I want to tell so, look at this factor. This factor has come from here this line here. And of course, this is not a Lorentz invariant quantity but what you see in this line? It is boost invariant alongside direction.

If you do boost only along z direction then it does not change. And I can so, this update you should not think that this is Lorentz invariant. Even though the weight is written, it is Lorentz invariant but then we should understand that this line is equal to this factor only when k 1 and k 2 are parallel to each other. They are collinear momenta, not otherwise. Otherwise, this is not equal to that.

So, it is equal only if only in that limit this collinear limit, where these k 1 and k 2 are in along the z-axis then so that invariant under boost along z but other than that all other factors are invariant Lorentz invariant. So, if you look at this, this kind of things we have seen in the first course also this is just invariant measure. Similarly, these are all invariant. This is anyway delta force, so that is invariant that matrix element squared is also invariant.

So, other than this factor, everything else is Lorentz invariant and this piece is invariant under boost along the z-axis. Another thing that if these regions R 1 R 2 are in the overlap, and because we are, we in this theory, we have bosons identical bosons. So, the same state gets I mean counted twice because if you have two particles in a given region and the regions overlap. Two particles in a given region then saying that this one has momentum P 1 and that one has momentum P 2 or vice-a-versa.

These are not two different states. This is the same state you can these, this particles are indistinguishable. But the way this has been set up it is counting as if these particles are distinguishable you are tracking the moment of each of these particles. So, I will be then over counting the number of such states. And I should divide by 1 over and 1 factorial n factorial if you have n particles in the final state.

So, divide by n factorial, if the regions overlap, if all of these r's are same identical. And then that factor you can include together with mod m square. So, you can write 1 over n factorial mod m square. That will be the case when you have identical particles. Some names this factor is called flux factor this first prefactor is called flux factor. Yes just a name it is useful to have names.

So that you can refer to these objects easily that is flux factor. And you get total gross section sigma if you integrate over the entire phase space and this thing is called phase space. If I integrate over this integral dp1 integral dp n with these objects, together with the delta function that is called phase space. I will write down in a moment if we integrate over the entire phase space.

And what is phase space is? Integral d cube P i over 2 pi cube 1 over 2 omega p i product over all is this entire factor times 2 pi to the 4 delta 4. This entire thing is called phase space, or phase space integral phase space. So, because it is Lorentz invariant it is also called Lorentz invariant phase space LIPS. And the differential elements are sometimes called de LIPS. Good then let us look at some examples.

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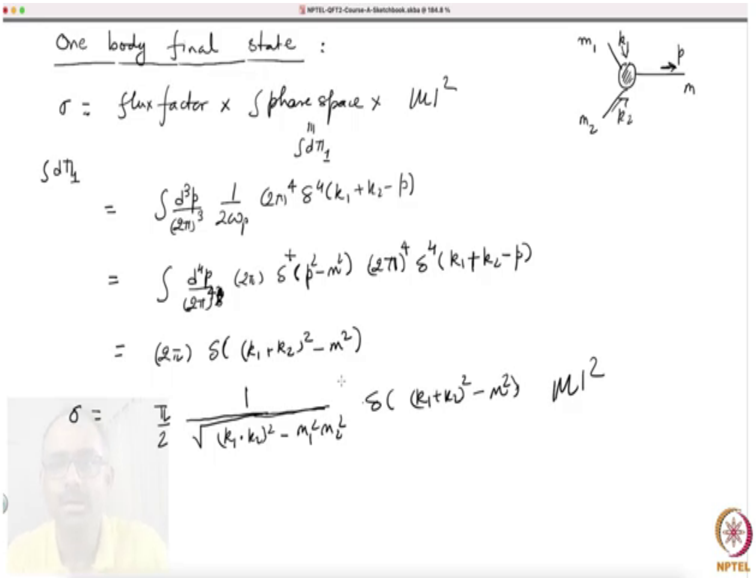

So, first, let us one body look at the first process which has one body final state. This will be useful when you are studying deep elastic scattering and we will not do that in this course. But at least I can talk about phase space here. So, here is the thing, so, you have a particle with momentum k 1, another particle with momentum k 2. And as far as I am concerned right now, it does not matter whether, what is the nature of these particles?

Because I am looking only at the kinematics the phase space part. I am not looking at the dynamics. Dynamics is contained in mod m square. That guy knows about which particle is interacting with which kind of particle and what are the interaction forces? But phase space does not know about those things. So, suppose we look at one body final state and then I want to write down the expression of cross section.

Of course, I will write it in terms of modern square. Modern square I will not specify because that requires looking at a particular process. But I will do the kinematic part and I will also allow for these masses to be different. So, let us say this particle has mass m 1. The momentum k 2 is carried by a particle of mass m 2 and this p is carried by particle of mass m. So, what is cross section?

Cross section is flux factor times integral over the phase space times matrix element square. So, let us do the phase space first, phase space let us write it as d pi 1. There is just a notation for this thing one stands for one body final state. So, this is integral d pi 1. That is what the notation is so, d pi 1 is equal to. What is that d cube p? There is only one particle in the final state 2 pi cube 1 over 2 omega p.

Where omega p is the energy of this particle p or carrying momentum p. And then you have this delta function. I will encourage you to do this calculation before watching the video and hopefully you will succeed on your own. But in case you have difficulty then you can refer to this video but try it on your own first. So, this is Lorentz invariant object and we have seen earlier that we can write this as d 4 p over 2 pi to the 4.

Then delta plus and the plus means that the energy component is positive delta plus of p squared – m square. And this is a factor of 2 pi the time machine, 2 pi times delta plus of something has happened again. Now, all the factors are fine I am still missing this 2 pi to the 4. I think I know what is happening p? Now because we have this k 1 and k 2 as these incoming particles with real energy. So, they are positive automatically.

So, I do not have to write it as an additional constraint. Because k 1 sorry k 1 + k 2. The energy components of; this is the energy component of p, p 0. And because k 1 0 + k 2 0 is positive p 0 is positive. So, it is ensured by this. So, I do not need to carry a plus with me I can drop it. It is automatically insured from here. And then I can do this integral d 4 p cube, sorry. So, what will that give? It gives some half vector of sorry that is the mistake.

That is why my factors were not matching 2 pi 4. So, this 2 pi to the 4 cancels 2 pi to the 4 and only this 2 pi is left so, 2 pi. In doing the integral we will take away this delta function and replace p by k 1 + k 2. So, I get k 1 + k 2 whole square – m square that is fine that is good. So, sigma cross section is now pi over 2 1 over k 1 dot k 2 whole square – m 1 square m 2 square times this delta of k 1 + k 2 whole square – m square times matrix element square.

So, this will be the expression and you can now for whatever process you want to get this cross section. You can supply the modem square for that process. And this line will provide you the result for that. So, next we will take a two body final state.

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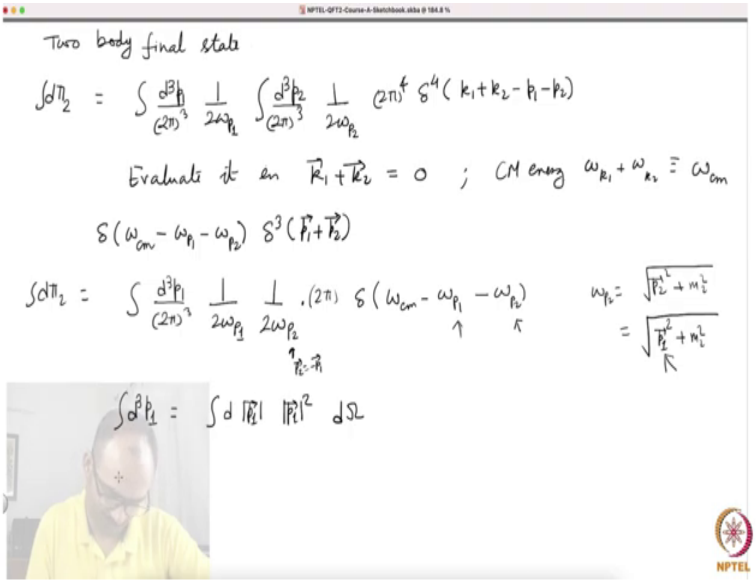

So, let us look at two body final state. So, what is the phase space? That is I am going to denote by dpi 2 to refers to the two body final state. And the phase space is integral d cube p 1 over 2 pi cube 1 over 2 omega I want to write p 1. So, the energy omega p 1 is the energy of particle carrying momentum p 1, an integral d cube p 2 over 2 pi cube 1 over 2 omega p 2. And then over and energy momentum, conserving delta function, k 1 + k 2 – p 1 – p 2.

Now, this object is Lorentz invariant, as I have said earlier also. So, let us go and evaluate this in the centre of mass frame of k 1 and k 2. So, let us evaluate it n k 1 + k 2 rest frame so, k 1 = – k 2. The moment of incoming particles are back to back and equal. So, if we are in that frame and the centre of mass energy would be omega k 1 + omega k 2 which I will call omega cm.

Now, if I look at this delta function in the centre of mass frame then I can write it as see these are 4 delta functions. I will write energy delta function corresponding to energy and the moment are separately, so, you will have delta of omega cm. So, k 1 0 + k 2 0 is what is omega k 1 + omega k 2? And because I am in central mass frame, this is equal to omega cm should be equal to omega p 1 + omega p 2.

This is what I should get and then the delta cube of now because of my choice, k 1 + k 2 = 0. So, sum of these three momenta is equal to 0 and then I am left with only p 1 + p 2. What does this delta function says this function? This delta function says that it hits only when p 1 p 2 = – p 1. Meaning the momentum of the final state particles are back to back. It does not tell you the direction but it tells you that they are back to back which is what you expect?

So, momentum, conserving delta function is giving you the correct result. Now, as I have said repeatedly the moment, you have delta functions, it is easy to integrate. So, let us integrate over d cube p 2 and use up this delta function. And doing so, we will put p 2 = – p 1, wherever p 2 appears which is this place 1 over 2 omega p 2 and also this delta function. So, these 3 delta functions will be used up.

One delta function will be left and all p 2's will get replaced by p 1. So, what you get is the following? So, 2 pi cube will kill this 2 pi will leave only one factor of 2 pi from here. And you will have 1 over 2 omega p 2 but now here in this p 2 you should use p 2 = – p 1. That is what you should do. But I will still continue writing p 2, it is easier to write that way. But we understand that this condition has to be supplied that p 1 + p 2 = 0.

Then you have factor of 2 pi and then a delta of omega cm – omega p 1 – omega p 2. And here also in this so, what is omega p 2 now omega p 2 was p 2 square + m 2 square. But now that is because of this delta function p 1 square + m 2 square. Because I put p 2 = – p 1, so that is omega p 2, good that is fine. Now, omega p 1 and omega p 2 they all contain p 1 square.

Now, we still have an integral left over p 1 but they all have the magnet, the square of the magnitude of vector p 1. Because p 1 appears here and appears in this exactly in this way P 1 square + m 1 square, square root. It appears here again you have p 1 square appearing here and these two places also. So, I can do this integral if I separate this into angular and radial parts. And then I can use one of these delta functions which are only one and do the integral.

So that is what I am going to do now. So, d cube p 1 I will write as d p 1 that is the magnitude of p 1 square. And then you have also the solid angle. So, this is what you have this is the solid angle volume element coming from the solid angle and this is the radial part. Now, I will do these integrals or probably not yet. So, I can write let us write once this expression is useful.

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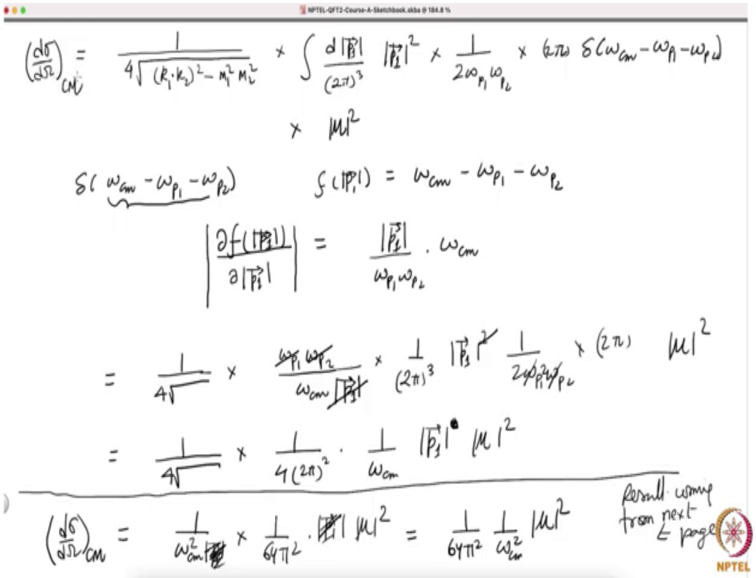

So, I am writing d sigma over d omega. So, I am taking this volume element to the left side of this expression. So, d sigma I have calculated d cube p 1 d cube p 2 that thing and I am I have mod m square. And the final integral with d cube p 1 I have a d omega that the omega I am taking to the left. So, I get this sigma over d omega. And this is equal to 1 over 4 k 1 dot k 2 whole square – m 1 square m 2 square test the flux factor times integral d P 1 over 2 pi cube.

Then you have p 1 mod square then 1 over 2 omega p 1 omega p 2 but remember p 2 vector is equal to – p 1 vector. That is what I should use 2 pi times delta of omega centre of mass – omega p 1 – omega p 2 times square. So, this mod m square has to be integrated over p 1. So now, we have to do the integral and for that I will utilize this delta function. I should find out the measure it provides. So that is easy.

So, you take delta of omega centre of mass – omega p 1 – omega p 2. And take this argument and define it to be f of P 1, what is f of P 1? It is this thing omega cm – omega p 1 – omega p 2. Now, I should find out the derivative of f with respect to the magnitude of p 1. What will that give? This gives a constant. And actually a modulus of this I will need one over modulus of this modulus of this gives 0.

This will give you maybe I will just write on the final answer because it is very easy. You will get P 1 over omega p 1 omega p 2 times omega c m. You can do this differentiation it is not difficult. So, what do you get right now? So, this is 1 over 4 times whatever that is in the square root times. So, this factor, I have to divide in the denominator, so that will give you omega p 1 omega p 2 over omega cm.

That comes from here and then you also have magnitude of p 1 so, 1 over this thing. Then when I do the integral over p 1, the delta function gets used up. And here 1 over 2 pi cube where I should use omega p 1 + omega p 2 is equal to the centre of mass energy. So, there is a factor of four coming from here and mod m square. Let me check if I missed something or looks good.

So, I will write it down 4 times that square root times 1 over 4 into 2 pi square into this 1 over omega cm and then this p 1 gets cancelled with this p 1 square. So, I have only P 1 modulus square and then I have matrix element square. Did I miss something? No, everything is here. So now, let us look at the flux factor.

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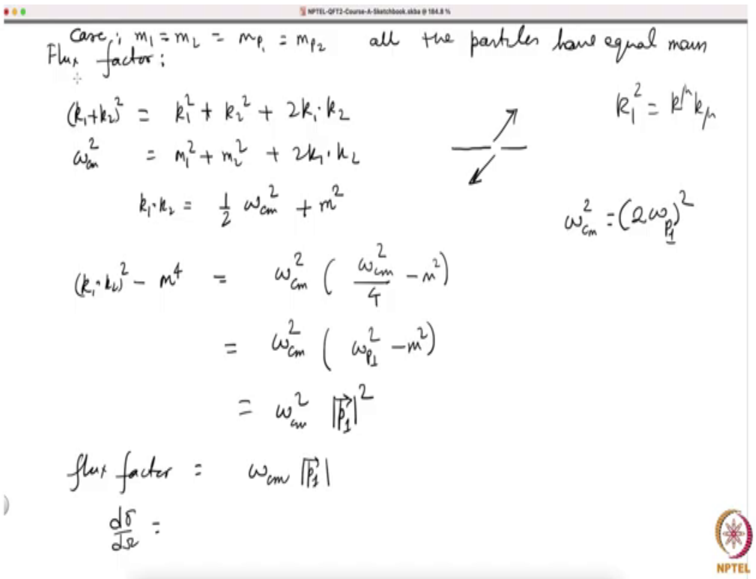

So, this involves k 1 dot k 2 whole square. So, I will let me first calculate that so, for that I will calculate k 1 + k 2 square. And this is k 1 square + k 2 square + 2k 1 dot k 2 where, when I write k 1 square, this is k mu. So, k 1 square is just m 1 square k 2 square is m 2 square + 2k 1 dot k 2. And we are in the centre of mass we are in the frame k 1 + k 2 = 0. So, this thing the special components of this sum is 0 and only temporal components are there.

So that is what I am going to use now and this is just omega centre of mass squared. So, what is k 1 dot k 2? Is k 1 dot k 2 is 1 over 2 omega cm square. Actually I have what I wanted to do was I want to take the case where m 1 = m 2 = also mp 1 = mp 2 meaning all the particles have the same mass. So, I am looking at that case now. So, this becomes plus m square.

Because this gives you 2m square and when you divide by 2 it gives you m square. So, with that k 1 dot k 2 whole square – m 1 square m 2 square which becomes m to the 4 that square root we need to calculate. And this object is equal to I will just substitute here. I will just skip some trivial steps. You can check that it should come out to be omega cm square omega cm square over 4 – m square that is what you should get.

And then you can use that omega cm – m square this you can check now omega cm square that is the centre of mass energy square, will be equal to the half of the energy of one of these particles. Because all particles I mean the incoming ones, they have equal energies, they have equal masses, equal momenta, so, equal energy similarly for the ones going out. They have equal momenta of I mean the magnitude of the moment is same.

But they are also carrying equal masses. So, the energies are also equal, so, it is twice the energy of p 1. So, what I am saying is the final state particles are going back to back because you are in the centre of mass frame. The momentum of this guy is equal to the momentum of this guy. And because the masses are equal, the energies are also equal because e square – p square is m square.

So because m's are equal, e's have to be equal or omegas have to be equal. So, this is 4 omega p 1 square. So, it becomes omega centre of mass square omega p 1 square – m square. And what is this e square – p square? That is e square – m square that is p Square and what is p? That is this. So, when I look at the flux factor, flux factor for this case, where all the particles have equal mass is the square root of this which is omega cm times modulus of p 1.

Now, I can substitute this in this expression. For 1 over 4 times the square root and what is that omega cm times p 1? What happened omega cm times p 1. So that will make it omega cm square below and 1 factor of p 1 will get cancelled. So, what do you get? You get d sigma over d omega is equal to: Let us write here. This is equal to this factor I said it is omega cm P 1 times 4 and there is some this 4 also 1 over 4. So, this 4 times 4 16 times 4 64, so, 1 over 64 pi square.

This omega cm makes it omega cm square this p 1 will cancel. So, I will just cancel this one and write p 1 and then mod m square. Somehow, I should have gotten a mod I have an extra p 1 how come. So, this should have been p 1 not p 1 square correct because I forgot to take into account this piece this in cancel this one so that you have only p 1. And this p 1 which I had will cancel this p 1.

So, you get 1 over 64 pi square 1 over omega centre of mass square times mod m square that is the sigma over d omega for a two body final state, when you are looking at in the centre of mass frame because we made this specialization to this fret. So now, I know something went wrong. No, nothing went wrong.

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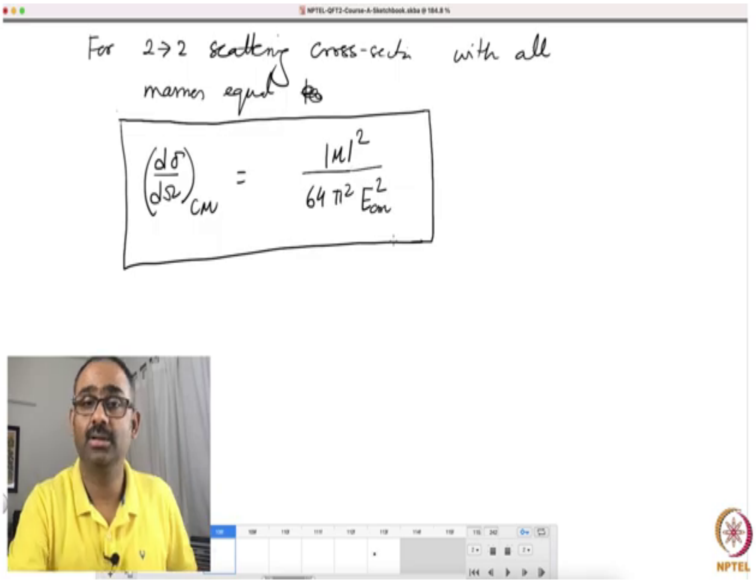

So, let me write down the final result again for 2 to 2 scattering cross section with all masses equal, the differential cross section. d mod d omega in the centre of mass frame, is 1 over 64 pi square centre of mass energy, I will just write E cm instead of omega. It is easier to remember times, matrix element square. So, note that this one is not proportional to some delta function.

As we had for it is not behaving very properly. So, for the case of one body we had gotten a delta function. Let us check one body it was the cross section was proportional to a delta function which was ensuring that k 1 + k 2 squared is equal to m square. But here you do not have a delta function left. But that is a useful formula to remember and we will continue our discussion in the next video.