Introduction to Quantum Field Theory (Theory of Scalar Fields) - Part 4

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Lecture No # 02
Module No # 01
Scattering Matrix Continued

Let us recall what we were doing last time. So we had started looking at scattering in interacting theories. And we defined in and out states which are states defined or basis states defined at time t = 0.

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S - MATRIX

Recap: 'in'-states & 'out'-states

$|\alpha\rangle_{out}$, $|\beta\rangle_{in}$

$|\beta\rangle_{in} = |\vec{q}_1,\ldots,\vec{q}_m\rangle_{in}$ for real scalar field theory

$|\alpha\rangle_{out} = |\vec{p}_1,\ldots,\vec{p}_n\rangle_{out}$

$S_{\alpha\beta} \equiv {}_{out}\langle\alpha|\beta\rangle_{in}$

$\downarrow$

$S(\vec{p}_1,\ldots,\vec{p}_n|\vec{q}_1,\ldots,\vec{q}_m) = {}_{out}\langle p_1,\ldots,p_n|\vec{q}_1,\ldots,\vec{q}_m\rangle_{in}$

So let me quickly remind you what we were doing so we had to find in states and out states. So let me write it as so in general in any theory I am writing I am not necessarily in a real scalar field theory but any arbitrary theory. So I will write alpha out so that specifies some out state where alpha are the appropriate labels and similarly we will write beta in. So these are in and out states.

So for example here beta will specify the particle content for T going to minus infinity so if you take the in state and you evolve it by folding it appropriately with folding function and you evolve backward in time. And in the far pass you are going to see it as free particle which are well separated. And particle content is described by beta and similarly the in far future this out state will evolve into a set of into a set of particles which will be labeled by this alpha.

And in case of real scalar field theory it is only the momentum which is the label and the total number of particles in the state that would be the label. So that is what it is and then we maybe let me just be more specific for the real field theory case. So this as I wrote last time it will be these labels for real scalar field theory. And alpha out p n out and we also define asymmetric element to be the following.

S alpha beta was defined to be an inner product of these 2 states so if first state beta this so the amplitude of state n to be found in alpha out. That is what the definition of S matrix in an for the case of real scalar fold theory this would be this thing I wrote as S. So here let us say beta so anyway these levels are not matching so n is no that is fine. So n is q 1 is labeled by these labels and out is labeled by this moment.

So that is how we wrote S alpha beta in this case and this was just this inner product p 1 to; p n out that is correct that is what we had written last time. So we will make some elementary observations this time and more detailed study will be done by next time. So let us ask first thing how what is the inner product of these states alpha out and beta in I mean within them not of outstate within state.

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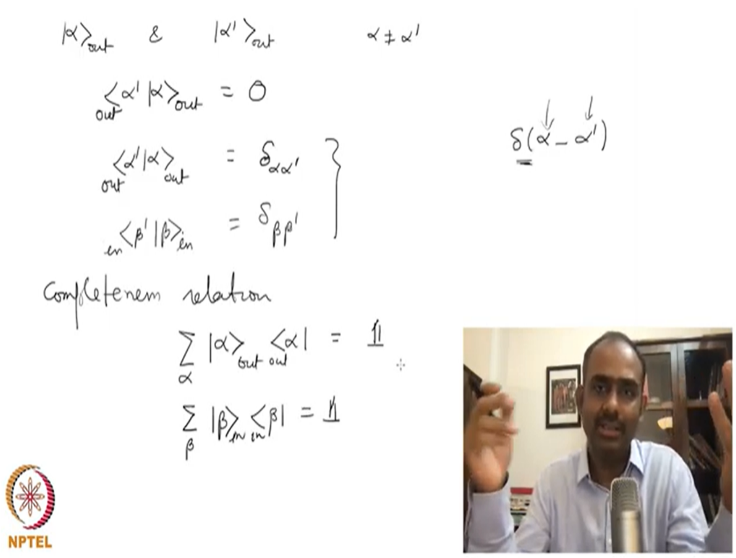

So let us say I am given a state alpha out and another state alpha prime out where alpha is not equal to alpha prime. For example if let us say both alpha and alpha prime let us say they both contain 5 particles I mean 5 levels p 1 upto p n and q 1 upto q n. Then alpha not equal to alpha prime would mean that these 5 labels are not the same as those other 5 labels that is what it.

So we would like to know what is this so it is easy to understand what this is so suppose you take a basis state at time t = 0 let us say it is alpha out and evolve it into far future. So it will corresponds to a set of particles which are labeled which are the particle content will be alpha. If you start with different state alpha prime out and far future the particle content will be alpha prime. Now since we are saying alpha and alpha prime are not the same they are different then these 2 states and far future are different.

And there is no possibility of state alpha out to evolve into to give a state in far future that corresponds to the state that you would have got by evolving alpha prime out. So that this is the possibility 0 that amplitude is 0 so we can write it that is 0. Now we can ask about the normalization of these states. So what is in general; where I am allowing now alpha prime to be equal to alpha also.

So remember these basis states alpha out evolve to these states which correspond to free particles and we have already talked about these free particles in the previous course and you know how these states are normalized you normalize them to Dirac delta function. So the same will be true here you can almost see that this is going to be proportional to I am this is just a notation shorthand notation but this is proportional to this.

Where in the case of where alpha is only the momentum labels you will have Dirac delta functions delta cubes. But let me use the shorthand notation and just like delta of alpha minus alpha prime. So clearly when alpha is not equal to alpha prime then you get 0 so they are orthogonal and when they are equal you get I mean they are normalized to delta functions. So and in fact instead of writing delta like this I will pretend as if these levels are discrete that is just I am going to pretend it.

But you understand that these are involving both delta functions and maybe deducted chronic delta functions. So I am just writing it like this but if you do not like this you can think of this in these terms. Anyhow so now that I have written down the normalization for out states based on some intuition and some hand waving will make it more precise later and we will convince you that indeed what I am saying is true unless it already appears obvious.

I can now also similarly write down for instance and remember when you are looking at the inside in the far future you have many possibilities it is not just 2 particles you collide. You can collide n number of particles so the all those possibilities are there and that is why you have lots of possibilities for the in state also good. So we have these things and then I had also remarked that these states form a complete set meaning any set any state in the final state can be expressed as a linear sum of or a superposition of the out states.

For the in states so this implies that I have a I should have a completeness relation and that would be alpha if I am using for out where I sum over alpha. So again alpha is not only discrete it would be continuous also but I am just using sigma and it is understood that if those indices which are continuous you have to integrate over not sum over. And this will be that and that will be identity and similarly for your in states. So this is the completeness relation.

Also remember these are 2 different bases for the same Hilbert space it is not that they are living in 2 different labeled spaces these in and out states. And clearly you can express out state for example in terms of in states and vice versa.

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Since the 'in'-states & 'out'-states are two different basis of the same Hilbert space

$$|\beta\rangle_{in} = \mathbb{1}\cdot|\beta\rangle_{in}$$
$$= \sum_{\alpha} |\alpha\rangle_{out\ out}\langle\alpha|\beta\rangle_{in}$$
$$= \sum_{\alpha} S_{\alpha\beta}\,|\alpha\rangle_{out}$$

coefficients are S-matrix elements.

So let me also write this as since the in states this is in states and out states are 2 different bases of the same Hilbert space I can do the following. So I take let us say beta in which I can write it as identity times beta in and for the identity I insert this completeness relation. So I get alpha out again here alpha out sum over all alpha and here you have a beta in and what is this?

This is just as alpha beta that is the S matrix element so I have written beta n one of the basis elements as a linear sum of the out states and the coefficients and precisely the asymmetric elements. So you understand what we are doing? We want to know these coefficients as alpha beta once you know these elements you will be able to predict what we happen in the future? So these coefficients are the S matrix elements.

Good now I will show you something else when you are going from in state to out state what you are doing is? You are changing basis one basis of you are going from one basis to another basis in the Hilbert space. And these are these basis I have as I have argued here they are these are orthonormal basis. So you have basically going from one set of orthogonal basis to another set of orthonormal basis this is vectors.

So if you do so then your transformation matrix will have some property it will be unitary and that is what I am going to show that your S matrix is unitary.

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$$
\begin{aligned}
{}_{in}\langle \delta | \gamma \rangle_{in} &= {}_{in}\langle \delta | \sum_\alpha |\alpha\rangle_{out}\, {}_{out}\langle \alpha | \gamma \rangle_{in} \\
&= \sum_\alpha {}_{in}\langle \delta | \alpha \rangle_{out}\, {}_{out}\langle \alpha | \gamma \rangle_{in} \\
&= \sum_\alpha {}_{out}\langle \alpha | \delta \rangle^*_{in}\, {}_{out}\langle \alpha | \gamma \rangle_{in} \\
&= \sum_\alpha S^*_{\alpha\delta}\, S_{\alpha\gamma} = \delta_{\delta\gamma}
\end{aligned}
$$

$${}_{in}\langle \delta | \gamma \rangle_{in} = \delta_{\delta\gamma}$$

S-Matrix is unitary.

So let us show this so I take some state gamma in and take its product with did I call it sigma now let us call it delta n. Let us evaluate this now what is this subject? This subject is delta of delta gamma where remember this chronicle delta is good would involved in general Dirac delta functions also because delta is not only discrete it also has continuous parameters like momenta. So this is this subject but I can write this thing by inserting a completing set involving only out states.

So this would give me so here insert set of out states and which is equal to the second take in. So you have delta in alpha out and you have here out gamma in. Which I will write as this I can write as alpha out delta in but then I have complex conjugation alpha out gamma in and this is what this we have been calling S matrix S alpha gamma. So you write this as S alpha gamma and this is what S alpha delta star.

And that is the condition of unitary that this sum is equal to let me write this thing here delta of delta gamma. I wish I had chosen instead of delta something else here but anyway so this is the condition of unitary. So we say that that the S matrix is unitary I know about this matrix it is unitary. Now let us try to have little bit more understanding actually very little understanding of the in and out states.

We are going to spend some time on it but there is something very trivial that I can already say about it. So imagine you have one particle state so you have only one particle so far in past you has one particle. Now since there is one particle only one particle and there is nothing else interacting with it that one particle is going to evolve in one particle state in future also so it is one particle state remains one particle state. So when you take that one particle state you can decompose into various momentum components.

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In : $|\vec{k}\rangle_{in}$, $|\vec{p}\rangle_{out}$

$$|\vec{k}\rangle_{in} = |\vec{k}\rangle_{out} \leftarrow \text{For the case of single particle states}$$

$$|\Omega\rangle_{in} = |\Omega\rangle_{out}$$

So the basis states at time T = 0 let us focus on the in states you will label them as; k in. And when you look at far future and you go backwards at 2 time equal to 0 you will have basis states as p out. And since single particle state evolves to single particle say there is nothing else going to happen it is clear that a basis state k in is same as k out. So there is no distinction between in and out state as far as single particle states are concerned.

See there is only one label k it is not k, p its one so even though the theory is interacting your single particle states your in states corresponding to single particle states and in state out states corresponding to single particle state they are the same there is no distinction between them and also you can imagine the vacuum so vacuum will evolve into vacuum will not turn into n particle states because that will violate all these conservation laws.

So vacuum is going to remain vacuum so it again clear that vacuum in is same as vacuum out. So these are few things that I wanted to say in this video and we would start a more careful analysis of these things and some of the things which I have written based on intuition or some hand waving I will make them more concrete next time. So see you in the next video.