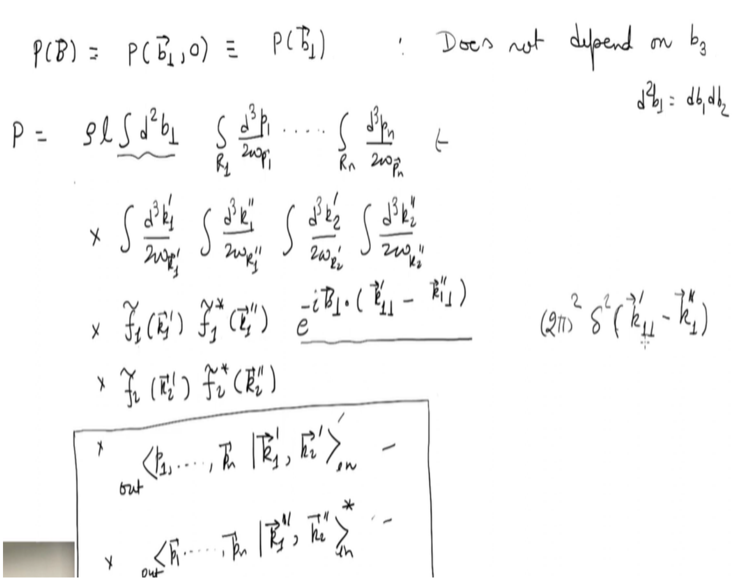
Introduction to Quantum Field Theory II
Prof. Anurag Tripathi
Department of Physics
Indian Institute of Technology –Hyderabad

Module No # 04
Lecture No # 19
Differential Cross - Section

(Refer Slide Time: 00:18)

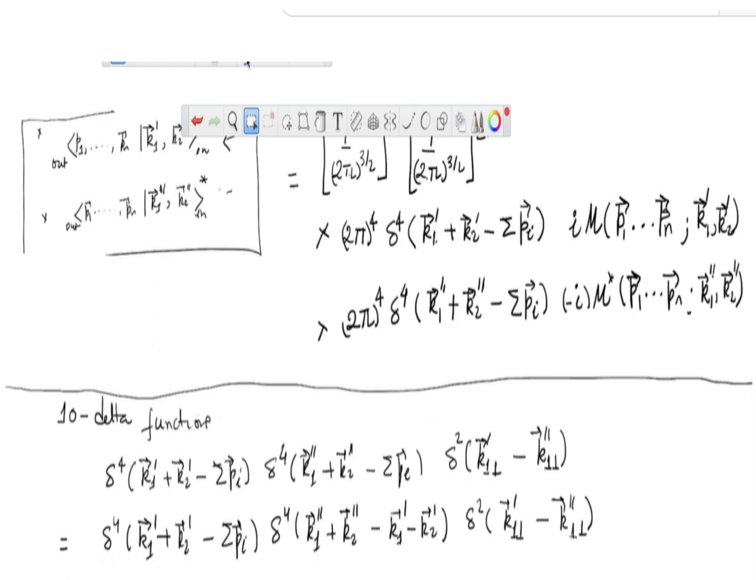


So let us resume there where we stopped and it was here so we had written down the probability for producing the final state which has the particles carrying momentum P1, P2 so and so forth up to Pn. So I am assuming I am looking at a final state which has n particles in it and they have momenta p 1 p 2 and so forth that is what you see here in this out state and this was the expression, for probability.

And last time we had also looked at these last 2 factors which are here in this box and we wrote down its expression in terms of these matrix elements which are called which are denoted by m here. And that those matrix elements we get by stripping off the factors that always accompany them and also these delta functions. Now as I have repeatedly said that if you have delta functions in your integrals then it is much easier to integrate things right because delta functions make very things very simple.

So let us first see how many delta functions we have what kind of constraints they are giving and then we will proceed.

(Refer Slide Time: 02:05)



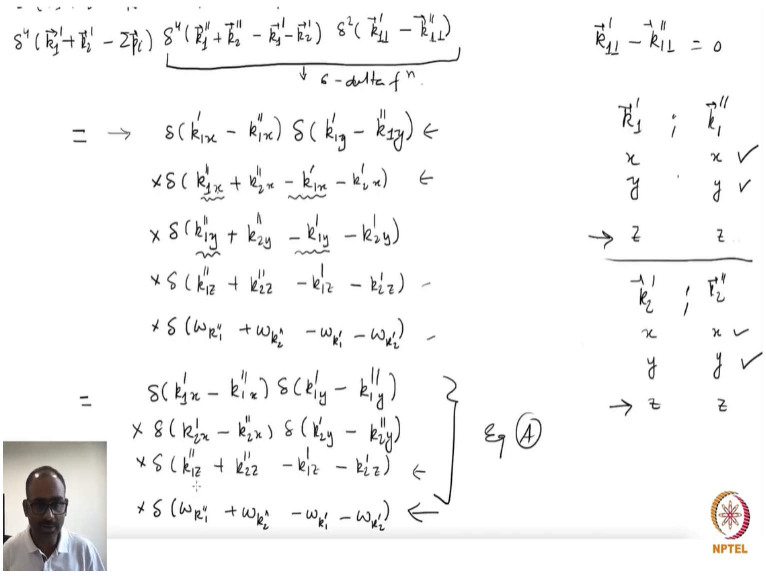
So that is the plan so you have 4 delta functions here delta 4 which says that k1 prime + k2 prime should sum up to Pi should add up to sum over all the Pi's. And another delta function saying k 1 prime k 1 double prime + k 2 double prime should also equal to sum of all the external momenta so these are 8 delta functions. Now if you go here and look at this vector you can integrate trivially over d to b part this integral over this exponential function gives you a delta function.

So it gives you 2 pi, square delta 2 k 1 per prime or k 1 prime per - k 1 double prime but per for short form for perpendicular. So this will give you this delta function so I will have then in addition to the z delta functions 2 more delta functions. So I will now write down all of these delta functions so delta 4 k 1 prime k 2 prime summation over p i then delta 4 k 1, double prime + k 2 double prime minus summation over p i and times delta 2 k 1 prime and k 2 prime the perpendicular components of them.

Let us check correct so this is what we will work with first now this I can write as again this delta function. And in this delta function the second one I will replace this summation over p I by k 1 prime + k 2 prime right. Because this delta, function these 4 delta functions click when summation over p i = k 1 prime + k 2 prime so otherwise this entire expression is 0. And because it clicks only for that value I can replace summation over p i by k 1 prime + k 2 prime.

So I will write this one as, k 1 double prime + k 2 double prime - k1 prime - k2 prime I have just put from here times delta 2 k 1, perpendicular q 1 prime perpendicular - k1 double prime perpendicular perp good. So that is what we have now I will just take it to the next page I hope I still remember how I read that.

(Refer Slide Time: 07:29)



So this is what we are looking at now let us look at this carefully. Let us look at these 6 delta functions first we will come to this these, 4 delta functions at the end so we will look at these first these ones. So this is what this is saying that the x component of k 1 prime should equal to the x component of k 1 double prime and the y component of k 1 prime should be equal to y component of k 1 double prime.

So let me write down this thing so these are6 delta functions so first I am writing what you get from this delta 2. So I am just expanding delta 2 instead of using vector notation I will write in components. So this is delta k1 prime x component should be equal to k 1 double prime x component. See this is a vector equation k 1 prime perpendicular - k 1 double prime perpendicular equal to 0 that is when the delta function hits that is the vector equation let me write it down.

These are, basically 2 equations one for x component and one for y component that is what I am doing here so times delta k 1 prime y - k 1 double prime y so delta 2 is written in this way. Now let me write down the delta 4 I will just re write down from my notebook so that I do not use have do not have to use my brain and tear it out. So delta k 1 double prime x + k2 double prime x - k1, prime x - k2 prime x that is the x component of this argument.

So that is what is here then I will do the same for y this is also good k 1 z + k 2 z double prime - k 1 z prime - k2 z prime then one more for the energy component. So delta of omega k 1 double prime +omega k 2 double prime k 1 prime sorry this primes or to wrong place k 1 prime + omega k 2, double prime - omega k 1 prime - omega k 2. So I have just written them down in components that is all I have done so if you see here let us look at k 1 and k 1 prime and k 1 double prime.

So this let us look at k 1 prime vector and k 2 sorry k 1 double prime vector these 2 so this delta function says that the x components of these 2 vectors match. So x components of, these 2 they are equal this delta function says that the x y components of these 2 also are equal how about the z components? Well none of these says that the z components are equal at least not directly.

Now let us look at k2 prime and k2 double prime how about them so if you look here in this delta function you have k 1 prime x minus sorry there is an issue of sine. No fine k 1 double, prime x is here k 1 - k 1 prime x is here. Now this delta function because of this delta function this entire expression will be non-zero only when k 1 double prime x = q1 prime x otherwise this entire expression is 0.

So I can put k 1 double prime x is equal to q1 prime x in this entire expression so if I do so then k 1 double prime x - k 1 prime x = 0 I can, put here because of this delta function and these 2 go away. And I am left with only k 2 double prime x - k2 prime x so x components of k 2 double prime and k 2 prime they are equal from this delta function once I use this first delta function.

So S components match how about; the y components the same thing here k 1 double prime y - k prime y that; is equal to 0 using this, delta function. And then you are left with k 2 double prime y = k2 prime y. So y components of k 2 prime and k 2 double prime they also match and again I cannot say directly anything about this z components I will have to think about it. So you cannot say about this that is one fixing that will require one condition fixing something about z will require another condition 2, conditions and right now we are also left with 2 delta functions this and that.

So what I am going to argue next is that in fact argue is that the z components also match so that is what the goal is. But before I do that let me now write this product of delta functions using the information that I have already used in writing making these tables. So I have delta k1 prime x then you, have delta of k 2 double prime x - k 2 prime x from this line times this line will give you delta of k 2 double prime y - k2 prime.

I am not being very consistent k2 prime x sorry - k 2 double prime y k 2 prime y - a2 double prime sorry I keep making mistakes that is why I should look at my notes k2 prime x - k2 double prime x k 2 prime y - k2 double prime y. Now this is better this is correct so this is fine and then this remaining 2 and we are going to next manipulate these 2 delta force let us see if I can do that thing again works.

So these are the things with which we have to work now if you look at the this delta function now this delta function will click if k1 double prime z = k1 prime z and that will, be nice because then it will mean that the z components also match and also when k 2 double prime z = k2 prime z. So all of these will the argument of the delta function would be 0 and delta function would click but then there is another possibility that k 1 double prime z = k2 prime z.

And k 2 double prime z = k 1 prime z that even then it will click so what I will, do instead is first I will look at this delta function and then we will come to the second last delta function so that is the idea.

(Refer Slide Time: 18:30)

Look at the last delta f^n in (A)

$$\omega_{k_1''} = \sqrt{\vec{k}_1''^2 + m_1^2} = \sqrt{k_{1x}''^2 + k_{1y}''^2 + k_{1z}''^2 + m_1^2}$$

$$= \sqrt{k_{1x}'^2 + k_{2y}'^2 + k_{1z}''^2 + m_1^2}$$

$$\omega_{k_1''} - \omega_{k_1'} = 0 \quad \text{if} \quad k_{1z}'' = k_{1z}'$$

$$\omega_{k_2''} - \omega_{k_2'} = 0 \quad \text{if} \quad k_{2z}'' = k_{2z}'$$

$$\delta(k_{1z}'' + k_{2z}'' - k_{1z}' - k_{2z}')\,\delta(\omega_{k_1''} + \omega_{k_2''} - \omega_{k_1'} - \omega_{k_2'})$$

$$\propto \delta(k_{1z}'' - k_{1z}')\,\delta(k_{2z}'' - k_{2z}')$$

Let us call this equation a there is no equation these are there is equality so fine and a so what does it say? So we have let me first write down I will I am writing down omega k 1 double prime. So let us write down omega k 1 omega prime and what is that? It is k 1 double prime square + M p 1 square I will physical mass for M p and one is a subscript not for momentum. But about this how about we allow we drop the p for physical mass and I will just write M 1.

So I am allowing 4 different masses for projectile and the target particles and what is this? It is just k 1 double prime x square + k 1 double prime y square, k 1 double prime z square + M 1 square. Now I have already argued that k 1 double prime x is same as k 1 prime x. So I will write k 1 prime x squared + k 2 prime y square + k 1 double prime z square + M 1 square. And this I have done using the previous delta functions because I have these delta functions which allow me to do this.

Now omega k 1 double prime - omega k 1 prime = 0 see that is omega k 1 double prime - omega k 1 prime these 2 together could give you 0 and omega k 2 double prime - omega k2 prime could also give you 0. But and in that if they do then this delta function clicks it gives you a non-zero value. But for that k 1 double prime z should be equal to k 1 prime z then only, this will be satisfied not otherwise.

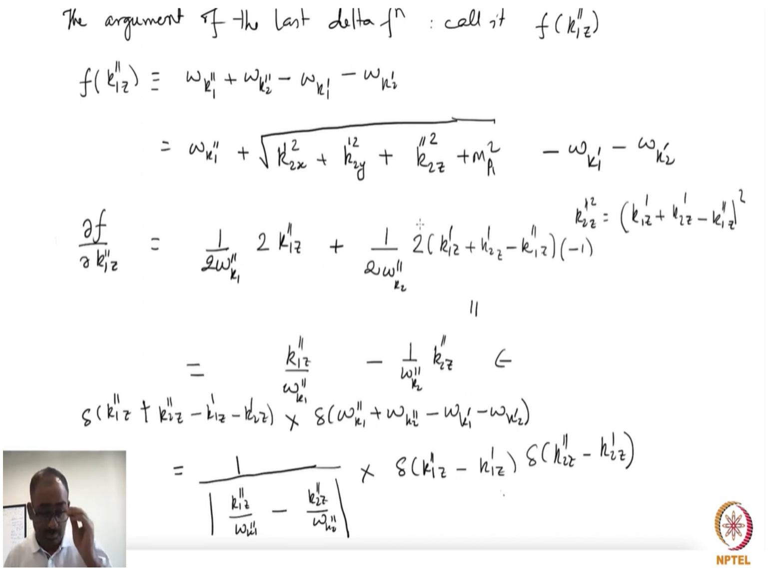
And also you see now that other way around that omega k 1 double prime equal to omega k 2 prime. So you might think that these 2 together give you 0 and these 2 together give you zero that is also another possibility in principle. But that is not going to work because we are allowing for different masses so it will not work in general so we are keeping, the masses different for k1 prime and for, k 1 prime and k 2 prime and q 1 dot prime and q 2 double prime.

So in general that is not going to work so we get this condition satisfied if k 1 double prime z = k 1 prime z. And similarly omega k2 double prime - omega k 2 prime = 0 f k 2 double prime z = k 2 prime z and which is nice. Now I can from this one, I conclude that the z components also match now the z components match s match and then I can use that information in this delta function and turn it to something is not very.

I will think about it later if there is something sloppy in the argument but I will just proceed now so what in essence you get is looking at these 2 last delta functions what you, get is? Delta k 1 everything is fine I guess so with what I have argued above from that we get that the product of these 2 delta functions. In the previous page should be proportional to should be proportional to delta of k 1 z double prime - k 1 z prime times delta of k 2 z double prime - k 2z prime that is what we expect and then we can say that these 2 match.

So, good because now I can replace all these delta functions by a delta function which is con saying that k 1 prime is k2 prime and k 1 double prime is k2 double prime. But now we should find out the Jacobian here when you go from here to the right hand side you will pick up some factors and that is the factor which we need to figure out. So which one are the last delta function what is that, we will now look at this one this delta function.

(Refer Slide Time: 27:09)



So the argument of the last delta function we will call it I will see it as a function of call it f k 1 double prime z. So let me write down I am saying f of k 1 double prime z is by definition omega k 1 double prime + omega k 2 double prime - omega q 1 prime - omega k2 prime that is what appeared in this argument that is, what I have defined as f. Now because we are going to manipulate these delta functions I will this delta function I will take a derivative of, f with respect to k 1 double prime where I mean to find out the Jacobian.

Now this is omega k 1 double prime plus this k 2 prime k 2 double prime omega k 2 double prime I can write as the x component k2 prime double prime x square. But the, double prime k the x component of k2 double prime is same as x component of q2 prime. So this is fine plus again the y component of double prime is same as y component of k 2 prime so this is I can write it this way but k 2 Z double prime square.

Because this is you have to be determined + m p 1 square and then minus omega k 1 prime - omega k2 prime. So let us take the, derivative what will you get you will get because this involves the square roots this just like this one you write a expression square root and then differentiate then you will get that is what you get from here. Then the second term gives you 1 over 2 times this exact thing but that thing is 2 k 2 double prime times the derivative of what you have in the square roots that gives, you.

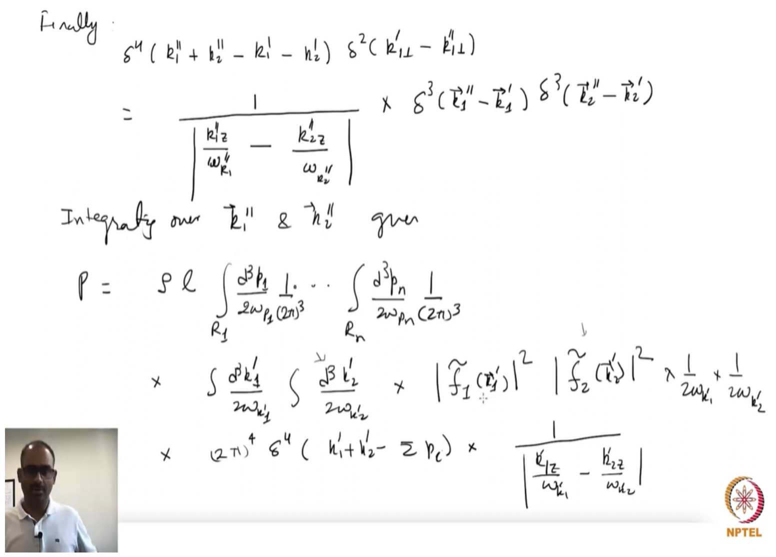
And for that derivative I should substitute k2 double prime z to be the following this is from the previous delta function this expression is coming from here so I am just substituting k2 prime has been left. So I am just substituting k 1 double prime k 2 double prime z in terms of k 1 double prime z and these things this expression here and what do you get? You, get taking the derivative you get 2 times k 1 prime z plus k 2 prime z - k 1 double prime z and then because there is a minus sign here you get a factor of - 1.

And what is this it is 1 over 2 let me if you look at this piece this is equal to 1 over omega k 2 double prime and then k 1 prime z you can use the previous delta functions and this will give you k2 double prime z. And there is a minus sign here and this one gives k 1 double prime z over omega k1 double prime. So you know that this is the modulus of this one over modulus of this thing goes into the Jacobian so what do we get?

We get delta of k 1 double prime z + k 2 double prime z - k 2 prime z times delta of omega k 1 double prime + omega k 2 double, prime - omega k 1 prime k 2 prime. This product which was here these 2 = 1 over modulus of this thing this expression times delta of k 1 z - q1 double prime z - k 1 prime z times delta of k 2 double prime z - k 2 prime z that is what you are going to get.

And this one I have written using previous delta functions so that is the Jacobian I have, obtained. And now I have earlier I had the equality of the x components of k 1 and k 1 prime and k 1 double prime and k 2 prime and k 2 double prime. But now also I have equality for the z components of k 1 prime and q 2 double prime. So all the 6 components are matching now so what did we get finally?

(Refer Slide Time: 35:14)



We get this delta 4 times this delta 2 which came from the b perp integral k 1, prime perp - k 1 double prime curve. We have shown that this is equal to prime k 2 double prime z over omega k2 double prime that modulus times delta q which enforces the equality of all the 3 components of k 1 and k 2 k 1 prime and q2 k 1 double prime and k 2 double prime and k 2 prime. This is very useful because now I can put this in the expression for the p the momentum, the probability.

So here I have now lots of delta functions and they are all nice and simple now and I can just do this integral. So let us first count all the delta function that we have 6 here and 4 previously see total we had 10. Let us go back total delta functions are 10. 4 here which says that k 1 prime + k 2 prime is equal to the sum of all external, momenta or the final state momenta and then 6 of these we have manipulated.

And turn them into apart from a Jacobian maybe I can write here delta 4 k 1 prime + k 2 prime minus summation over p I times this Jacobian which written times this delta cube of what was that? K1 double prime - k 1 prime time's delta cube of k 2 double prime – k 2 prime and this J is what, this J is 1 over this modulus thing coming from next videos. So this is what we have done now and now I can do these integrals very easily.

So I will do that thing now so let me first write down the result and then I can explain integrating over k 1 double prime and k 2 double prime gives maybe I should just tell what will happen? So I am going to integrate over k1 double prime and k2 double, prime let us see what happens? I am going to integrate over this and this and you have a corresponding delta functions.

So when you integrate over k1 double prime and use the delta function which says k 1 double prime is equal to k 1 prime the 3 components of them then what will happen here is? This one will turn into f1 star k1 prime and this one turned into f 2 star k2 prime, and then this and this together will become f1 k1 prime mod square right because this will be then complex conjugates of each other. And similarly these 2 will give s2 tilde k2 prime mod square that is one thing that is going to happen.

And then these omega k 1 double prime and omega k2 double prime they will also turn into 2 omega k 1 prime and 2 mega k2 prime because I will be using delta, functions and what else that is one thing that will happen and also. So there is one more place where you will have k 1 double prime which I can replace by k1 prime and which is this Jacobian factor right. So here also k1 double prime z will become q 1 prime z and omega k 1 double prime will become omega k1.

Because of these delta functions, so finally you will get the following and so let me write down that thing. So p = rho times l d cube p r p 1 over 2 omega p 1 and this is integrated over region R1 d cube p n 2 omega p n it is integrated over region R n. And I should have also included a factor of 1 over 2 pi cube I missed times d cube k 1 instead of k 1 prime that is what I am left with I will write k 1. K 1, prime is a dummy variable so instead of using k 1 prime I will use k1 because there is no need to carry a prime.

Now so I will just use k1 instead of k 1 prime and k2 prime I will use k1 and k2 they are dummy I can do so times f 1 tilde k1 mod square. As I told you they will convert to mod squares mod square times 2 pi to the 4 delta 4 k 1 + k 2 minus summation over P i that is this delta, function which we never touched. And times the Jacobian factor which is 1 over k 1 prime z over omega k 1 no primes now k 1 z over omega k 1 - k 2 z over omega k 2 modulus of that.

Looks like I have taken care of all the no I have missed factors of 1 over 2 omega k1 2 omega k 2. This is the result that I have now my bad I should not have dropped the primes I should still keep the, primes let us otherwise I will be inconsistent with my notation I will tell you why I have put it back? This is k 1 prime and that is also prime let us keep this prime careful the reason I should not use k1 and k2 because in the beginning I said that the momenta of these incoming particles are localized at k1 and k2 somewhere here.

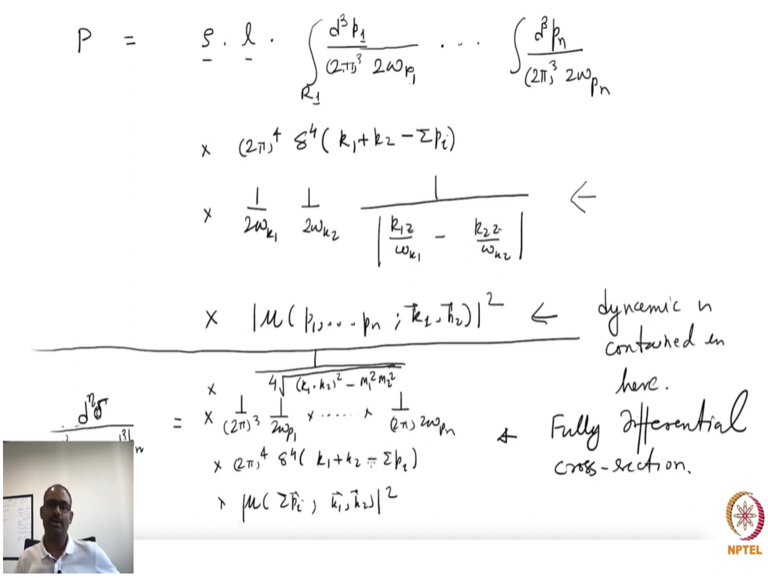
See this k 1 is not, dummy because I told you that the momentum of here these wave packets are such that they are concentrated around k2 meaning they have a the peak is around k2 so k2 is not a dummy variable I should I should not have said that I can replace by k2 and similarly k1. The wave packet the wave function; of this particle that peaks around k1 in the momentum space.

So here that is why I, have kept these things now what I will do is? I will use the fact that f1 tilde and f2 tilde they are sharply peaked around k1. So you can think of them as Gaussian function speaking at k 1 k 1 prime = k 1 and k2 prime = k2. And we have already talked about the normalization of these functions so what was the normalization? Here the normalization was this if you take f2, tilde and Mod Square it and integrate over the momentum divided by divided by this 2 omega then that gives you one.

So in the limit that it is sharply peaked I can where is it in the limit it is sharply peaked I can replace k1 prime by k 1 and k 2 prime by k2 because you can think of these as delta functions. So they hit only when k 1 prime = k 1 and k 2 prime =, k2 so in that case wherever k 1 prime and k 2 prime appear in this expression I will replace by k 1 and k2. Because it picks only those values from because of the delta function or a very sharply bit caution Gaussian.

And then I use the normalization that you just saw and replace this integral d cube k 1 prime over 2 omega k 1 prime f tilde the 1 k 1 prime mod square by 1 and similarly this one, these 2 together by 1 and then you get the following final result.

(Refer Slide Time: 48:12)



P = rho times l times 2 pi to the 4 delta 4 k 1 just a second minus summation over P i that is this delta function, where k 1 k 2 have been k 1 prime and k 2 prime have been replaced by k1 and k2 times. 1 over 2 omega k 1 over 2 omega k 2 1 over k 1 z over omega k 1 - k 2 z over omega k2 modulus, of that times m of k 1 k 2 giving p 1 2 p n and we have to do a Mod square of it.

Because of here I had missed writing down not square I will not write the arguments now this should this vector should have been there. So this is what you get finally it has become very nice and simple I will do a little more of massaging of this thing later. But for now let us see what we, have got so in apart from these factors which are basically one divided by energy of the incoming particle k 1 and 1 over the energy of the second particle k2.

And then you have Mod m square that is what you have to calculate so the entire information about the dynamics of what process is happening and what kind of interactions are there depends on is contained in this, amplitude this matrix element m and the Mod square of it. So this is the piece which really knows about the dynamics or the interactions the remaining factors are kinematic factors and you are integrating over all the final state in here.

And this rho l and l they do not have anything to do with the nature of interaction or the dynamics it is just that there are too many particles and the, these densities and all these things are involved. So the real content is contained in the factor which is multiplying rho time cells all these things. And this is what we will look at carefully actually I will I can give you an exercise it will be better.

So exercise number one look at this line and argue that this is object which is invariant under boost along the z direction. So, is not invariant under complete Lorence transformations but only if you restrict transformations along the z direction then this line is invariant. And all other quantities are invariant in this expression so p is invariant under boosts along the z axis and z axis is the beam axis.

So I will define d and sigma is the definition d cube p 1 to d cube p n is equal to so I am writing down these factors, now. So this is this is basically integral let me first write down then it will be easy to understand 1 over 2 pi cube 1 over 2 omega p 1 times 1 over 2 pi cube 2 omega p n. So these things these factor here times 2 pi 4 delta 4 k 1 + k2 minus summation over P i.

This is this factor times Mod m square and there is one more factor which is going to come from this line and you can show that this is 1 over 4 k 1 dot k 2 whole square where n 1 and m 2 are the masses of particle with momentum k 1 and particle with momentum k2. So that is the expression this expression which is really what you have here if you were to take these differentials on the right and integrate over them and multiply with rho and l you will get p.

So stripping out rho and, l whatever is left behind the differential of it is called the differential cross section. So this is a fully differential cross section we will do more with these differential cross sections later in the course. But now we have made a full connection with the experiment I can go and calculate these differential cross sections or I can do some integrals over these cross sections, differential cross sections and get some integrated over cross sections and that I can calculate.

Because I can calculate modern squares and Mod M squares I can calculate because they are related to the greens functions remember these are Mod M squares or m the matrix element is basically s matrix element from which you have removed some delta functions and some other factors of 2 pi's. And we, have already seen that asymmetric elements are related are (()) (57:15) we had related them to amputated greens functions.

So now you can write down green functions using phenomena diagrams and calculate m. Once you have m for a particular process you multiply with these things and integrate over some momenta if you wish and that gives you after multiplication with rho and l gives you, probability of producing that particular final state. So you have a calculation from theory side now which you can match with the experiments.

So we have now completed our connection making the connection with the experiments and we have an expression in general for a fully differential cross section. And you see in these steps I never made use of the fact that I am, looking only at scalar interactions. You have this result is general the fact the differences that will come from having different kinds of particles will all be contained in the calculation of Mod M square. But this formula is general otherwise so I will say a little more maybe next time so we will stop you.