Introduction to Quantum Field Theory II
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Module No # 04
Lecture No # 18
Scattering Cross - Section

Let us consider an experiment a typical high energy physics experiment in which we collide a beam of particles with another beam of particles and see what the outcome is. So what I am going to do is I will start by colliding one particle with a bunch of particles and then ask the probability of producing a particular final state and then we will add up all these probabilities to determine, what is the total probability of finding a particular final state?

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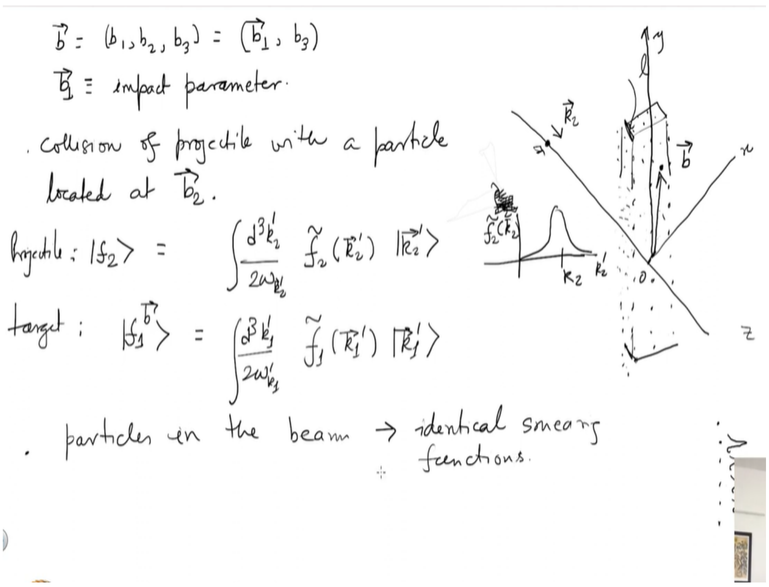

So a typical setup is like this so this is the z axis so along this direction you fire a particle let us say the particle that you are firing is a going to move in this direction and I will label it as k 2. So the momentum of this is k 2 and I am not going to assume that I am working only with scalar fields whatever, I say here is general it applies to any theory though I will tell you where the differences will come.

But the expressions that I give in this in this lecture will be applicable to any theory. So it does not matter whether it does not cause any hindrance that we have only studied phi for interaction real scalar fields in this course. So that is the direction of the projectile, so you project it towards a beam of particles I will show it like this. So lots of particles here you can assume them to be stationary but you do not have to this is fine. So this is perpendicular to the z axis maybe it will be better if I draw a little bit like this.

Now it looks perpendicular so you have this beam it has some thickness so you have also so it is, basically sorry for my bad drawing. So it has some thickness so this is along the z direction now so basically I am trying to draw the following this denotes the thickness and this is how it looks. So let us say the thickness is l so this thing is l and I have written already this is z this is x direction and the perpendicular direction is y direction.

You have all these particles, here so what do we do? We take this projectile of momentum k 2 and collide it with this beam and this is what happens to plane and let us see experiments. So there not only this one will be a single particle but it will also be a bunch so you will have a bunch coming from left a bunch coming from right and they will collide. So let us take a particle of some particle here, which is inside the inside this beam and let us say its coordinate is b.

So this position vector here is the origin disposition vector I call b so I am looking at this particular particle so b is equal to it has this components b 1, b 2 and b 3 which I will write as. So b 1 and v 2 together I will call as b perpendicular and then we have b 3 and b has dimensions of length because we, are looking at this coordinate space b is the coordinate of the particle in the physical space. And b perpendicular this vector which is basically the component of b vector in the x y plane perpendicular to the direction of collision that is what I am talking about.

So b perpendicular is defined; as impact parameter or the magnitude of this is defined as the impact, parameter. So first we will look at collision of projectile with a particle located at b 2. So what we should do is first we should construct the wave functions and for k 2 and also for the particle located at b and for the particle located at k 2 located here sorry not at k to here and that is what I do.

So the wave function the state I will denote as f 2 for this one and as, before f 2 is d cube k 2 prime over 2 omega k 2 prime that is what we have done in previous lecture. So here is the smearing function so k 2 prime is dummy of course and this f 2 tilde is peak around k 2. So when you integrate over k2 prime it will pick up a value close to k2 in the vicinity of k2 this is non-zero otherwise it is almost zero.

So that is the overlapping function sorry, smearing function k 2 and this axis s k 2 prime so this is your f 2 tilde sorry something happened does not even do undo f 2 tilde k 2 I do not know how I got this so that is projectile. And the target particle which is located at b I will write the state as f 1 b so that label b tells you that I am talking about this particle located at b. So you have d cube k1 prime f 1, tilde k 1 prime and in this momentum state given prime.

Now in this beam which looks very ugly here you should have drawn it neatly but in this being all these particles for them, we can choose the wave function of them this f 1 tilde or the coordinate space function which you obtain by doing a Fourier transform all of them to be having the same form. So let us say if I am looking, at all these particles in the beam then let us say this is the one at location b if this is the wave function of this one in the coordinate space then for this one it has identical one and this one also has identical one and so forth.

And also for all the other ones also within the slab of these particles we will assume that they all have are described by the wave, function which looks identical. So that is the choice we can make no harm if you make a different choice provided you are still localizing these particles properly. So we will but how we will make that a simplifying choice so particles in being in the beam they have wave functions or the identical smearing functions they looks the same that is what I mean.

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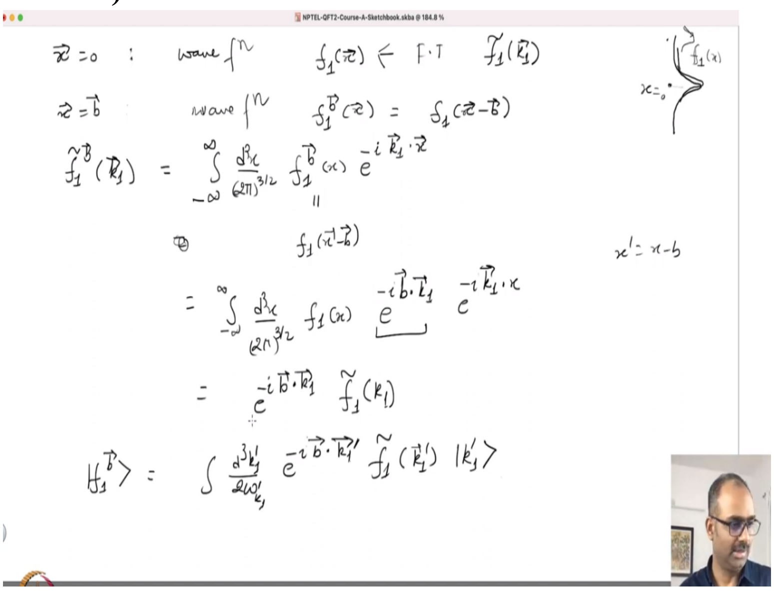

So if you have a let us say look at x = 0 the particle which is located here at the origin so right now you should be so this particle at the origin which is at x = 0 the target particle one of the target particles. So let us write it is the smearing function for it or the wave function for it let us call it f 1 x. So this envelope which i, made this one is given by f 1 x for this particle here and then we have another one here we should find out what it should be that one.

So if I look at a particle located at x = b then its wave function would be given by f 1 b of x that is the notation I am using and what is this function? So if you take this f 1 of x whatever that function is it goes to 0 and takes this entire function and transports it to the location b that is what you have to do. So you take this function and move it and put it where the point b is so instead of the peak being located at x = 0 it you take this entire thing and move so that the peak is located at x = b and how do you do that?

Well that is equivalent to changing the coordinate to shifting the, coordinate to by an amount – b. So f 1 this functional form then becomes f 1 of x - b when you shift the entire thing to b it is equivalent to changing the coordinate and shifting it by - b so that is why you get this functional form. So now I know the function f 1 b in terms of the function which I had the wave function which I had written down for the particle located at x, = 0.

And we can also calculate now what will be f 1 tilde b k 1. So if you take a Fourier transform here and it gives you f 1 tilde k 1 then what is the corresponding Fourier transform? If you look at this function and that is easy you have d cube x over 2 pi 3 halves f 1 b x e to the - I k 1 dot x that is just the Fourier transform. But this object is f 1 of x - b, so now I will do shift in the in the variable and I will write.

So I will write x prime equal to x - b so this becomes f 1 of x prime which I am again writing as x and this will gives you this will give you e to the - I k perpendicular. And so b dot that is what I want to write b dot k perpendicular and then - I y sorry not perpendicular it is one I am sorry k 1 dot x and, then you have d cube x over 2 pi 3 halves all I have done is change a change the variable.

And this is a constant factor it comes out b dot k 1 times f 1 tilde k one so you see the effect of this translation of this functions is merely a phase factor that is picked up so you just pick up a phase factor when you look at the Fourier transform. So good that also we know which means that, f 1 be this get the state I will write as integral d cube k1 prime q 1 prime. So let us call it here e to the - I b dot k 1 f 1 tilde k1 prime so there should be a prime here and k 1 prime (()) (17:17)we will use this later.

So now recall that even though the in and out state the basis states which we had defined in terms of which we construct all the other states we could, normalize them only up to delta functions right we could only normalize them to delta functions. So here let us see somewhere here like here these were because of the delta functions right. But now because I am smearing it out there is a finite momentum range I can unite not unite I can normalize these states to unity that is what I will do.

But I should first pause the, recording so what I was saying so unlike the in and out states which we use as basis state which can only be normalized to delta functions. We can normalize these states this f and b, are get f 2 or this cat f 1 b this takes to unity because these are not states of definite momentum these are spread out the momentum is spread out. So I can normalize them to unity so here is the, normalization.

So right now I am looking these as single particle states right you have one particle which is coming and it is far away from every other thing. So I can construct a single particle state there is another one which is in the bunch but also it is highly localized so I can also construct a single particle state for that.

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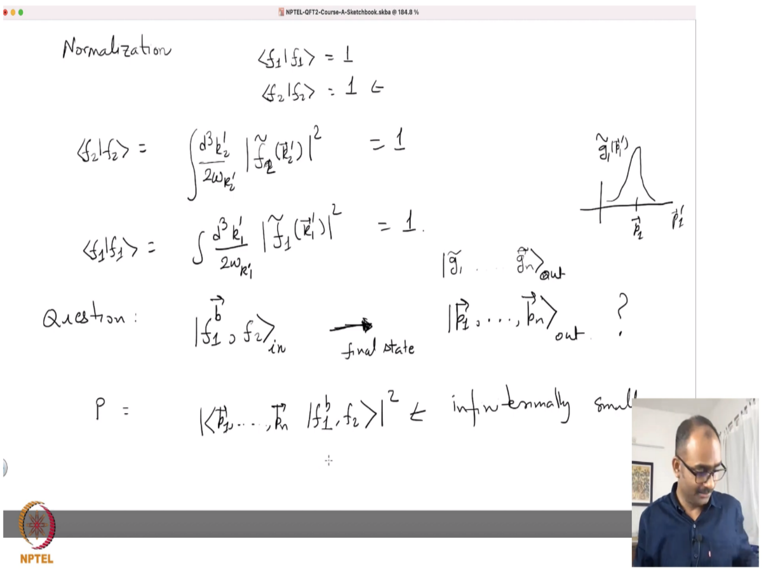

So we will normalize these states and the normalization is, and similarly f 2 now let us see what is the consequence what is the requirement that we should satisfy if we want this normalization. So what is f 1? F 1 is or let us say f 2 is this so when I put a bra f 2 next to it I will have another factor of f 2 and I should use a different argument let us say k 2 double prime. And then you will have bra k 2 double prime get k2 prime and, that is what is going to give you a delta function with a factor of 2 omega k 2 prime.

So let us do that with a factor of yes that is correct with that factor and we will have 2 such factors because one such factor over 1 over omega k 2 prime will come from this integral and from the corresponding (()) (21:21) integral you will have 1 over 2 omega k 2 double prime. But then this delta, function that you get from here will force both omega k 2s to have to be omega k 2 prime so let us do that.

So f 2 will be equal to d cube k 2 prime over 2 m omega k 2 prime time's f 2 tilde k2 prime mod square and this by this condition is 1. So let me just tell you how that has happened? So here when you look at the corresponding bra so f to this one this will be about d, cube k 2 double prime over 2 omega k 2 double prime f 2 tilde k 2 double prime. But then you will have a complex conjugate right you are these coefficients they get conjugated when you go from kit to bra.

So you have this and k 2 double prime and then I am using k 1 double prime sorry k 2 double prime k 2 prime = 2 omega k 2 prime delta cube k 2 double prime – k 2 prime that is, what I have used. So when you put all these things together you get this condition so this is the condition that has to be satisfied on the smearing that should be satisfied by this mirroring function. So you should take f tilde 2 such that if you were to mod square it and divide by 2 omega k 2 prime and then integrate over you should get one.

And of course similar argument will give, you the following there is no need for putting these primes here I could just leave them out. But I will still keep so these are the conditions to be satisfied by these smearing functions good. So what is it that we want to ask in the experiment what we want to ask is? Given the 2 particle state the initial 2 particle state where which is given by f 1 f 2 sorry which one, I am calling 2 which one f 1.

So f 1 b and f 2 the one in the beam I am calling f 1 and the one in the one which is the projectile I am calling f 2. So this one so you have a 2 particle state an initial 2 particle state which is this let us put n for initial. So question is what is the probability that such a state which includes these 2 particles which are going to collide evolves, to a final state can right now thinking in terms of in the Schrodinger picture.

But you understand these are of course constructed in the Schrodinger picture but the way I am speaking is the way you would speak in Schrodinger picture but I think I would I believe that you understand what I am saying. So this instead gives you a state which has n particles in the final state. So, that is the question that we want to ask what; is the probability of such an event.

Ideally I should not even put this basis state here I should put some smearing functions for each of them and then ask what is the probability of producing this final state right where g1 will be peaked around p1 and gn will be peaked around j tilde and will be picked around p n. But I will not do that thing I will, just say that I am sorry I am looking at the probability of producing this.

Because the effect of putting the smearing functions I will carefully keep in this the next steps and then you will see that at the end these smearing functions just disappear from the calculation. So then you can also believe that you know even in this case this g tilde will disappear so for at least for this part, for the out part I will just keep this these labels as momentum labels instead of the smearing functions. But if you want you can put this given tilde etc.

So let me nevertheless write down that if you were to if you really wish then you could do this. So this will be your g 1 tilde even prime or p 1 we have pm prime so it will pick up the momentum at p 1 so that is what we want, to ask and it is immediately clear that the moment you ask this thing the quantity that probability will be vanishingly small right. Because what you are asking is given this state what is the probability that I get particle which has momentum precisely p 1?

Another particle which has momentum precisely p 2 and that and so forth and the nth one has momentum precisely p n. If you are so precise, with these requirements if you are so specific with these requirements then most of the events that you produce are not going to fall into this category. Most of them will have some moment momentum where the particle one particle has momentum different from I mean none of them is same as p 1 none of the same none of them has momentum as p 2.

So that will be vanishingly small or an, infinitesimal quantity so it would make sense that we are some probability of an event which has a finite probability associated to it so that is what I will do. So as I argued just now that if you were to ask the probability that f 1 b f 2 this state giving you this momenta p 1 to, p n the probability is given by modern mod square. Then this will be infinitesimally, small right because it just almost impossible to produce exactly that configuration with exact those momenta in the final state there will be it will be 0.

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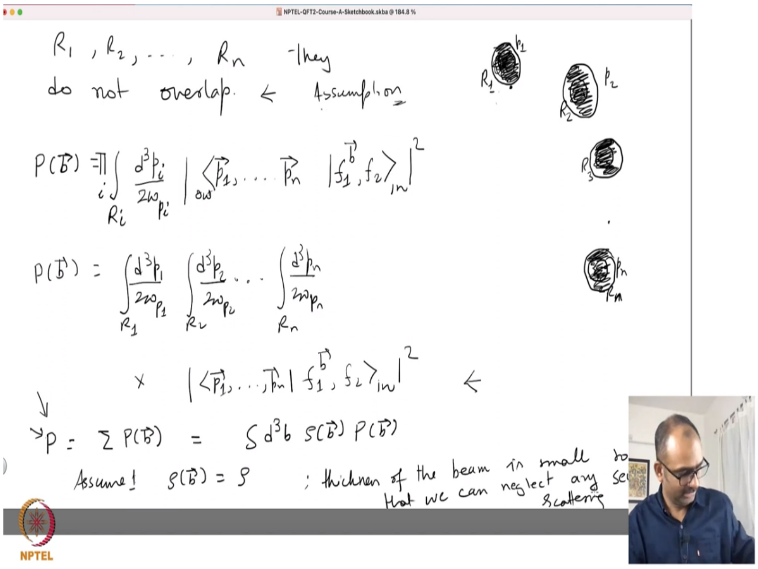

So what we should ask instead is that I want the final state particles which are produced after the collision to have not momentum p 1 but some moment they should have 0. So what I precisely mean is? I will count the event as an event, which I am looking for if one of the particles has a momentum which lies in a region R 1. Another particle has momentum which lies in the region R 2, R 3 and so forth that is something I can ask where sorry r n where this is some region not in physical space first momentum space.

So if an event occurs and the particle has you measure the momentum of the particle and the, particle has a momentum which is lying somewhere here you count it as. So let us say event number one so you collide these initial state particles and it produces a final state where the particle has one of the particles has momentum which is here another one has momentum here another one has momentum here and so forth here.

So that counts as an event because the momenta lies in this region I, collide another one another bunch of particles in the initial state. And suppose this time one of the particles has momentum here another one has in here third one is in here and fourth one another one here this is not counted right because the momentum is outside. So I am going to add up all those event events in finding out the probability in for which these particles fall in this region.

And, this region can be big I am not assuming that this momentum region is small it can be big it does not have to be a small region I do not care where it falls in within this region it can be a big region and in principle not in principle. I can do the following that I take the region R 1 to be the entire momentum region. Let us say, this is entire momentum space then R 1 could be filling this entire momentum space R 2 could also be filling this entire momentum space.

Meaning I will be counting all those events in which one of the particles has whatever momentum another particle has whatever momentum and so forth. So I count all of them as the right events for which I am trying to find out the probability but, let us not do that for a moment. For a moment I am make an assumption that not an assumption I am looking at an probability of an event a probability of a fan producing a final state.

In which the particles have moment are lying in range R 1 in a region R 1 which is locate and p 1 is one of the momenta within this region and let us say you have another, region where you have momentum p 2 inside it and there is some region given and like that. So that is what we want to ask an assumption is one assumption I make is that R 1, R 2, R 3 and all of these they are there is no overlap between these regions that is an assumption I make for now.

But we can relax that assumption there is no need to so for a moment I will just, make this assumption so R 1, R 2 and all of an R n they do not overlap and I want to ask the probability of producing such a final state. So probability which will be a function of among many other things a function of b and remember what is b is the location of this particle. So it will depend on this right see if you take b to be to have a magnitude very large.

So suppose you have a, particle somewhere here then the probability of collision of this guy with that guy will be small because it will have a wave function which will be localized around here. And this guy will have a wave function when it which when it reaches close to the center to the origin it will have wave function localized around here and there is not much overlap.

So the probability of them, scattering and giving some final state which is different from having the same this particle and that particle will be very low. So probability will depend on b as you move away from the origin the probability of a non-trivial scattering becomes small. So you have a b dependence that is what I am making explicit so the probability of producing this final state when you have a, scattering with the particle located at b is given by f 1 b f 2 that in state.

And then I am producing this final state I should Mod Square and then I have to integrate over this entire region R 1 and R 2 and so forth. So I have I should have let us see d cube p I over 2 omega p I and region R i and then I should take product. I do not like writing it this way so I will write it, again the same thing I am just repeating d cube p 1 over 2 omega p1 is correct d cube p2 2 omega p 2 d cube p n 2 omega p and R n times.

So and you should convince yourself that this is the right factor that you should have you should have 1 over 2 omega p 1. And this you should be able to argue based on the normalizations that we have used. So for now this is assumption for now, and with this assumption in place this is the probability for producing this final state. This is the probability of producing this final state when you are looking at a collision with the particle located at b.

But if you are if you now ask what; is the probability of producing that final state when this particle collides with any of these particles in the bean. Then that probability is just the sum of all the above probabilities. So that is what you should do now let us assume that before that I should add d cube b. So let us say you have a volume density rho b so that gives you the number of particles per unit volume and then that is d cube b is volume element.

So I integrate over all of them and then I am, integrating this probability multiplied with this density. So now I am going to get the probability p of producing the final state so I will make an assumption that rho of b is a constant so that I do not have to worry about the b dependence of the density. So I am just assuming that this is a constant and there is another assumption that I should make.

So imagine what, will happen typically so you have a particle coming with enormous amount of energy it collides with one of the particles in the slab in this beam and it produces lots of final state particles. Because the energy is very large of the incoming particle it will produce final state particles which will also have large energy. So let us say what happens, is that this is the thickness this is of the beam and it collides with a particle which is really on the surface of this beam this slab.

It produces final state particles which are coming out with large energy and these particles that then collide with other particles in the beam. And they produce some other final state because it is a different collision it will have some different final, state. And then they may collide with the remaining particles now that becomes difficult. Now we are not the probability you calculate using this is not going to give you what is happening in the experiment in that case where these other collisions are also possible.

So we assume that such collisions are not possible they do not happen or their contribution is very minimal. Meaning once a, collision has happened with one of the particles in the beam whatever comes out goes to the detector basically that is the assumption and we can show assume we can make this assumption if we let the thickness of the beam to be very small. So that we; reduce the probability of the secondary scatterings.

So we assume that the thickness of this beam is small so that we can neglect any secondary scatterings because if you do not make that assumption then what you are calculating here will not match with the experiment because this will be 2 very different things. So we will let it to be very thin only one particle collides once and that final state is what is seen in the detectors.

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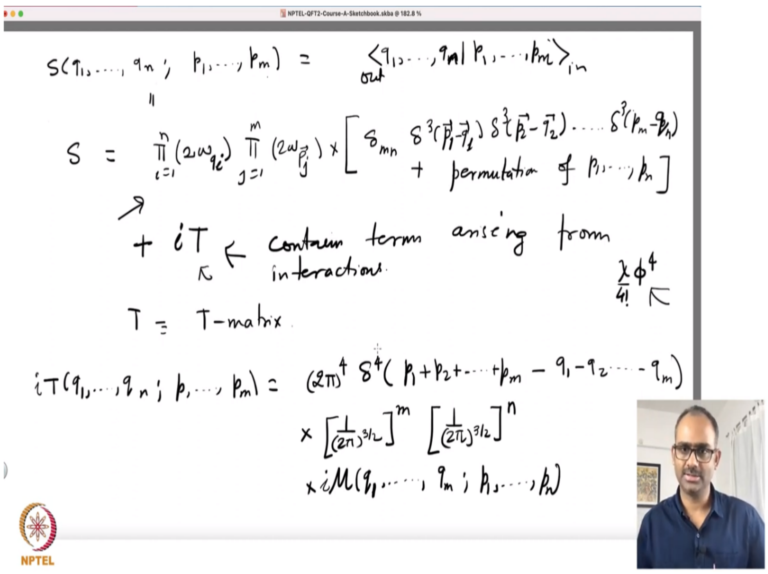

So applause so once we have made that, assumption then in equations it means that P of B is basically P of B perpendicular and it does not depend on b 3 just a second. So that is the assumption of small thickness which is what I will now denote as P part so this does not depend on b z upon b 3 and if you were to think what will happen if you had really thick beam. Then most of the time what will happen is, these collisions will produce final state and nothing will come out eventually so you will not get anything outside.

So this probability P is rho times l and then you are left with only integral over d to b part meaning d to b for is. And then you have integral d cube p1 over 2 omega p1 over region R 1 and then all these remaining integrals and finally d cube p n over 2 omega p n. Then I should include this f 1 tilde and it comes with these vectors let me show you so you have this state instead and for f 1 tilde I have these factors so this is what I have to include.

So you have integral d cube k 1 prime omega k 1 prime then integral d cube k 1 double prime over 2 omega k 1 double prime that is for f 1. And then I should I will also have for f 2 so d cube k 2 prime 2, omega k 2 prime d cube k 2 double prime k2 double prime times f 1 tilde q1 prime. Let me write down then I will explain why I am writing this how do I get these functions?

Star k1 double prime e to the - I be perpendicular that is perpendicular B perpendicular dot k Perp prime k1 purple 2 men indices k1 per prime - k 1 curve double prime f 2 tilde k 2 prime f 2 till last star, means complex conjugate double prime. And then we have k 1 prime k 2 prime in p1 to, p n out then we have k 1 double prime k 2 double prime in and p 1 2 p n out but this time complex conjugated let us see how I got this expression.

So let us go back P B this one so I am now integrating over d cube b and I have already said that P B is only a function of b perpendicular so I, will set b 3 = 0.So that is why I have once I said b 3 = 0 I am left only with b perp in this factor and this factor comes from where from f1 tilde here f1 tilde is a function of b. And you recall in the Fourier space f1 tilde has a phase factor e to the - i b dot k so that is where I put b3 to be 0.

So that is why you have only b perp in appearing here. So let, us go back here these integrals are here in this line then I have integrated over b 3 that gives me l rho has been pulled out. So we understand this first line that is coming from here and this piece these 3 these integrals and putting b 3 = 0 because function does not depend on probability does not depend on b 3. Now if you look at this function this is mod, square so you have this thing times the conjugate of it that makes the mod square.

And you have f 1 b f 2 but f 1 and f 2 they are written in terms of these integrals here this is for f 1 and similarly for f 2. So when I write f1 I will have 2 factors of, f 1 tilde and 2 factors of f 2 tilde but one of them will have a complex conjugate right. Why because this itself has a, complex conjugate at this times the same factor with the complex conjugate so that gives a complex conjugation. And then you have integrals over this f 1 tilde's this factors d cube k 1 prime etc., and similarly for f 2.

So now you understand why I have f 1 k 1 prime and then f 1 conjugated one but with k 1 double prime because I should use different variables. So this is what has been, spelled out so k 1 prime k 2 prime in state times the complex conjugated one and these are bases in states originally I had f 1 tilde sorry f 1 and f 2 this. Now this 2 particle state I am expanding in the basis states so that is why there you see not those wave functions but then you multiply with those smearing functions which are here.

So I hope it is, clear that you should have complex conjugates for f 1 I should have 2 f tilde f 1 tildes or f 2 also to f 2 tilde's coming from this and that they should come with complex conjugates. Also I should use different labels in this one and that one so this is why you have f 1 tilde k 1 prime and f 1 tilde k 1 double prime then I should integrate over all of those as you have seen in the previous pages so we understand this expression it is just the expression for b.

So we have to now work with this so let us first look at this part we will worry about these all other factors later but let us first look at this part. I think there is a we, to copy let us see if I can do and maybe not sure whether it will work let us see now one more try it worked. So I was lucky, I could do this so this is the object we want to look at which we had here and this is equal to.

We have long back written these elements the asymmetric elements in terms of what we call m the matrix element m let us check whether you can find it easily here. You see this it is unfortunately let us try so here you see this is what we have now equivalent of this right, we have some labels p 1 p 2 p n here and some k 1 k to here in and out. So that is what we are looking at if we ignore the forward scattering in which we are not interesting this term interested then this term is gone and I am looking at i T what?

That is after removing the forward scattering whatever is left of s is i T and i T let me put i T, here. And so that there is an I here, also this I t which is the asp matrix element what we had done was we had pulled out for each momentum of factor of 2 pi 3 halves that is why you have M factors of 2 pi 3 halves one over 2 pi 3 halves here and n factors of 1 over 2 pi 3 halves overall energy momentum conservation times 2 pi 4 and then this i M.

So that was the definition of i M and this, element this product so that is what I am going to use now I will just pull out these factors pull out a delta function and write it in terms of i times M.

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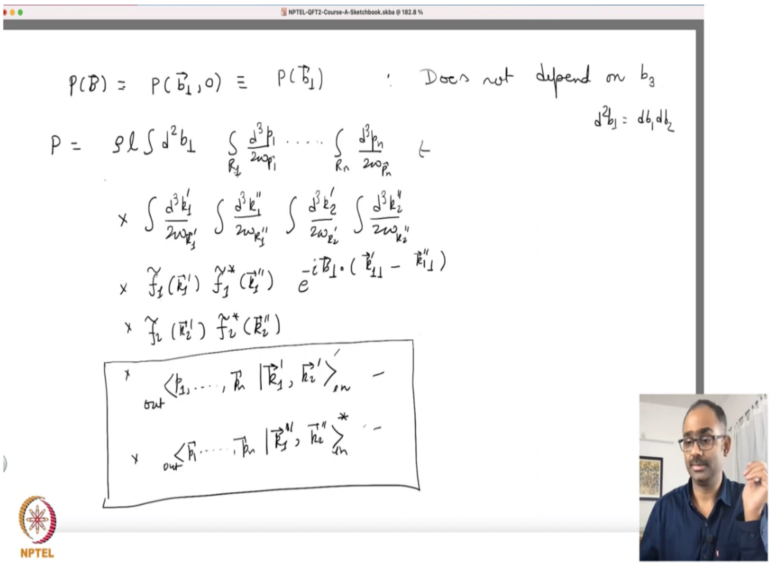

So this is the first one this one it has n factors of it will bring in factors of 1 over 2 pi 3 half. Then k 1 prime and k 2 prime they will become bring 1 over 2 pi 3 halves each of them so there are 2 of 2 labels they, bring 2 factors times 2 pi to the 4 some problem. Now delta 4 of k 1 prime looks like my laptop is slowing down sorry minus summation over all this moment appear n of them that is what you have and then you get I times m k1 prime k 2 prime and they give you what p1 to, p n that is what comes from the first factor.

Then the second factor gives you the following so each of the p n's, will bring this entire factor again so that makes it 2 n these 2 also bring the entire thing again. So that makes it four 2 times 2 then again you get a delta function which is 2 pi it is not working at all 2 pi to the 4 delta 4 k 1 double prime + k 2 double prime minus summation over p i times - i M. Because we have a complex conjugate so - i times M star p1 to, p n k 1 double, prime k 2 double prime.

We should I think I need to stop because it does not work anymore so anyhow that is the expression and we should now look at all these delta functions and also do these integrals. Because we have lots of delta functions doing these integrals will be easy that is what I will do next time.