Introduction to Quantum Field Theory - II (Theory of Scalar Fields)

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Module - 6
Lecture - 17
High Energy Experiment Setup - 2

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So, let us continue our discussion on scattering experiments in high energy physics. So, let me say a few words before I go to this topic of constructing the initial and final states.

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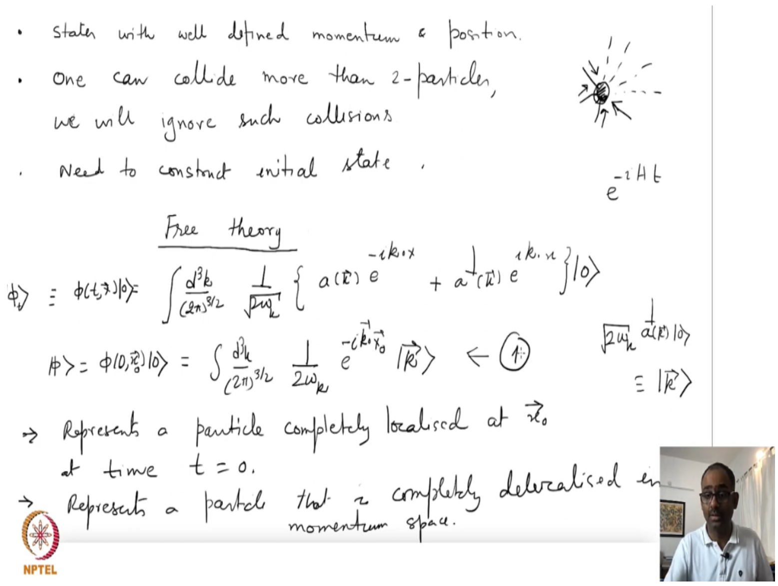

First thing is that when we collide particles, these particles will have some reasonably well-defined momentum and also they will be reasonably localised in space also. By this, I mean that you do not deal with states which are eigenstates of momentum for example; you rather deal with states; I am thinking of a single particle state; which have momentum which are very well-defined.

So, you can say that this is an electron or a positron or whatever particle is moving with this much of momentum, but when you make that statement, you do not mean that it really has that momentum; you mean that it has a spreading momentum which is much smaller than the mean momentum that it has. So, when we say that it has a reasonably well-defined momentum, what we mean is that the spread in the momentum is very small compared to the centre value of the momentum.

And also when you say that I have fired an electron, it is coming, going to come out of the electron gun or whatever and you know that it was within the gun; it is not like you do not know where it was in the entire universe. It was there at that time, but of course, you have uncertainty relations, so, there is uncertainty in position, but then uncertainty in position is very small. So, it is well-localised at some point in space and also well-localised in momentum space.

So, it has a well-defined momentum, which means that when we construct states which scatter, I should construct states with well-loop, well-defined; by well-defined I do not mean precise, exactly 1 fixed momentum, I am allowing for some spread; well-defined momentum and position. Also we are doing quantum field theory and we have interactions. So, in principle, you can have any number of particles coming together and interacting to give whatever final state you are looking at.

So, you may have several particles coming together and interacting in a region. So, these are all particles which are participating in collision and then they give out some final state. Let me draw the final states by dotted lines. So, let us say they give out some final state. So, you can have several particles that collide together, but typically what will be relevant is that we study only collision of 2 particles.

So, we do not consider an incoming state which has more than 2 particles which are undergoing a collision. So, we are going to look at a scattering of initial state in which you have 2 particles. So, that is an approximation because such events can also occur, but we are not going to include them. So, that is an approximation. So, even though one can collide more than 2 particles, we will ignore such collisions and we will only look at collisions of 2 particles.

So, now, what we should do is, now we should construct an initial state that has 2 particles; and by this statement what I mean is that I should construct a Heisenberg state, a state in Heisenberg picture. And remember, a state in Heisenberg picture is same as a state in Schrodinger picture at time t = 0; we have been referring to time t = 0. So, these 2 are same things, but of course, states in Heisenberg picture do not evolve with time; only operators evolve with time.

So, the interpretation should be that if you were to take that state at time t = 0 or equivalently the Heisenberg state and then look at the same state in Schrodinger picture; because Schrodinger picture, this is a state at t = 0; and evolve with Schrodinger equation or the unitary operator e to the -i h t and evolve it backwards in time, then when you go far back in time, that state represents 2 well-localised particles which are far apart from each other.

That was the interpretation that we gave. So, that is a state which we want to construct now. So, we need to construct initial state and that is what I am going to do. So, before I construct the initial state which contains 2 particles, I will start looking at first a single particle and I will do so in free theory; it will be easier to understand that way. So, let us begin with free theory, but remember, there is no collision happening in free theory.

Free theory, nothing happens; even if you have 2 particles, they come close to each other, they just pass through each other. This particle, if it is not interacting with this one, there is no way it can tell that I am here and do not go through me. It will just go through it, because there is nothing else that can happen. So, free theory, that is where first we will look at things.

So, the field phi, the operator phi is d cube k over 2 pi 3 halves, then a factor of 1 over 2 omega k, then you have a of k e to the -i k dot x. So, that is 4 vector k mu and that is x mu and that is the k mu x mu; plus a dagger k e to the i k dot x. Now what we do is we act this field operator phi; remember, a and a daggers are operators, so, phi is also an operator. We add this on the free vacuum and this is what I define as the state phi.

So, I will just; this is a state phi acting on ket 0 gives you a state, and that state I am labelling is ket phi. So, what is that? That is, a annihilates the vacuum. This is free theory remember, a annihilates the vacuum, so, only a dagger acting on ket 0 gives you something. And now I am interested in; so, let me put a label t here. I am interested in phi, ket phi which is basically you put t = 0.

So, the definition is; I want to look at phi and it is defined as; so, I am looking at the field at time t = 0 and then this x on the vacuum, and this gives you integral d cube k over 2 pi 3 half. Then this term does not contribute anything; it hits the vacuum, gives you 0; a dagger acting on vacuum creates a single particle state with momentum k; and recall the normalisation; we had normalised them according to this.

So, we had root 2 omega k a k dagger acting on vacuum. That is what we called as ket k that was a single particle state in free theory. So, you have to have a root 2 omega k above, so, I should divide by root 2 omega k and that gives you 1 over 2 omega k. Then e to the time component; because I am putting t = 0, so, the time component goes away and you have e to the -i k dot x and then you have ket k.

This is something which we saw in the previous course also, first course. And you see that ket phi, this state, is actually a state which is a linear combination of all the states, all the eigenstates of the momentum operator. This ket k has a fixed momentum k and we are summing over all such states. So, and what does this represent? This represents, this ket phi represents a particle which is localised at point x. So, it is really there; it is not like it is around there, but it is really at the point x and you see that you have spread over entire momentum range.

And that is something also we saw in the previous course that we can interpret this as a state which localise completely a particle x. And actually, I wanted you to do a little different thing. So, I am going to change; I will put a nought here, x nought. So, ket phi is defined as this. It will be helpful to do this, so, here this. So, my particle is localised at x nought; that is the definition of ket phi.

So, let me write down, represents a particle completely localised at x nought at time t = 0; I put t = 0 already. And how about what happens to this in momentum space? As you see, it has completely delocalised it; it is a sum over all momenta. So, it also represents a particle that is delocalised in momentum space, meaning you cannot ascribe any momentum to it, you cannot assign any momentum; but that is not what we want, that is not very useful for us because this is not how particles in the real world behave.

In the real world, they are localised in space and also they have well-defined momentum, which right now this state does not have. So, you want to also have localisation in momentum space. So, that is what I am going to do. That is the goal.

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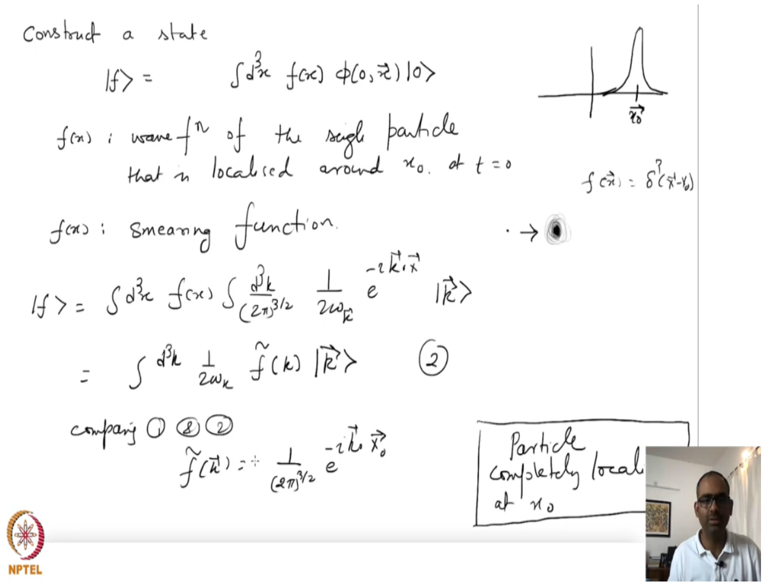

So, I construct a state which is ket f, and what is that? So, I take phi 0 x, so, at time; I am constructing the state at t = 0, because we are looking at Heisenberg states. So, I take phi 0 x and act on vacuum. So, this creates a particle localised completely at x, but now I want to delocalise a bit around x nought. So, what I do is, I integrate over all x, but I should multiply with a function f of x, some function f of x.

And what should that function do? That function should localise the particle at x nought. So, what I should do? I should do the following. This is a 2-dimensional picture, but we are looking in 3-dimensions; so, x nought vector. Now, what I want? I want to localise here. So, I think of a function which looks something like this. You can take it to be a Gaussian, but does not have to be. So, what happens? This f is 0 away from x nought.

The moment you go a little further away from x nought, it is vanishingly small, it is very small. So, the major contribution to this integral comes from the region where x takes the value x nought, so, only from this region. So, then it is phi only when x is around x nought contributes to this. So, you are constructing a state which is localised around x nought. In the limit you take f x to be a delta function.

So, if you take f x to be delta cube of x minus x nought then of course, this integral will give you f 0 x nought. So, then you completely localise this, but we are saying it is not a delta function, but something sharply peaked around that. So, now, this f x is the wave function of this particle, of the single particle state; so, that is what is localising it around; that is localised around x nought at t = 0.

So, you see, if you have at x nought, a particle which is sharply localised here, then it is given by phi acting on ket 0; but what we have done is, we have multiplied with the function f and we have smeared it around; what f does is, it smears it around. Smear in Hindi means pothana **(FL)**. So, that is what it means; you are smearing it around like with your thumb. So, now that thing which was initially a sharply localised thing has now turned into this.

So, that is why f of x is also called a smearing function. So, now let us write this ket f by putting the expression of phi of x and we get integral d cube x f of x. And what is phi? phi is d cube k over 2 pi 3 halves 1 over 2 omega k and e to the -i k dot x ket k. This is an expression what we had on the previous page, this one. That is what I have substituted here. So, I can write this as; I will separate these k and x integrals, and this I will write as d cube k over 2 pi 3 half 1 over 2 omega k; these I have taken care of.

Then the x dependence is only in this exponential and f of x, and there is an integral over this which is clearly a Fourier transform of f. And because I am using 2 pi 3 halves in doing Fourier transforms, so, I am keeping 2 pi 3 half both in the transform and the inverse transform. So, I should be careful. So, d cube k times 1 over 2 omega k and times d cube x 1 over 2 pi 3 halves f of x e to the -i k dot x that is f tilde of k.

That is the Fourier transform. That is right. So, I want to label this equation as equation 1 and this one as equation number 2. So, if you compare 1 with 2, what do you get? That f tilde of k is, apart from some factor which is not very important for us right now; this is what you get. So, f tilde of k which is the wave function in the Fourier space or the momentum space is given by this e to the -i k dot x nought for when you have constructed this state where the particle was localised at x nought; and this gives you a particle completely localised at x nought.

That is the corresponding wave function in the momentum space when you have complete localisation. Now, if we go to the other extreme where I want the particle to be localised in momentum, then we have the following situation.

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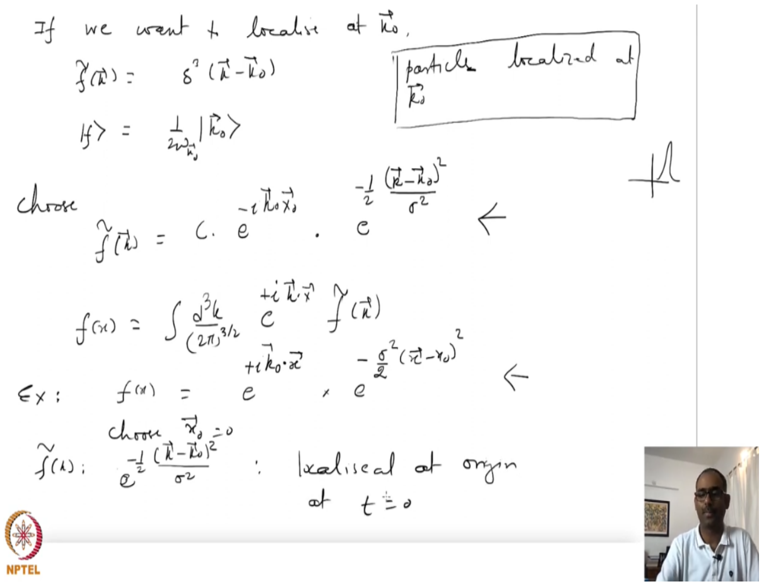

So, if we want to localise at k nought, then what will be the f tilde of k? So, here, if you want ket f to be a single particle state with 1 precise momentum, then f tilde k should be delta function. If you take delta cube of k minus k nought, then only k nought will be picked up, and that is what I am doing. So, apart from some factors, that is what I should take and in that case, your f, the state would be, you can put factor of 2 omega k nought here and then you can get rid of this 1 over 2 omega k nought but this is how it should look like.

f tilde of k should be proportional to a delta function, and this gives you a f tilde k; this gives you a particle localised at k nought, but these are the extremes which you do not want, they do not correspond to real particles. So, what should we do? So, let us choose f tilde of k to be e to the -i k dot x nought if you have only this; where is it? Here; then that corresponds to complete localisation at x nought, but that is not what we want.

Then, if you have complete localisation in k nought, then you have delta function, but instead of taking a delta function, I will choose a sharply peaked function like this at around k nought and in the limit you take the width to be 0 that goes to delta function. So, let us choose this thing. Any function will do which has the similar feature, but let us take Gaussian to be specific.

Some constant you can multiply to take care of normalisations, but that is what the functional form of f tilde of k would be. So, now you see, this one, this wave function, if you look at f tilde of k in the momentum space, this is almost like a delta function. This factor f sigma is very small but it is now, has a finite spread depending on the value of sigma and you pick up value around k nought.

So, this is the situation, but then of course, it is delocalised in space also and it is spread around value x nought. So, if you take f tilde k mod square that is the wave function for this particle in the momentum space, then this goes away, this factor goes away and you see that the f tilde k is just the square of this exponential. So, it tells you that the momentum is localised around k nought.

Now, if you want to know, if you want to see how things are in the position space, then you find out the wave function f of x. And how do you get f of x? f of x is an inverse transform, so, you should have e to the +i k dot x f tilde of k. That is the inverse transform; and calculate this and you will find the following. You will find that f of x will be e to the +i k nought dot x. So, this is a factor of k nought; it is not x nought, it is k nought; times e to the minus sigma square over 2 x minus x nought whole square.

So, you see, if you take mod f square, then this factor goes away and you have a square of this which is a sharply peaked function around x nought. So, the square modulus of the wave function gives you the probability of finding that particle in space. So, you see that it is almost 0 everywhere because of this function and it is non-zero only around x nought. So, you see, by choosing this f tilde of k or this f of x, you construct particles which resemble real particles.

That is a good starting point. Now we should look at how to construct 2 particle state, and I am interested in looking at a 2 particle state because when you are colliding particles, your initial state contains 2 particles which are going to collide. So, as my state has these 2 particles, and that is the state I want to construct.

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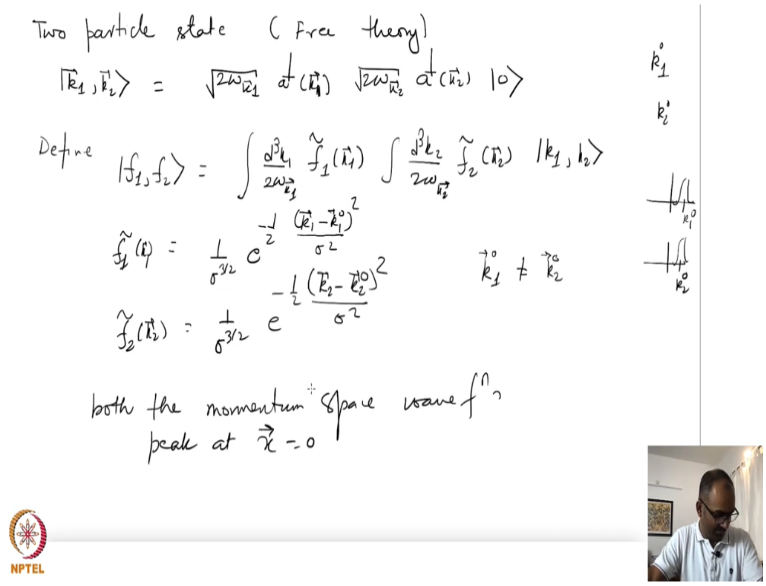

So, and again I am working in free theory. So, what is a 2 particular state in a free theory in which the particles carry momentum k 1 and k 2? That is 2 omega k 1 coming from the normalisation, and then you have a dagger k 1, then you have square root of 2 omega k 2, a dagger k 2, and these operators act on the vacuum of the free theory. So, how do we define a 2 particle state in which both the particles are localised around some point in space and around some value in momenta?

That is what we want to do; and before I do that, I should say that here, for f tilde of k, I will choose x nought to be 0. So, I want that at time t = 0, the particle be localised at origin. So, this factor goes away then. And then your f tilde k is just some constant times e to the minus half whole square over sigma square at origin at t = 0. So, coming back here, I will define like; for the single particle case, I will define a state f 1, f 2. And what is this?

This is; all I am going to do is smear around the value k nought. So, I want to delocalise a bit. d cube k over 2 omega k 1, f tilde 1 k 1. So, this is a function of k 1 and of course also contains k nought, k 1 nought if we are localising the first particle around k 1 nought so that it has a momentum k 1 nought. And then you have d cube k 2 2 omega k 2 f tilde 2 k 2 and then you have k 1 k 2.

So, this is a state which has precise momenta k 1 and k 2; these particles have these labels. Now, I am doing a smearing around point k 1 0 and k 2 0. So, f 1 tilde k 1 also contains k 1 0 and f 2 tilde k 2 contains k 2. So, these are basically functions of this form in the momentum space. This is around k 1 0 and that is around k 2 0, or you do not have to take a Gaussian but you can take that also, and it will be, apart from some normalisations, it will be minus half k 1 - k 1 0, some factors; I can put a factor here but I am not worried about the normalisations right now.

f 2 tilde k 2; and you see here, just like here, I had removed this factor containing x nought by putting x nought equal to 0 because that is what localised this particle at the origin at t = 0, and I have done the same thing, otherwise you would have had such factors but now we want to localise particles at the origin. Now, so, we will choose k 1 to be different from k 2 0; k 1 0 and k 2 0 are different.

So, both these wave functions, momentum space wave functions peak at x = 0. That is because I have put x nought to be 0. So, this is what we have in theory. Now, our wall is interacting wall and we want to construct similar things in interacting theory, and it is not much different.

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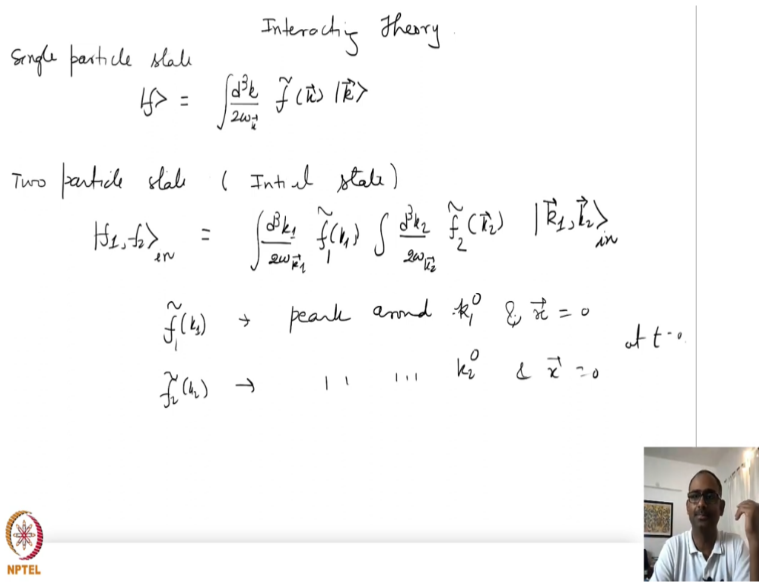

So, in interacting theory also for single particle states, I will have the same thing; I will have ket f to be d cube k over 2 omega k, and then you multiply with some smearing function ket k and you can choose similar to exactly like this if you want to use the Gaussian's. And how about 2 particle state? Now, here, so, this is the state which we will have before the scattering. This is what we are constructing initial state.

For the initial state, again I will have f 1, f 2 and I will call it initial state in. And this will be what? d cube k 1 over 2 omega k 1, just like before and then you have f tilde k 1 smearing function, then you have d cube k 2 over 2 omega k 2 f tilde k 2; and then what? I should have k 1 k 2 this ket; but remember, we are constructing an initial state, an in state, and when we want to write in initial state, I should be using the basis which is what we call the basis of in states. So, I should put this label in.

See, at t = 0; all these are Heisenberg states, so, at t = 0, I have this vector space, the Hilbert space and I can choose to work with the in states. And in states are those states which when you evolve with Schrodinger equation with appropriate folding functions, this is like f 1, f tildas, they give you a particle which are well-separated in the far past, but the same in states do not give you particles which are well-separated in the far future.

So, I cannot use out here; I can only use in state. If I were to construct a final state just like this one with f 1, f 2 out, then I should be using out states, because they have this property that they give you well-separated particles in the far future when you evolve a Schrodinger equation and when you have folded them with right kind of functions. So, that is the only thing which we have to take care of when we are doing interacting theories that I should use in states.

So, what are the requirements? I want f tilde k 1; in fact, this is f 1, f 2; f 1 k 1 this should peak around k 1 0 and I will use a Gaussian without this exponential factor of this kind so that in the coordinate space, in the position space, it peaks around x = 0. And similarly, f 2 tilde k 2; I want to again use a Gaussian so that it peaks around k 2 0 and x = 0 at time t = 0. So, now we have constructed these states. Now we have to see how to get the information of scattering using our quantum field theory, interacting quantum field theory. So, that is what we will do next.