Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 6
Lecture - 15
Kallen-Lehmann Spectral Representation Continued

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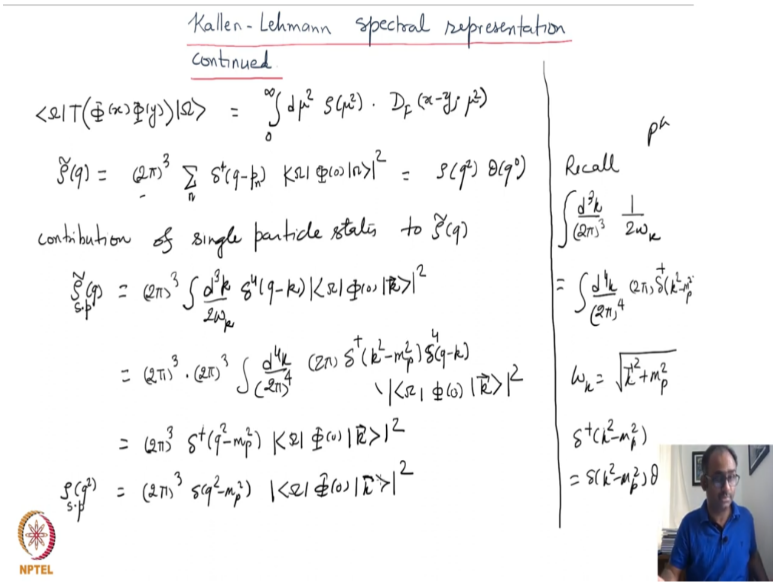

So, let us start our discussion on Kallen-Lehmann's spectral representation and we will continue from where we left last time. So, just to remind you, we were analysing this object and where phi could be elementary field, the field that appears in your Lagrangian or phi could be a composite field.

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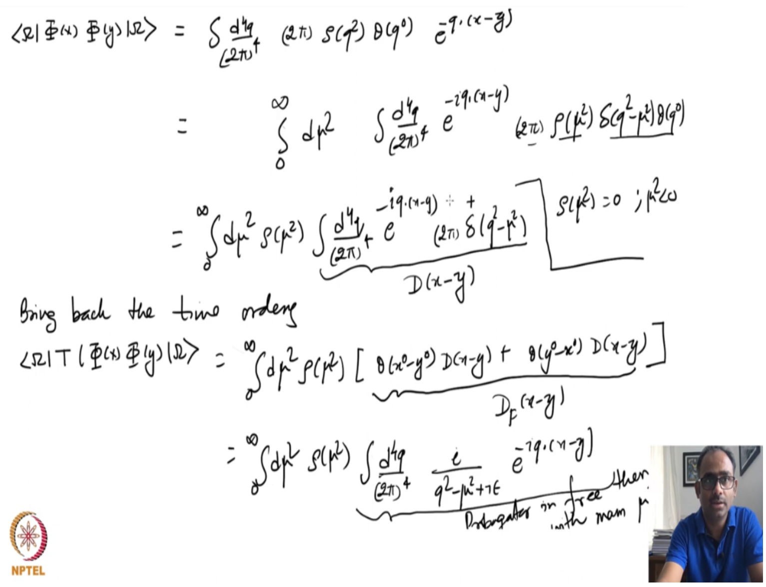

And we showed that this can be written as this spectral density times free propagators, but this time, the mass that appears in the free propagator is the mu square and mu square is not a mass in the physical theory; it is something which you integrate over. So, mu square takes all possible values from 0 to infinity. So, this object takes this form. That does not mean that you have particles of mass mu square in your theory; that is not something it means.

It just means that this quantity can be written as an integral over mu square of the spectral density times the Feynman propagator where this propagator, instead of having mass m, it has mass mu square that is the notation here, and you have to integrate over all of these. Let us see; correct. Also let me write down the definition of rho tilde of q; rho tilde of q is this.

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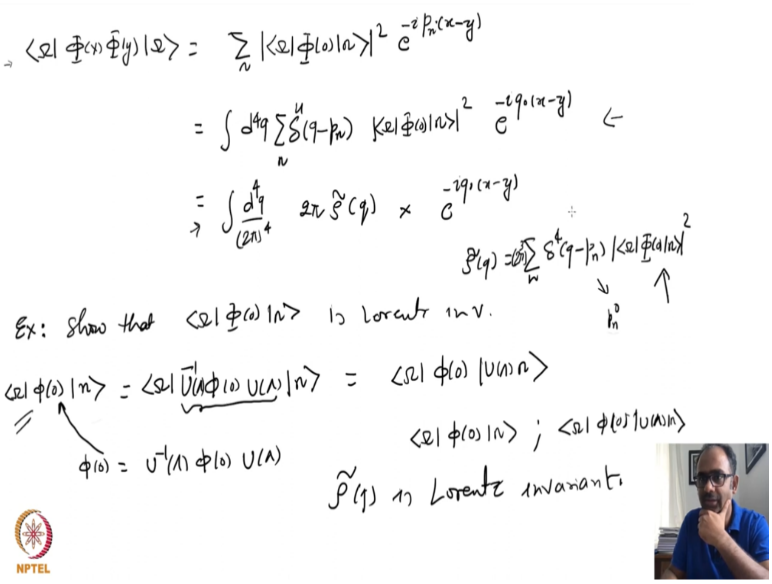

Let me write it down. So, rho tilde of q is 2 pi cube times; you are summing here over all the states in the physical theory. By all the states means I have taken the states n to be the eigenstates of energy momentum of p mu, energy momentum vector p mu; and this is what? delta 4 q - p n times omega; what was it? phi 0 ket n. And we had argued that rho tilde of q is a Lorentz invariant object because this factor here, this is Lorentz invariant and this factor is Lorentz invariant.

So, this entire thing is Lorentz invariant and also it is non-vanishing only if q nought is positive, and that is why we also wrote it as rho of q square times theta of q 0. And then let us see what else; perfect, that is all. Now we will continue from here. Now we want to; you see, rho tilde of q or equivalently rho of q square, that takes, that gets contributions from all possible states n, all the states like single particle states, multi-particle states.

So, single particle state, 2 particle state, 3 particle state; in general, everything will contribute. So, what I want to do now is to filter out the contribution of single particle states. So, contribution of single particle states to rho tilde of q and these contributions I will call as rho tilde q single particle. And similarly rho of q square I will define rho single particle with this argument q square. So, that is what we are looking at.

So, what is that object? That object is 2 pi cube. So, that 2 pi cube factor here; then summation over n, but now we have to appropriately replace the sigma over n by the measure that you have which we have discussed earlier, is d cube k over 2 pi cube; No factor of 2 pi cube, we had just 2 omega k. You remember, we had long back discussed that this is the appropriate measure for single particle states because the n, label n is not discrete; it is continuous.

The single particle states have continuous momenta, so, you have an integral, and this is what gave you the correct measure. That is fine. Now I should put delta 4. So, delta 4 q minus. Now p n gets replaced by k and then you have omega phi 0 k mod square. That is fine and then other contributions not to this but to rho tilde q coming from multi-particle states. This is nice. This is good. Now what I will do is, I will utilise the following result.

So, recall that we had shown d cube k over 2 pi cube 1 over 2 omega k. This we did in the previous course, in the first course; is same as d 4 k over 2 pi to the 4 times 2 pi delta plus k square minus m physical square. This is what we had done, where here m physical is what appears in omega k, where omega k is k square plus m p square, and this m p square is what is appearing here; and here, omega k is the same object.

So, this I can write as 2 pi cube times; you have 1 over 2 pi cube multiplied here, so, I have another factor of 2 pi cube times d 4 k over 2 pi to the 4 times 2 pi delta plus k square minus m p square and then remember, delta plus; delta plus is this, k square minus m p square is delta of k square minus m p square times theta of k 0. Let me close the window there. So, this factor is here and then I have still a delta 4.

Looks like again a problem. So, now I will do integral over k and this 2 pi cancels this 2 pi 4 which leaves a 2 pi cube, and that 2 pi cube, I can cancel with this 2 pi cube. So, I have a 2 pi cube here, this one left. And then when I integrate over k, the k gets forced to become q. So, here in this delta plus which is same; let me write with a plus; this becomes q square minus m p square times omega phi 0 k mod square and remember delta plus has a theta q 0 in it.

So, from this I can then read off what rho of q square is coming from single particle. So, that is 2 pi cube times this delta of q square minus m p square because rho tilde is rho q square times theta nought, so that the theta q nought, that theta q nought I am taking away; and then you have this factor, this Lorentz invariant factor. So, that is nice. When I look at rho of q square coming from single particle states, it is proportional to delta of q square minus m p square, and it will have a constanised consequence for the g tilde of q when I take the Fourier transform of this object.

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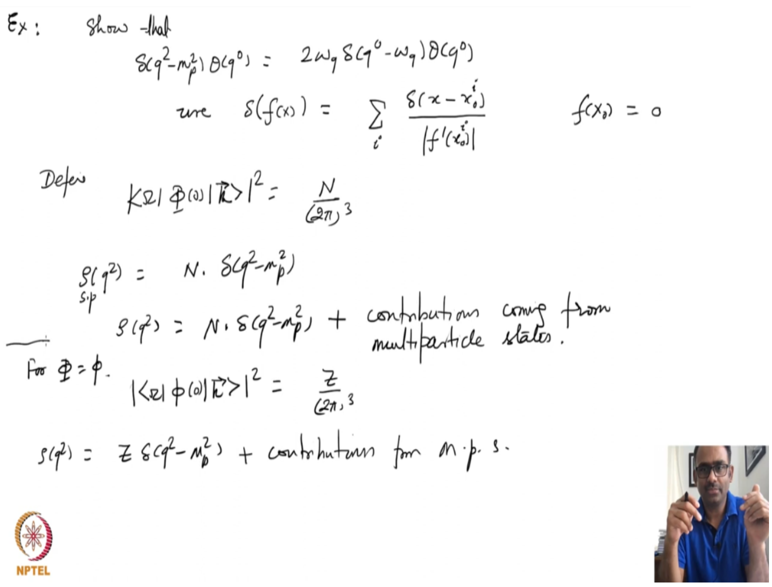

I will give you an exercise to do the same integral which I have done in a different way. So, show that delta of q square minus m p square times theta of q nought is equal to 2 omega q delta of q nought minus omega q times theta of q nought. So, this time you will not utilise this result here, this one which I used, but you should now show this and then you can utilise to arrive at the same expression of rho of q square.

In proving this, you should use, you will have to actually, delta of f of x where f is some function of x and this is delta of x minus x nought over f prime, the first derivative at x nought, and you have to take a modulus of it. So, signs are not counted here; so, where x nought is a 0 of f x, meaning, if you put f, if you put x equal to x nought in the function, it gives you 0, and you may have several zeros.

And in that case, we should put a label i to label those zeros and I should sum over all the i's. So, that is something you should do and arrive at the previous result. So, what next? Now this quantity rho phi nought k, I will define it to be n over 2 pi cube. That is a constant and I will define coming from single particle states, I will define it to n over 2 pi cube where n is a constant. So, what do you get?

You get rho coming from single particle states is n times delta of q square minus m p square. That is what you get here. So, here, 2 pi gets cancelled and you have only delta function times n. Just to be sure that I am not leaving, I will just write as rho of q square is; so, now, this is for the full thing, not just for single particle states, q square minus m p square plus contributions coming from multi-particle states.

Now let us look at the case where your field is an elementary field phi that appears in the Lagrangian, but the structure is the same. This, you have n here and we had earlier defined for the case of elementary field phi that this omega phi of 0; now I am not using a boldface phi but I am using this phi k. This is what we called as Z; Z over 2 pi cube. So, n gets replaced by Z when you are looking at the elementary field phi. So, what do we have then?

In that case, when I am looking at spectral density for the case of this elementary field phi that rho of q square is Z times delta of q square minus m p square plus contributions coming from multi-particle states where for that is what we are doing now. So, we got the rho of q square which is nice; let us analyse a little more. So, let us also ask what happens to the contributions from 2 particle states. So, rho of q square, let me try to plot this.

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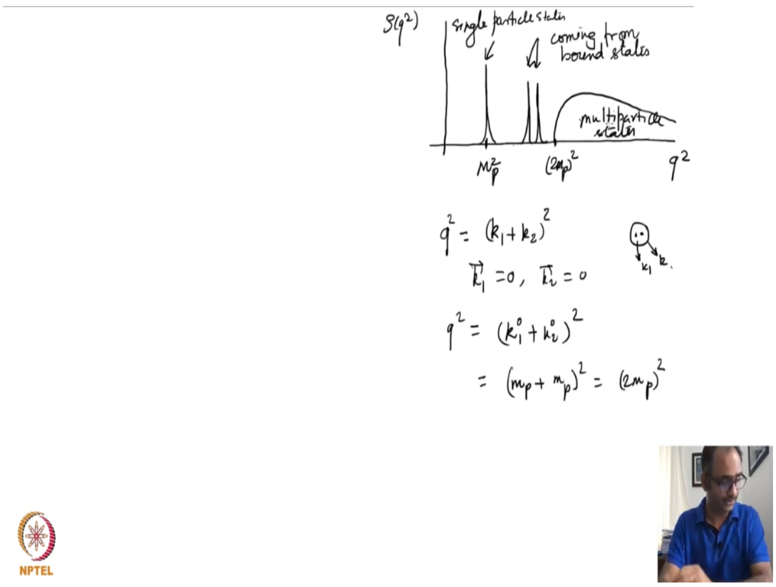

So, rho of q square at moment looks like this, that when you are at the physical mass, the square of the physical mass meaning q square is equal to m p square, then you have a delta function here, and I am just drawing it like this, but of course you have to take this the limit in which it becomes infinitely sharp. So, that is the delta function. So, there is no contribution to rho of q square if q square is less than m p square.

Now let us ask what happens when you go above m p square. Do you have contributions coming from all these values? And the answer is no. Why? Because, so, you can see this way. So, look here, hear we had a sum over all the states of which are eigenstates of p mu and we single out first single particle states, and this brought in d cube k. So, you have 3 integrals and you have a delta 4 which basically 4 delta functions, delta of q 0 - k 0, delta of q 1 - k 1, delta of q 2 - k 2 and delta of q 3 - k 3, so, 4 delta functions; and integral over 3 variables, d k 1, d k 2, d k 3.

And that leaves behind 1 delta function and that is what you are seeing there in the final result. That is what you see here. Now, when you look at the multi-particle states, let us say 2 particle state, the delta function, you still have 4 deltas but now you have integral over d cube k 1 and d cube k 2. Let us say the momenta, individual momenta are k 1, k 2. Then, when you integrate, you have 6-dimensional integral but the delta function is 4-dimensional, so, it gets used up, and you are still left with 2-dimensional integral.

So, there is no way you are going to get a contribution to rho of q square from multi-particle or from 2 particle states, which will be proportional to a delta function; that is not going to happen, because of this counting; 6 integrals d cube k 1, d cube k 2; 4 delta functions; so, you are still left with 2 and that you integrate over these functions or these objects. So, one thing is clear, it is not going to be proportional to delta function.

The another thing that you can observe is when you do integral over k 1 and k 2. We are labelling the momenta by k 1 and k 2; so, p n takes those values. So, you have 2 particles and we ask. So, this state will have some invariant mass, some q square or k 1 + k 2 whole square is what we are calling as q square and we ask what is the minimum value that q square can take.

And because you have a 2 particle state, the minimum value that q square which is the sum of the momenta squared will be, the minimum value will be taken when both these particles are produced at rest. So, imagine you give a momentum for momentum q and then, these two, both the particles are just produced at rest; they are not moving at all. So, they just have energies due to their masses, but there is no energy carried due to the momentum, and that is the configuration which will provide the lowest value of q square.

So, what is that configuration? So, if q let us write as k 1 + k 2; these are the labels; because I am looking at a 2 particle state, so, it carries 2 labels k 1 and k 2. And what is the minimum value? The minimum value is when you have the 3 momenta of both of them equal to 0. And in that case, q square is just k 1 0 + k 2 0 square, but k 1 0 is m p, k 2 0 is also m p squared. So, that is 2 m p square.

So, you see that there is no contribution to rho of q square coming from 2 particle states if the value of q square is lower than 2 m p square. So, here you have 2 m p square which is basically 4 m p square, and then you start getting contribution from here. There is no contribution from the intermediate values of q square, and as I said, it will not be some delta function; it will be some distribution.

So, contribution from multi-particle state starts from here. So, it will look something like this. Now it is still possible that in a general theory which you are doing, you have bound states. For example, if you are doing electrodynamics, you can make atoms, you can have an electron and a positron. You can form bound state. Now bound state is more like of a single particle state, because that bound state is going to move as a whole; it is not, these two are not; you can assign a momentum to that entire atom; you can assign an energy to that entire atom and you can see that bound state as a single particle.

Given that whatever I have said before this implies that we should also get delta functions at those bound states. So, if there is some bound state in the theory, for example, if you can make some atoms with positron and electron or whatever, then it will also lead to some delta functions. It could be more than one depending on how many bound states you have. So, these delta functions come from bound states. This is from our single particle states and this continuum is coming from multi-particle states. This is good. I will talk about looking at the Fourier transform of this object in the next video.