Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 6
Lecture - 14
Kallen-Lehmann Spectral Representation

So, let us look at two-point function in an interacting theory. I will be looking at two-point function in scalar field theory and this analysis will be non-perturbative, meaning I will not use any Feynman diagrams, I will not make use of any perturbation theory, and will show you what I have already told you earlier that if you look at the two-point function in the Fourier space, then it takes the following form that it has a pole at physical mass m p.

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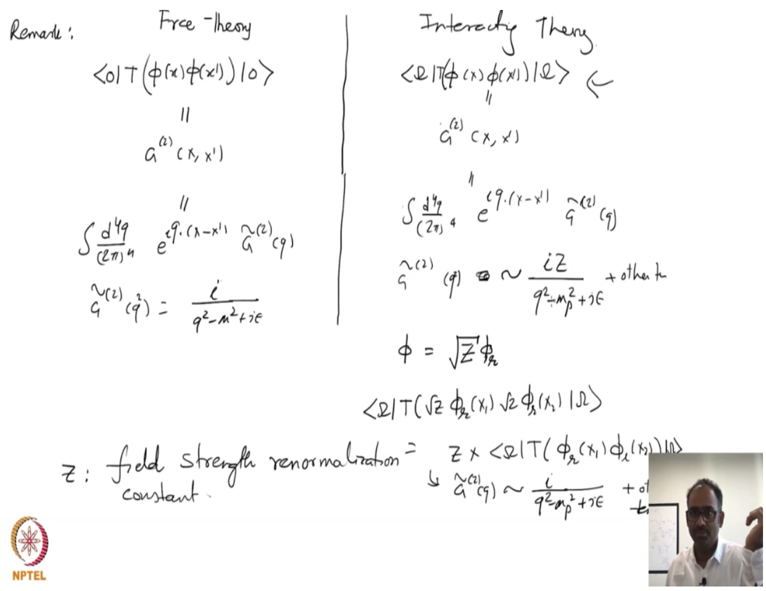

You see that when q square is equal to m p square, this blows up; it is a simple pole. And of course, you remember that in free theory also you had a simple pole at the physical mass and their physical mass was same as the mass parameter that appeared in the Lagrangian, but it was only i over q square minus m square plus i epsilon in free theory, but here you also have a Z sitting here and that Z is what is the residue of this pole; and then the other terms which are not similar when q square goes to m p square; and this is the result which I want to arrive at now and this is what we will start now.

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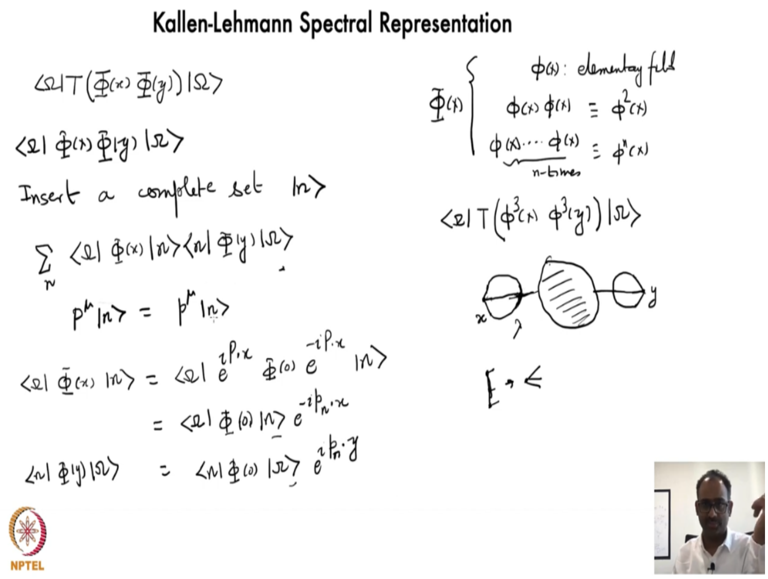

So, let us go here. So, the goal is to look at; that is vacuum; setup is not so good here; and then time ordered product of these fields phi x and phi y which are your fields that appear in the Lagrangian, these elementary fields, but I will do it in more generality; I will not take them as to be the elementary fields that appear in the Lagrangian but I will use, allow these fields to be also composite.

For example, let me denote by phi the elementary field. This is the one which appears in your Lagrangian, so, phi of x, but you could also construct operators like this. You could have phi x times phi x. So, at the same space time point x, you have this product of 2 fields, which is also written as phi square of x; and you could have phi q f x and so forth which would be just product of 3 fields at the same space time point.

So, let us see, these are n times and this is what is phi n of x. So, that is what I am allowing here. So, I am saying phi of x could be any of these. It could be an elementary field phi or it could be a composite field where you have several elementary fields at the same space time point. And if you were to look at this object in perturbation theory and draw a Feynman diagram, suppose I want to look at this object omega time ordered product of phi cube of x, phi cube of y; we have done this kind, this analysis in the previous course.

So, you remember, if it was just phi of x and phi of y or phi of x 1, phi of x 2, phi of x 3 and so forth, then at each space time point for you draw this dot which labels the space time point x and then you have 1 tick coming out of it and that tick corresponds to the fact that you have a one field acting on that phi. That is what we have learnt in the previous course when we were analysing the Green's functions and drawing the Feynman diagrams, but here, because I have 3 fields at the same point, instead of 1 point coming out, I should have 3 points coming out.

Or you can think of it as x 1, x 2 and x 3, 3 points each with 1 tick and then in the limit that x 1, x 2 and x 3, they go to x, they all merge into 1. So, this gives you basically this. This is what I am drawing here. So, for point x, there will be 3 such ticks coming out; for point y, there will be corresponding 3 ticks coming out; and then you connect them in all possible ways putting vertices; depending on it, what order in perturbation theory you are looking at, and draw Feynman diagrams.

For example, one of the Feynman diagrams would be, because I am doing phi 4 theory, would be this. That is a four-point vertex here; 1, 2, 3 and 4; 4 lines coming out. So, that is a factor of Lambda here and you could also have the same thing here, another factor of Lambda, and then something else happens, whatever diagram you wish to draw, according to the Feynman rules.

So, it would have this kind of structure, and similarly for other things, but I do not wish to do perturbation theory as I said and I want to look at this object in a non-perturbative manner. So, I will proceed with that and keeping in mind that phi could be a composite field as well. So, what I do is, I first insert a complete set and I will not look at the time ordered product to begin with; I will just look at this.

So, I have dropped the time ordering that I will bring in later and I will analyse this object first. So, I insert a complete set which I will denote by a ket n. So, it becomes summation over n omega phi x then this identity phi y omega. I have inserted a complete set. Now, I can take these states ket n to be eigenstates of the momentum operator P mu, and I can do so because I have space time symmetry in this problem, so, there is a conserved momentum that exists.

And because it is a Hermitian operator, its eigenstates would form a complete set. So, what is this ket n? Ket n is an eigenstate of operator P mu. So, that is what I choose for eigenstates, choose for these states ket n and insert in here. Now, I will appeal to this space time symmetry, this invariance under space time translations and write phi of x in terms of phi of 0.

So, let us look at this object omega phi x ket n; this is omega e to the i P dot x where P is the momentum operator phi at 0 e to the -i P dot x, then we have this ket n. Then we have assumed that our vacuum does not change or it has an eigenvalue 0; that also you can see. If it has an eigenvalue 0, then P acting on vacuum will give you 0 and this is just omega because this gives identity. Then you have phi of 0.

Ket n is an eigenstate of operator P, so, it gives you e to the -i P n dot x. This subscript n tells that this P n is the momentum of ket n and because that is a complex number, I have just pulled it out. It does not have to be sitting in here; it is not an operator anymore. So, that is what I get. And the same thing you can do for this one, this factor, n phi y omega is n phi 0 omega e to the i P n dot y using the same argument. So, what do I have then? For this object, this becomes; it is not convenient, I will go to the next page.

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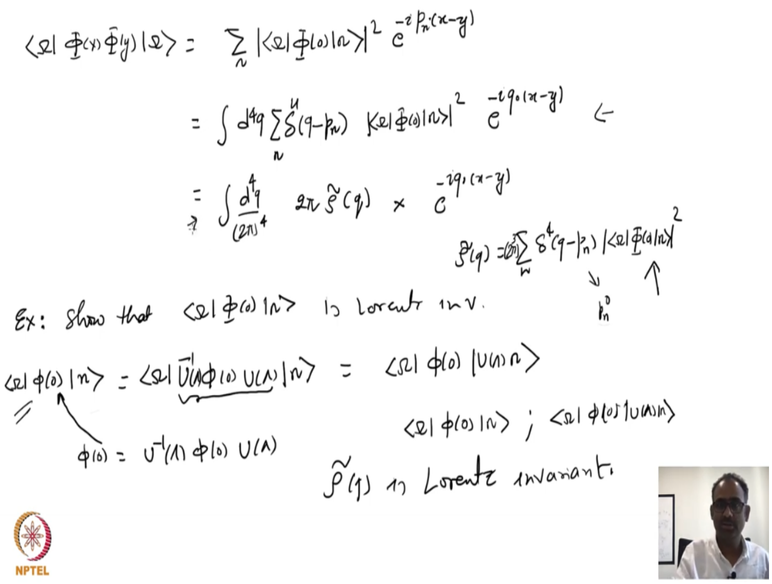

So, I will just insert that, insert all the results which I found. So, omega phi of x phi of y omega again is summation over n. And then what do you have? You have omega phi of 0 n and then you have again omega phi of; I can just put a mod square; these are complex conjugates of each other, this one here and that one here. These vectors are complex conjugates, so, I will just put the mod square times e to the -i P n dot x - y. That is correct.

So, you see that because we have translational invariance under, we have the symmetry, translational symmetry, this object on the left-hand side does not depend independently on x and y, it only depends on this difference x - y. It is not a function of x separately and y separately, but the function form depends only on the difference between x and y that is due to our translational symmetry.

Now, what I will do is, I will write this down in the following form. I want to write it as an integral. So, I just put it like this, integral d 4 q where you have integral over q nought, q 1, q 2 and q 3 running from minus infinity to plus infinity. So, d 4 q and then times e to the -i q dot x - y. That is the form I want this object to have and you can already see why, because this will be like a Fourier transform.

Whatever I write here in the blank space would be the Fourier transform of the object on the left-hand side. So, I am trying to give it that form. Now to arrange that, I should have; this piece is there anyway, so, I should write it here, omega phi of 0 n modular square. Now what should happen when I integrate over d 4 q? When I integrate over d 4 q, this q should get replaced by P n and that integral should also disappear, this measure.

And how do I arrange that? I can arrange that by putting a delta function d 4 q - P n. Now let us look at this. When you do the integral over this expression, the delta function will force q to become P n, and then you recover what you have on the above line provided I put a summation over n. That is good. And this I will write as d 4 q over 2 pi to the 4, then we have e the -i q dot x - y.

And then whatever is here, the remaining factors, I call them as 2 pi times rho tilde of q. That is the definition. So, what is the rho tilde of q? Let me recall it here; rho tilde of q is summation over n delta 4 q - P n omega phi 0 ket n and this modular square. I have missed the factor of 2 pi cube; because I have a 2 pi 4 in the denominator, that 2 pi cancels 2 pi 4 and makes it 2 pi cube and that 2 pi cube gets cancelled with this 2 pi cube so that we get this line.

So, that is definition of rho tilde q and as you see rho tilde q or 2 pi times rho tilde q is basically the Fourier transform of this object on the left-hand side. So, now I am going to argue that rho tilde q is Lorentz invariant. And the argument is same as we have used earlier in the case of this factor, when we were analysing this thing for a single particle, where ket n were single particle states. It is the same kind of argument; I will just say in words.

So, you take omega phi 0 ket n and insert or maybe I will do it towards the end instead of breaking the flow. So, let me give it as an exercise and these 2 lines I will write at the end to argue that this is indeed Lorentz invariant. And you see delta 4 is anyway Lorentz invariant; that we already know. So, this part is Lorentz invariant. I just should argue that this is also and I will leave it as an exercise for the moment. Let me do it anyway.

So, what will you do? So, you do a Lorentz transformation and that Lorentz transformation, you will have some operator u of Lambda or u inverse of Lambda, u of Lambda. Lambda is the, Lambda matrix contains the parameters that parameterise the transformation. So, you know you have 3 rotations and 3 boosts. So, those parameters which parameterise rotations and boost are contained in Lambda, and here is this operator which does the transformation and you have ket n, but then our vacuum omega is invariant under Lorentz transformations.

So, that remains omega and then you have phi of 0. Then u of Lambda acting on ket n is, let us denote it by this state. This is fine; but also remember that one thing I should use is that this object or rather here, phi 0 is a scalar field, so, it does not change under Lorentz transformations. That is something I have to use, otherwise I do not get anywhere. So, this is true, meaning, if you take the field phi and do a Lorentz transformation, you get the same thing; you do not get anything different.

So, this is what I am using. So, this is same as; so, once I substitute this thing in here, I arrive at this first equality; and then from first equality, I have arrived at the second one; so, which says that omega phi 0 ket n is same as omega phi 0 u Lambda ket n where u Lambda ket n is a state that you get by doing a Lorentz transformation of state n. So, you see that this, whatever this complex number is, that is same as this complex number.

So, this number is not changing under Lorentz transformations. All I am saying is omega phi 0 ket n; if you look at this object and if you look at this object, they have same values. So, this is also Lorentz invariant object. So, rho tilde q is Lorentz invariant. So, I write rho tilde q is Lorentz invariant. That is good. Now, you see here, the P n here. What is P n? P n is the 4 momentum of ket n which means the zeroth component of P n that is P 0 is the energy of the state n.

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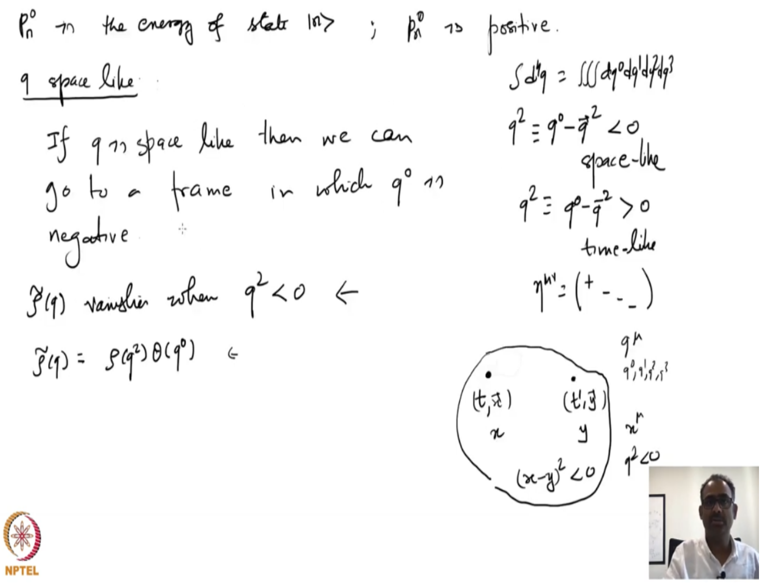

So, P n 0 is the energy of state ket n. Now if that is the energy, energies are positive, so, P n 0 is always positive because I have taken the ground state to have the lowest energy. So, any other state would have a positive energy. So, P n 0 is always positive. Now let us look at this again. You have d 4 q, meaning you are integrating over all values of q. So, integral is a sum, so, you are summing over all possible q nought, q 1, q 2 and q 3.

Sum of these will correspond to space like q; sum of them will correspond to time like q and we want to analyse this rho tilde of q and say what happens if q is space like and if q is time like. So, let us look at space like first. So, q space like; let me remind you again why I am doing this, because I have integral d 4 q which is d q 0, d q 1, d q 2, d q 3. I am summing over all this.

So, when you are summing over all possible configurations, there will be configurations for which q 0 is less than 0; that is what is defined as q square; meaning, for sum of these, it will be space like and then there will be others for which it will be time like and also light like. That is time like given the metric that we have used in our; our metric is eta when u is + - - -. So, let us see what we have to say if q is space like.

Now if q is space like, then I can do a Lorentz transformation and change the sign of q nought; you understand. So, if q is space like, then we can go to a frame or we can do a

Lorentz transformation frame in which q nought is negative. So, even if you begin with q nought being positive, you can do a Lorentz transformation and go to and make q nought negative.

If this is something new to you, you can also see it in this way. You have seen that if you have 2 space time points x and y which are separated by a space like interval, then which one; let us say 2 events occur at this space time point x and space time point y, and if the interval between x and y is space like, then which event occurred first and which event occurred later depends on the frame from which you are looking at.

And what is the zeroth component of x and y? These are the corresponding times; the time at which this event happened at position x and the time at which this event happened at position y. I should have been drawing here. So, let us say you have t x and t prime y. This is what I am calling is x and what I am calling as y. Now, if these 2 are separated by a space like interval, then you know that you cannot say which event occurred first; it is not absolutely determined; it depends on the frame of reference.

So, you can go to a frame of reference in which the event, this event occurs earlier, meaning t is smaller than t prime and you can do a Lorentz transformation and go to a frame in which t is later, meaning the event at y happened earlier. And because q mu is a 4 vector, then its components q 0, q 1, q 2 and q 3, they transform exactly in the same manner in which the components of x mu transform.

So, whatever I am saying here in for this, here I am utilising the fact that this interval is space like and then I can go to a frame in which, which one happens first. I can always go to a frame in which x happens first or t is smaller. The corresponding statement here is because they transform in the same way x mu and q mu. If q is space like, meaning q square is negative, then I can go to a frame in which q nought is negative.

So, I can do that and because I can do that, this delta function which you see here ensures that that contribution vanishes. Why? Because P n or the energy component P n 0 is always positive because ket n is a physical state with a physical energy. So, that is always positive and if you can make the q 0 negative, then that delta function vanishes. So, you see that rho tilde of q vanishes when q is space like.

So, what I have argued is rho tilde q; first I argued that this is a function only of q square because it is Lorentz invariant; so, let me write like this. So, rho tilde of q vanishes when q square is less than 0, that it is space like, and so I can write using this, rho tilde of q is equal to rho of q square. Here I am emphasising that rho can only be a function of q square; rho tilde q can be only a function of q square because it is Lorentz invariant.

But then you get a contribution only when q nought is positive, not otherwise; and you might worry that this theta of q nought is not Lorentz invariant, but it is because you also have this statement together that rho tilde of q vanishes when q square is negative; so, which means that rho tilde of q is non-vanishing only when rho tilde q, when q square is positive, q square is time like. And when q square is time like, then the sign of q nought can be changed.

Then, which happens earlier, which happens later is fixed. And similarly, using the same arguments like here, it is fixed that the sign of q nought cannot be changed. So, this right-hand side is indeed Lorentz invariant. So, I can write it in this form and I will put it back here, in this expression.

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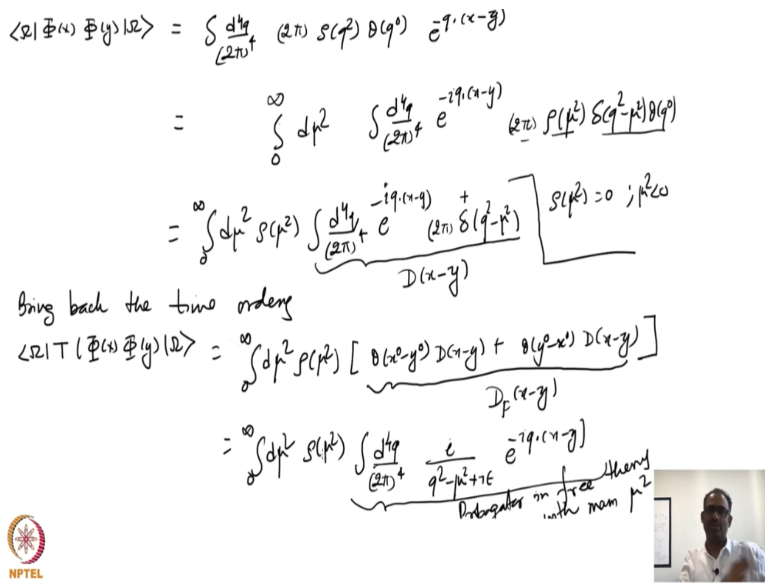

So, let me write it down; omega phi x phi y omega, this is d 4 q over 2 pi to the 4. Then I have 2 pi times rho of q square theta q nought, and rho of q square vanishes when q square is space like. This is rho tilde of q that is what vanishes. Now I want to write it as the following. Let me write it in the following manner. So, you have the same thing d 4 q over 2 pi to the 4 and then you have e to the -i q dot x - y that is this factor times 2 pi rho of mu square.

I had a rho of q square but I am writing rho of mu square; delta of q square minus mu square theta of q nought. And because I have written rho of mu square and I should turn it to rho of q square, I have integral over d mu square 0 to infinity, and I am starting from 0 because rho of mu square is equal to 0 if mu square is negative. That is what I argued a little while ago that if you have space like q square, then this vanishes.

That is what I am imposing to make this limit 0. It has a support only over the positive values of mu square. That is fine. I can write it slightly better, 2 pi to 4 e to the -i q dot x - y 2 pi. I have taken this, this. What is this? This is what we have been writing earlier as delta plus q square minus mu square. This is a familiar object. We have seen this earlier in the previous course in the case of free field theory. We saw that this is what we called as D of x - y.

That is the propagator in free theory, not really the propagator, it is the D; propagator, you will get when you have the time ordered product here, but that is what we had. Look, so, let us now bring in the time ordering. So, bring back the time ordering and then I have omega time ordered product of phi x and phi y; and remember, these could be composite operators, not necessarily elementary fields.

Then this is what integral 0 to infinity d mu square rho of mu square, then you have d x y. So, you have a let us say theta x nought minus y nought, meaning if x nought is greater than y nought, only then this is 1, otherwise it is 0, it does not contribute; plus theta of y nought minus x nought. So, this is this expression, and what is this? I have just inserted the time ordering; this is the same object.

So, now this you remember, this is the Feynman propagator D F x - y for the case of free theory. That is the Feynman propagator. So, what do we have? This is integral 0 to infinity d mu square rho of mu square and let us write down the expression of Feynman propagator. It is integral d 4 q; I am writing with q; over 2 pi to the 4 i over q square minus m square plus i epsilon; not m square.

So, you see here, it is delta plus of q square minus mu square. So, if you go to the place where we had found this propagator, it was delta plus of q square minus m square, where m was the mass parameter in the Lagrangian in the free theory. So, instead of m square, I have mu square. So, I should write q square minus mu square plus i epsilon, and what? e to the -i q dot x - y. This is the free propagator in free theory with mass mu square.

So, what you see is that the two-point function or this object is equivalent to sum over propagators in free theory where each propagator is multiplied with a weight factor rho of mu square and then you sum over all the mu square. So, you are summing over all of it and where the sum is over all the masses. You see, the mass is in quotes; there is no mass like mu square in this theory but the result is as if this object on the left-hand side is made up of free propagators and each propagators carries some mass mu and you have to sum over all those masses.

But you should multiply each of that propagator with an appropriate weight rho of mu square; and of course, rho of mu square will be given by the theory that you are looking at; only what kind of interactions and what kind of fields you have will know about rho of mu square, because this part is same for whatever theory you are looking at. This is going to be the same, identical, whether you are looking at phi 4 theory or whatever other theory, this is going to be the same result.

And what changes is rho mu square when you go from one theory to another theory. This rho of mu square is called the spectral density. So, let me write that down and then we will see how to further get information about rho of mu square.

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$\rho(\mu^2) \equiv$ spectral density.

Find the contributions to $\rho(\mu^2)$ from the single particle states.

So, rho of mu square is called spectral density and just for later use, this is; I have written this earlier, so, no point. Now, next thing to do is to analyse the contributions coming from single particle states to the spectral density. So, our goal is now to find the contributions to rho of mu square, the spectral density coming from the single particle states. So, we will analyse this next.