Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 5
Lecture - 13
Pole and Residue of the Propagator

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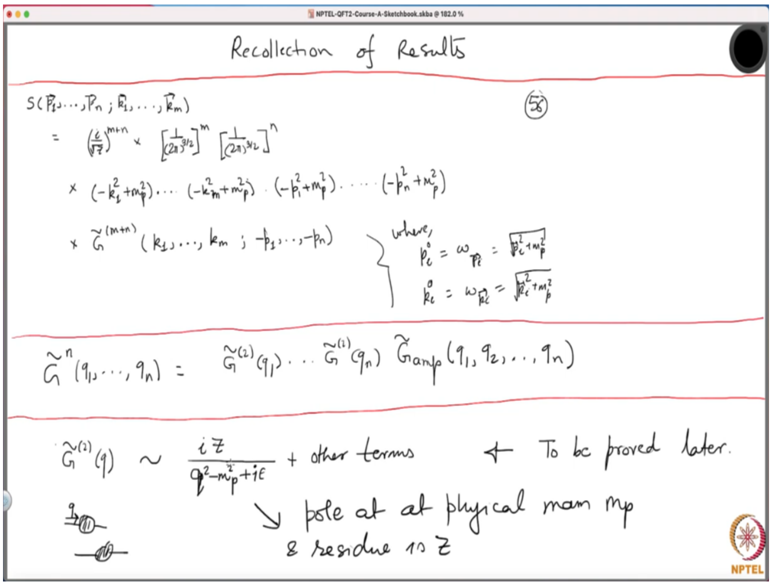

So, here I have summarised all the results that we have obtained so far. So, here is the formula for the S matrix and this contains the Fourier transform of the Green's function G tilde m + n. And then, we also analysed what G tilde n is and we saw that in general we can write it as a product of two-point functions G tilde 2 q 1 to G tilde 2 q n, so, for each leg times the amputated Green's function where this G tilde of q, I had defined here.

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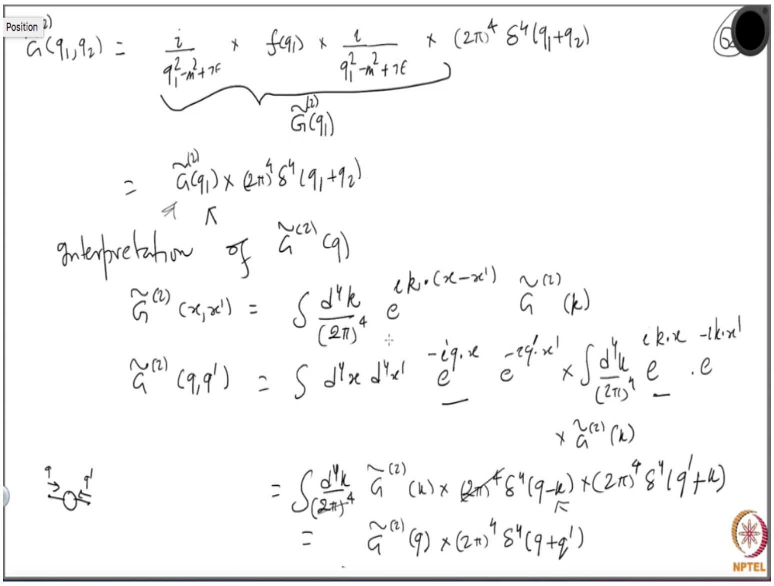

So, this is Green's function in the coordinate space and then you take a Fourier transform. And G tilde q is related to G tilde 2 q q prime by this relation, and that is what we had seen last time. And then, I also made a claim that if you look at G tilde q or G tilde 2 q, this object which we are going to do later that we will show that it has a structure which is almost the same as that of a free propagator.

So, if you had a free theory in which the particles had mass m p, then you would have had i over p square; this should be q, because I am using a q here; i over q square minus m p square plus i epsilon but now the difference is that you also have a factor of Z and then other terms which are not singular in q square becoming equal to m p square. So, that is the difference, but this is almost like what you would have in free theory as far as this term is concerned. So, before we take up this function in a non-perturbative manner, let us just take this result and analyse this object in a perturbative manner. So, that is the plan. So, let us see.

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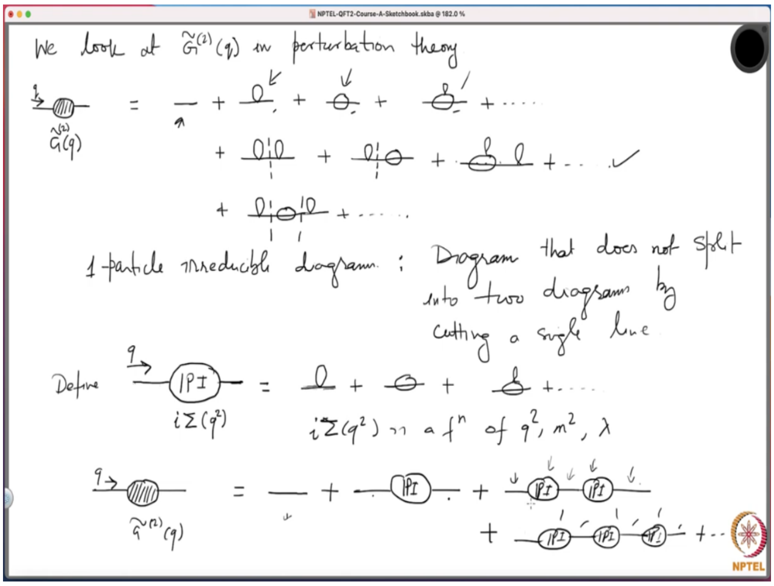

So, we look at G tilde 2 q or p, whatever you like, in perturbation theory. So, here, this is two-point function when I put a blob like this, a shaded blob, then this is a two-point function or rather a G tilde 2 or G tilde with 1 argument q. This is a two-point function, but this has 2 arguments, this has only 1 argument, and right now I am just denoting that blob; that blob denotes G tilde 2 with 1 argument and this is what it is.

And we saw that for 5, 4 theory, this is sum of all these terms, all these diagrams. This is order lambda diagram, because you have 1 vertex. This is order lambda square diagram. You have such diagrams and many more and still more. You have this also at order lambda square; this at order lambda cube and so forth; and of course, you also have terms of this form. There are infinite number of diagrams and I am just showing some of these.

Diagrams of this kind, these are called one-particle irreducible diagrams. So, let us look at this one. This one is not a one-particle irreducible diagram. Why? Because, if you cut this line here, if you cut it with a scissor, this falls apart into 2 diagrams; similarly, here. So, these are reducible. You can cut and they will fall into 2; but here, there is no single line which you can cut and you get 2 diagrams.

So, the thing is that you should be able to, upon cutting one line, you should be able to get 2 or more diagrams or 2 diagrams, then it is reducible, otherwise it is irreducible. So, these ones, these first 3 are irreducible and these ones are reducible, because here also you can cut this line, you get 2; or you can cut here, you again get 2 diagrams. So, one-particle irreducible diagram is a diagram that does not split into 2 diagrams by cutting a single line.

So, now let us look at these irreducible diagrams. I will define one-particle irreducible diagrams. So, by this blog with 1PI written in it, I mean the collection of all one-particle irreducible diagrams. So, you have this plus this plus and so forth. There are infinite of them. So, this blob will denote this and but remember, when I am drawing 1PI these lines, this one and this one, there is no propagator factor for this.

So, for example, when you draw this one, you do not write a propagator for this and a propagator for that, you just write what you get from this loop. Similarly, here, you do not write anything for these 2 ends, but only what you get from this loop, these 2 loops and so forth. So, these 2 end, 2 lines, these 2 lines will never include in 1PI. So, that is the definition, and I will denote this by i times Sigma; so, here is q entering into it; i times Sigma of q square.

So, you see, whatever happens here, whatever function you have, this integral, you will have a 1 loop integral here; here you will have a 2 loop integral. All those, that function which you get here will be a function only of q square, because that q enters into this loop and then other things are just dummy variables; other loop integrals are going to be completely integrated over and what will be left behind is a function of q square and that is what I am emphasising by writing i times Sigma of q square.

Why I have chosen a factor of i here? You will see the relevance of it, but that is the definition. I am just going to denote it by like this. So, that is fine, and remember that i times Sigma of q square which is this one-particle irreducible diagrams, the sum of them, is a function of q square, will also be a function of m square and lambda. So, Sigma q square is a function of q square, m square and lambda, because these are the parameters, m square and lambda are the parameters that enter into your calculations.

So, of course, it is going to depend on all of this, but I am going to suppress m square and lambda and just write q square, only q square. Now it is clear that if you look at the two-point function, this object; should have put a 2 here, a superscript 2; and this is what? This is first just a propagator. So, I write a propagator plus this plus this plus this and all other one-particle irreducible diagrams. So, let us say first row has all those irreducible diagrams.

So, it will have; and so, when I write this, I have to include i Sigma q square for this, but now remember I am writing a two-point function. So, these 2 propagators are part of it. So, here I write propagator for this and here I write propagator here, propagator there, and then 1PI is only -i Sigma q square. So, in this expression, in this line, I have to include propagators but here it is defined one-particle irreducible diagram, they are defined without the propagators.

So, what else? Now let us look at this one. You will have this variety. Let me include one more diagram and others. So, you see, this one will have a structure of the following form. Here, if you cut this line, it falls into 2, and this is how these diagrams are. Then of course you will have, and so forth. So, let us write it down. You have momentum q flowing into it. So, this will give you a factor of a propagator. That gives you a propagator.

This will give you a propagator times a Sigma, i Sigma times a propagator. So, a factor of Sigma and a propagator here; similarly, a factor of Sigma, a propagator here, a factor of Sigma, a propagator here; so, 2 Sigmas, 2 propagators and then one additional propagator from this; and here, 3 Sigmas, 3 propagators and an additional propagator from here. So, the structure is the following.

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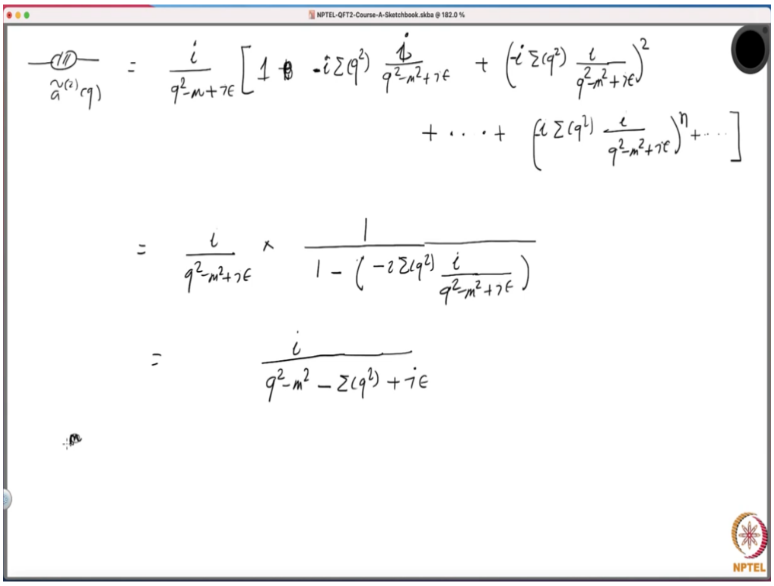

Each of them gives; not m p. Did I have m p before? No. This is not the physical mass; this is the parameter in your theory. This propagator I am writing. So, here, each of them has one propagator which is on the left. That is what I am taking out, I am factoring out, and then you have 1 + i Sigma q square times a propagator which is on the right, plus; so, you have 2 factors of these.

So, I have already taken this one common, factored it out, and you have 2 factors of Sigma. They both have the same q square entering into them, and then these 2 propagators which are again identical. So, I write this. Remember, this is m square, not physical mass. So, that is the series you have and you know that this we can sum up. So, this is a form of 1 + x + x square and so forth.

Remember, Sigma is a function of lambda and actually it starts at order lambda because this is the first diagram and the diagram appears at order lambda. So, Sigma has a perturbation expansion, perturbative expansion, and the expansion starts at order lambda and we are treating lambda to be small. So, you have a series here, which you know how to sum. And this is what? This is 1 over 1 minus; here I have a small issue, I should put a minus here.

So, that is how I define a -i Sigma. So, this is -i sigma. That is -i, that is -i. So, that gives you minus of -i Sigma minus q square times and I have used a factor of i also here. That is the sum of the series. Let us see minus times minus that gives plus and i items i gives -1; i square is -1. So, I get an overall minus sign here and you have this factor. So, these 2 cancel and you get i over q square minus m square plus i epsilon.

I am multiplying this vector here and then you get the minus. This plus epsilon I will write at the end and you get minus Sigma of q square plus i epsilon. Now, you see why I had kept; this is fine; do not need to do anything more. So, you see a Sigma q square gives you a positive contribution, then the way we have arranged everything by including a -i here, we get a positive contribution to the mass.

The shift in the mass or the physical mass is related to this bare mass by a positive shift if Sigma of q square is positive, and that is the reason behind choosing those factors of minus i in front of Sigma q square, but you do not have to. You can keep it to be +i also. So, that is the structure of G tilde 2 of q and now we have; so, this is fine. Now, we told that there is a pole in this propagator, this two-point function.

This is also called sometimes as propagator. So, one calls this two-point function also as a propagator. So, if you look at the propagator, it has got a pole at the physical mass; or equivalently, if you look at the denominator, it has a 0 at physical mass m p. So, let us use that information and say that when q square is m p square, the denominator vanishes meaning m p; I should write it on the next page; meaning m p square minus m square.

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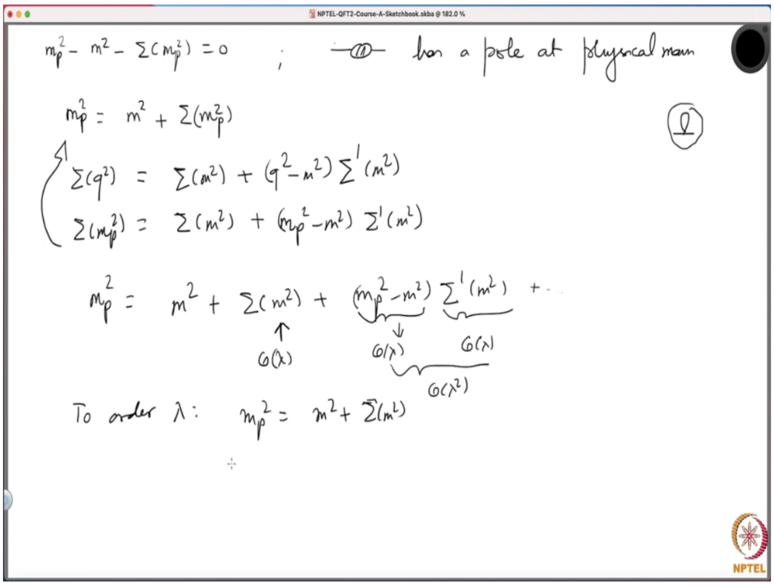

Why? Because, this has a pole at physical mass. So, I have just put q square equal to m p square in this, in the previous expression here in the denominator, and then I get this condition. And with the help of this, I can determine physical mass and also the factor of Z to whatever accuracy I require. So, let us do that exercise. So, I will write m p square is equal to m square plus Sigma as a function of m p square.

It says that the physical mass to the lowest order in lambda is same as the bare mass; they are equal; and the correction starts at order lambda because Sigma starts at order lambda. So, that is the correction. Now, what I will do is, I will iterate it and in here, for m p square, I will put m square plus Sigma m p square. So, I can write Sigma m p square as Sigma of; so, here I am writing m square plus, or I can do something else.

That of course I can do, but I want to do something else. Let us take Sigma q square and expand it around, do a Taylor expansion around m square. So, I will have Sigma of m square plus q square minus m square which gives Sigma of m p square as Sigma of m square plus m p square minus m square Sigma prime m square and of course we have other terms as well, higher order terms.

So, when I take this result and put in here, I get physical mass square is equal to m square plus Sigma of m square plus m p square minus m square. Now, you see this term is of order lambda, because Sigma starts at order lambda because this is the first diagram it has. Now, what is m p square minus m square? If you take m square that side, m p square minus m square starts at order lambda.

So, m p square minus m square is an order lambda term. What is Sigma prime m square? That is a derivative of this and this starts at order lambda. So, that derivative, this Sigma prime is also order lambda term. So, this is order lambda square. So, this way you can proceed and find out what is the physical mass in terms of, as a function of m and lambda, and this is a perturbation series in lambda.

So, this is how you can figure out what is the physical mass and to order lambda, m p square is m square plus Sigma m square. So, all you have to do is look at the one particle irreducible function, irreducible diagram, calculate its lowest order and that gives you the shift; and that shift together with the m square gives you the physical mass square. So, the next question will be how to obtain Z as a function in perturbation theory.

So, now I know how to get the physical mass in perturbation theory at least to order lambda; I can keep iterating this and get higher order terms as well but let us look at the Z factor. So, here you see that, the claim is that if you look at two-point function, then it has a pole at the physical mass and the residue at this pole is Z, because that is the residue. It is i times Z but I will not say i times Z; I will just say Z. So, the residue is Z or atoms, Z. So, let us find out Z from the perturbation theory. That is what we want to do.

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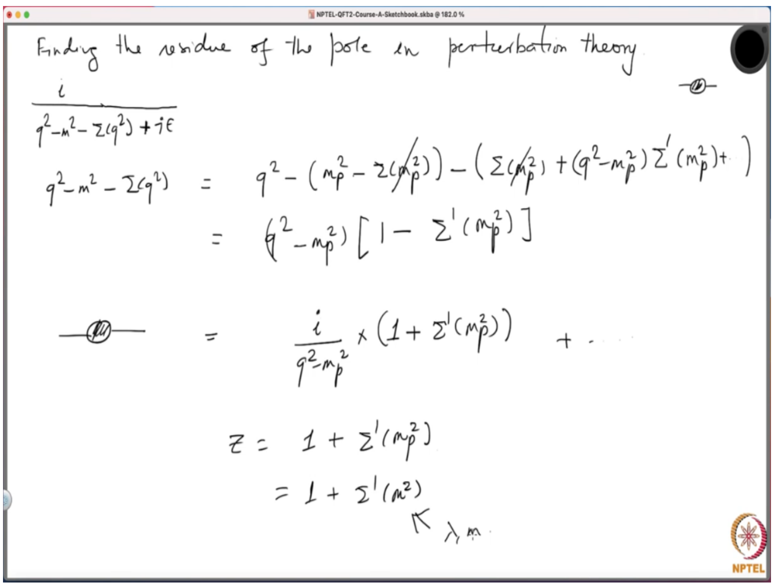

So, finding the residue of the pole in perturbation theory. So, what do we have? We have, this is the propagator or the two-point function. Remember, this is also called as propagator. In the interacting theory, the full, this is also referred to as propagator. So, what is the residue? So, let us look at the denominator q square minus m square minus Sigma q square because we are looking at the 0 of this. I will not worry about i epsilon; that is not relevant now.

So, what is this? This I can write as q square minus; what is m square? m square, we have seen that it is m p square minus Sigma of m p square, right, because, to the lowest order, m square is same as m p square. So, m square is m p square minus Sigma m p square. That is what I want to write. That is what here m square is, then minus, and I do an expansion around m p square physical mass.

I should have done something else, but nevertheless, let us proceed with this, because it is perturbation theory, it will not matter at least to this order. So, this cancels and we have q square minus m p square that is what I take common, and I get 1 minus Sigma prime m p square. This is to the lowest order. Is that fine? 1 minus Sigma prime valuated at m p square. That is fine. So, what have I obtained?

I have obtained that to the lower order lambda, this is same as i over q square minus m physical square times 1 over 1 minus Sigma prime of m p square which is 1 plus Sigma prime of m p square because I put in the numerator and we are doing a perturbation in lambda, of course, and there are other terms. So, you see that you can obtain Z if you calculate Sigma and take a derivative and evaluate it at q square equal to m p square which is same as, to the lowest order, it is 1 plus Sigma prime m square.

So, this is a function of lambda n. So, we can figure out what the residue is from perturbation theory. I want to make a remark here, and the remark is the following.

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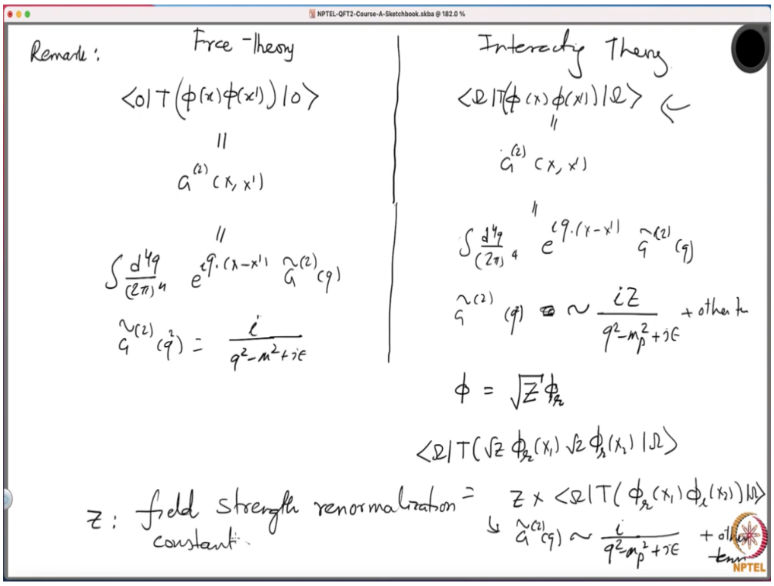

So, let us look at this. This is in free theory. And what is this? This is G 2 x x prime. And interacting theory, you have this object which we denote again by the same symbol. And here, if you write this in terms of G tilde; I should write q now; q dot x minus x prime q. Then you know what that G tilde 2 of q is in this free theory. G tilde 2 of q is i over q square minus m square plus i epsilon.

And remember, in free theory, m square is the physical mass; but if you do the same thing here, here the leading behaviour is near the, when q square is near mass shell, then the leading behaviour is this. Let me denote that by this factor, of course, the other terms. Now what you can do is, if you redefine the fields here; see, the normalisation of fields is not fixed; you can choose to normalise it differently.

So, if I take instead of phi I take phi r as the fields and multiply a factor of Z here, or rather square root of Z, then what happens? Then, if you do the same thing on this correlation function will be, which is just Z times the original correlation, the correlation function in terms of this phi r fields. And now if you calculate G tilde 2 with this normalisation of fields, then your G tilde 2 will not involve Z, because they have got a Z here and that Z will cancel.

So, you see that you can get rid of Z by choosing a different normalisation of the fields, and that is why this factor of Z is also called field strength renormalisation. So, here, let me write first this answer. Then for this case, if you calculate G tilde 2 of q, then it will, its leading behaviour will be i over q square minus m physical square plus i epsilon without a factor of Z plus other terms.

And then, that Z disappears because I have changed the normalisation of the field and that is why Z is called field strength renormalisation, and sometimes you also add constant, or renormalisation constant of field strength renormalisation constant. We will stop here and see you in the next video.