Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 5
Lecture - 12
S Matrix Continued

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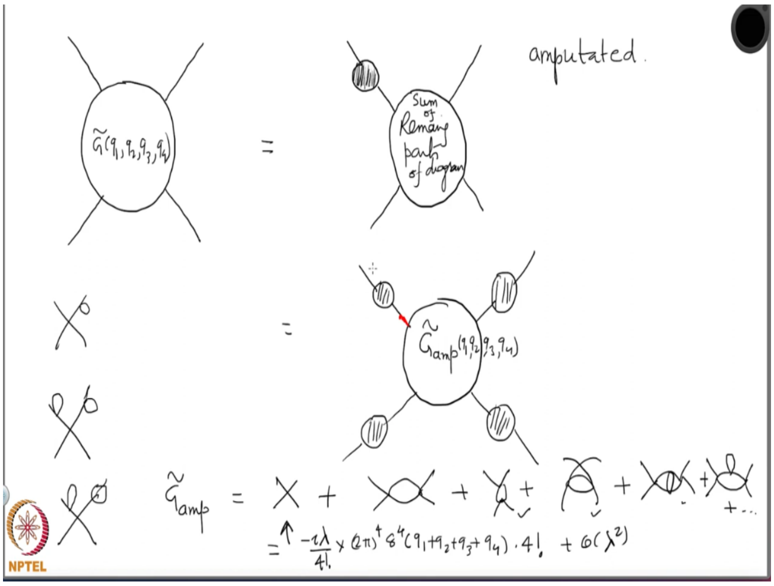

So, let us continue our discussion and the goal is to identify what exactly this blob is. We already have an idea what this is.

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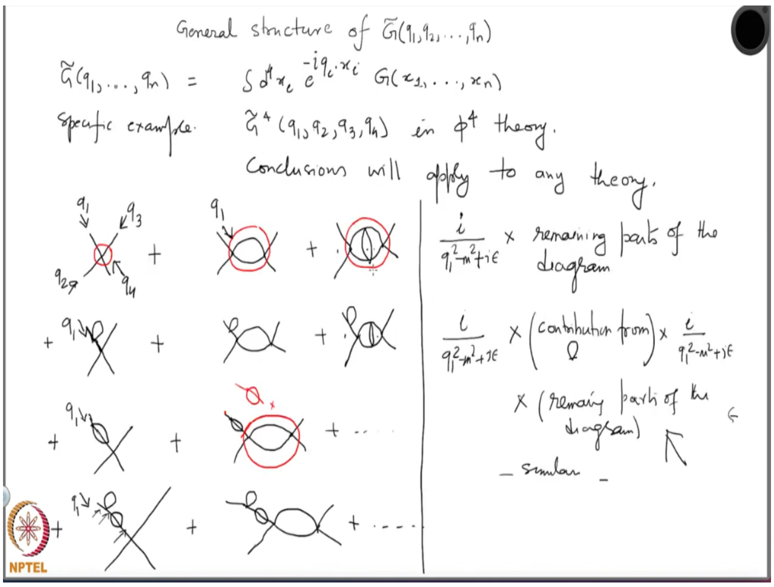

This looks like all these factors but we want to give a precise expression for this.

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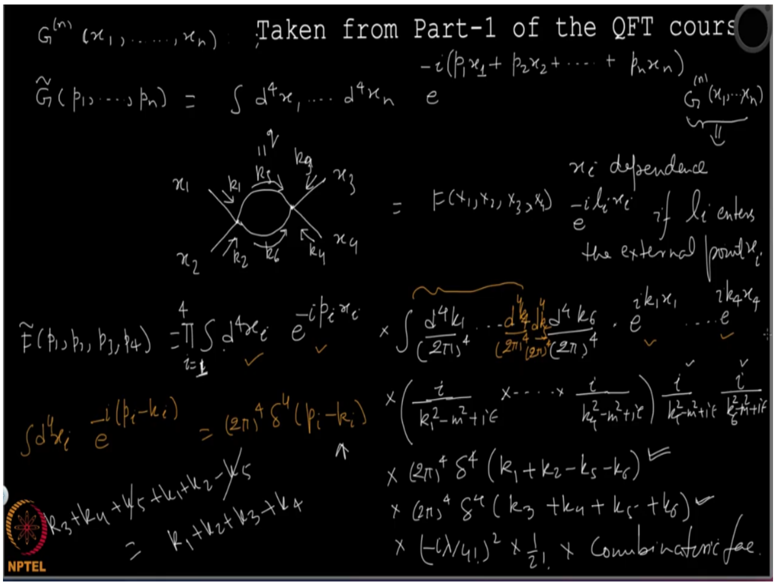

So, what I have done is, I have pulled out 2 sheets from my previous course, the first course on quantum field theory. And here let us see the way we had written down the Feynman rules for n-point Green's function. So, here you have G n x 1 to x n, and here this is G tilde. So, that is the Fourier transform where doing a Fourier transform involves multiplying with these exponential factors, e to the -i p i x i. So, there is a Fourier transform.

So, I am calling this particular diagram as F and the Fourier transforming F tilde, this is a four-point function. And you see, you have these d 4 x i to the -i p i dot x i, because we are doing a Fourier transform times the Feynman diagram in the coordinate space using momentum space Feynman rules. So, recall what we studied at that time that you label each propagator by some momentum k i.

In this case, you have 6, so, you have to integrate over all the 6; and at each external point, you include a factor e to the -i times the momentum that enters into that external vertex times x i. So, here, minus k 1 is entering into x 1. So, that is why you have e to the +i k 1 x 1 because the sign gets reversed. And then you have propagators; then delta function, momentum conserving delta function at each vertex, and then all these factors.

And let us see what happens then. So, if you take this exponential coming from Fourier transform and that exponential coming from the Feynman rule for g in x coordinates, you get a delta function 2 pi to the 4 delta 4 p i minus k i.

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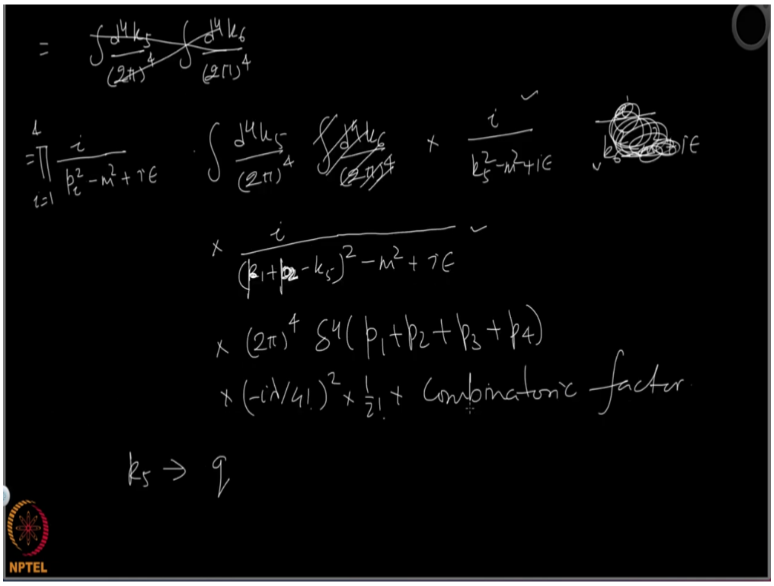

And then, further manipulating this and doing all the integrals; let us say we do the integral over k 6. We still keep them integral k 5, this one. You have k 5 and k 6 running in the loop, and I should have only one loop momentum left, because this is a one loop diagram. So, I do the k 5 integral that is trivial. And I am left with a; apart from these propagators and this loop momentum and other propagators, I am left with an overall momentum conserving delta function times 2 pi to the 4.

So, you can integrate over k 5 also. That is not so easy but you can; but here we are just integrating over all those momenta which are easy to integrate because of the delta function. So, that is why I integrated over k 6. So, the thing that we should carry from here is that I get at the end a 2 pi 4 2 pi to the 4 times a delta function, and delta function involves all these momenta.

In this case, because we were looking at a four-point function, I get sum of all these 4 momenta. That is what we should remember from here, and also the Feynman, the way this is written, the Feynman rules. So, now we will utilise this and let us revert to our original thing.

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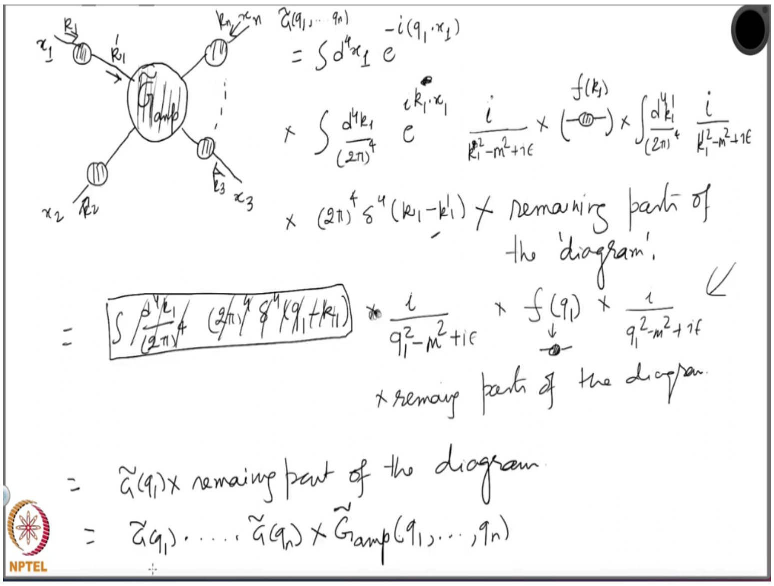

So, here I have; let us look at only 1 leg and for each leg, you will have similar thing, but I will not worry about it now, but I will draw it nevertheless. We are trying to interpret or find out what exactly we have multiplying G amputated or G tilde amputated. What are the exact factors that multiply in generic terms? We know exactly in terms of propagators and all this, but generically what is that object?

So, let us concentrate only on this one. So, here is the Fourier transform integral d 4 x 1 e to the -i q 1 dot x 1. This factor is coming because I am looking at a Fourier transform, because I am looking at G tilde q 1, q 2 and so forth and I will write only for leg 1. And this is right now drawn in the coordinate space; this is G tilde x 1, x 2 and so forth. Then if I write the expression of this G tilde x 1, x 2, x 3, x n, then it will be, remember; so, here I should assign momenta k 1.

I will not worry about these but let me nevertheless write. And this one, let me call k 1 prime. I will put the arrow this way. So, that is Fourier transform of this update. You have d 4 k 1 over 2 pi to the 4, then e to the i k 1 dot x 1 or k 1 prime dot x 1. This is exactly this factor you see coming from here. So, that is this factor and I should write all the propagators. I have i over k 1 prime square minus m square plus i epsilon.

Remember, here it is not the physical mass. Here it is the bare mass that appears in the Lagrangian, times all those things that are contained in the blob, all those contributions coming from there. So, I will just denote it like this, which I will write as f of k 1. So, this object is f of k 1 and it will depend only on this momentum k 1, because that is what enters in here and all other momenta that you will have inside, they will be dummy, they will be integrated over.

And because of the momentum conservation at each vertex, the only momentum on which this object will depend on will be k 1. So, that is why I write f of k 1; f of k 1 is fine; times you have integral d 4 k 1 prime 2 pi 4 i over k 1 prime square minus m square plus i epsilon times 2 pi to the 4 delta 4 k 1 minus k 1 prime. Why do you get this? You get this because of momentum conservation at each vertex.

So, that momentum conservation will ensure that k 1 is equal to k 1 prime and that is why you are going to get this factor when you write down the expression for this. When you have, once you have integrated out over all the other delta functions arising from the vertices inside this blob, you will end up with only one delta function which will be k 1 minus k 1 prime, and then times all the remaining things which arise from the remaining parts of this diagram or sum of diagrams.

So, I will just write remaining parts. I am writing diagram a bit loosely, because you have to actually add up all the diagrams. So, it is not just a diagram but rather a sum of all the diagrams. Now, this is G tilde basically. So, what do you get? You collect x 1 and integrate over it, and that gives you a 2 pi to the 4 delta 4 q 1. I have made slight mistake. This is not k 1 prime, this is k 1; q 1 - k 1.

That is correct; times, now if you; let me change; this is not right way. I should write integral d 4 k 1 over 2 pi to the 4; and then integrating over x 1 gives me this delta function which I was telling, q 1 - k 1 times this; this should also have been k 1. Now k 1 is q 1, so, I can just write, because of the delta function, I can write q 1 q square q 1 square minus m square plus i epsilon times f of, you can write as q 1, times this integral, times integral d 4 k 1 prime k 1 prime square minus m square plus i epsilon.

And actually I can; see, there is a delta function here which says k 1 prime which puts k 1 prime equal to k 1. So, if I integrate over k 1 prime, I will get this expression with k 1 prime replaced by k 1 and this delta function 2 pi 4 will cancel with this and this integral will be done; it will be taken care of. So, I will just do that, but also because of this delta function, k 1 is forced to q 1.

So, eventually, this k 1 prime get replaced to first k 1 because of this delta function and then eventually to q 1 because of this delta function. So, let me just remove this and write; I will just remove all this and write i over q 1 square minus m square plus i epsilon times remaining parts of the diagram and I should also remove this. I have done integral over k 1 as well. So, this is gone. So, this entire thing is gone. This is correct.

So, this is, the previous equation is equal to this. See, you can also check this way that what I am writing is correct. You have one delta function here and one delta function coming from doing x 1 integral over these 2 exponentials. So, at the end, I am left with 2 delta functions and 2 integrals, d 4 k 1 and d 4 k 1 prime. So, when I do those integrals over 2 delta function, neither I should have those integrals left nor the delta functions and that is why you have this result.

So, this is fine; no problem here. I should also again; this I will be drawing like this. Now, this is something we have already seen, but now I want to recall the general expression of G tilde which we saw here. So, if instead of this four-point function, if I was looking at a two-point function like; so, if I look at instead of this four-point function, if I look at two-point function, so, any diagram which contributes to two-point function, then if you repeat the same steps, then at the end you will have all these propagators, so, in this case, 2 propagators; and then, apart from other things, you will have a 2 pi to the 4 delta 4 q 1 + q 2.

So, this if I call this as q 1 and this is q 2, then this will have in addition to other factors, in addition to the propagators and other factors, we will have delta 4 q 1 + q 2. That is what you are going to get and that is what I want to utilise now.

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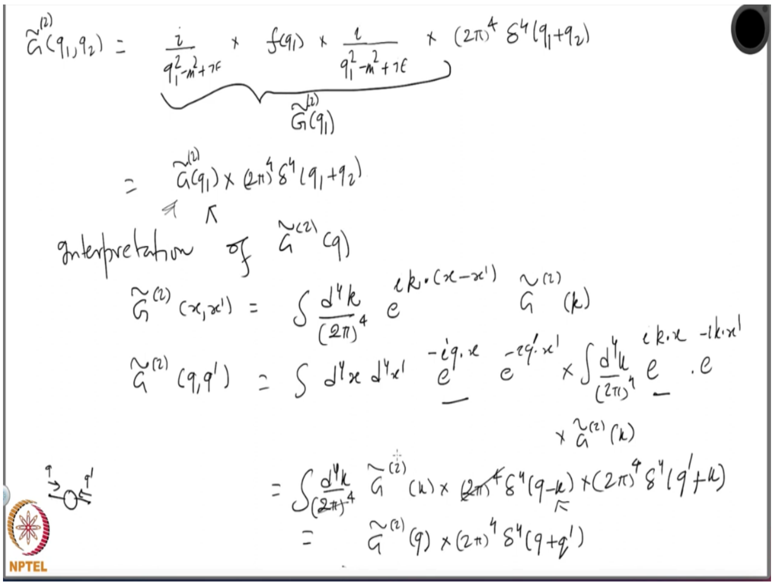

So, as you can see, as I just said, delta G tilde of q 1, q 2 which is just the Fourier transform of 2 points green function will have as we have seen above i over q 1 square minus m square plus i epsilon times this f of q 1, all those things which come from here, from this blob, all the internal vertices and internal propagators times again you get i over q 1 square minus m square plus i epsilon, and then at the end you have 2 pi to the 4 delta 4 q 1 + q 2.

I will denote this factor as G tilde of q 1. So, this becomes G tilde of q 1 times 2 pi to the 4 delta 4 q 1 + q 2. Now you see that here we had exactly this when I analysed the general thing. So, contribution from the leg 1 was i over this propagator times f q 1 times again the same propagator multiplying the remaining parts of the diagram. And here what do you have? If you look at a two-point function, then it has exactly the same form, this propagator with q 1 momentum in it, then f q 1 which is exactly the same f q 1; then again this propagator but another factor of 2 pi to the 4 delta 4 q 1 + q 2.

So, you see that what appears here, this piece, this is G tilde of q 1. So, I will write this as G tilde of q 1 times remaining part of the diagram. And if you do the same on each of the legs, then you get G tilde q 1 times G tilde q 2 so and so forth up to G tilde q n times what? Times exactly what we call as G tilde amputated, G tilde amputated Green's function with all these momenta.

So, that is what we have achieved and I know what exactly G tilde n is; G tilde, I should put a superscript 2; that is G tilde 2. And what is G tilde 2? This is exactly this object. Now, I will show you that G tilde 2 here with only 1 argument q 1 is basically a Fourier transform of

Green's function, G tilde 2 x 1 x 2 but involving only one variable that is; or instead of q 1, I will just write q because I am doing it in general.

So, G tilde 2 x x prime is; I am defining, I am doing a Fourier transform but with only one variable. So, this is what defines G tilde 2 of; I could write; that is fine. Now, also if you look at the Fourier transform involving both the variables transforming both x and x prime, then it was defined to be G d 4 x d 4 x prime e to the -i q dot x e to the -i q dot x prime times G of x 1 x 2, but that I will substitute from here.

So, instead of G tilde 2 x 1 x x prime, I will just substitute this. So, this is integral d 4 k 2 pi to the 4 e to the i k dot x into e to the -i k dot x prime times G tilde 2 of k. Let us do x integral. So, this involves x; that piece involves x. So, that gives you; and similarly, for x prime you do. You will again get a delta function just like for x, and then you are left with integral d 4 k. So, you will be left with d 4 k over 2 pi to the 4 times G tilde 2 k times these delta functions that you get after doing x integrals.

So, you get 2 pi to the 4 delta 4; this one and this one gives q - k and x prime integrals give q + k; something wrong, some primes are missing; so, here should have been prime. So, this one is fine and that one should be q prime minus k. Now, there is something wrong, q dot x k dot x. So, there is a relative sign, so, fine. Here, k minus k minus q prime, so, it should be a plus. Now, let us do the integral over k and utilise this delta function.

2 p i 4 cancels here and there and you get G tilde 2 k; not k, because k should be forced to become q. So, G tilde 2 of q, and this should give 2 pi to the 4 delta 4 q plus q prime. So, what we see is that the Fourier transform of this Green's function or G tilde 2 q q prime, that is just G tilde 2 q with 1 variable times an overall momentum preserving delta function. And remember here you get a q plus q prime because the way we have been doing is, if this is q and that is q prime, then you get q plus q prime.

And remember we had at the external vertices we had the momenta always assigned either entering from all the sides or exiting on all the sides; that is why you get a delta 4 q plus q prime. So, now we have an understanding of what that factor G tilde 2 q is. Why? Because you see here, this entire factor is same as what appears in here. So, clearly this factor times the delta function is G tilde 2 with q 1, q 2. That is something we argued.

We argued some time back that two-point function, if you take the Fourier transform, then it is 2 pi overall momentum conserving delta function times t tilde 2 of q 1 where this is this object which is what appeared in the previous expression here. And now we have an interpretation for this factor. What is that? That this is; so, if you were to take a two-point function in the Fourier space, then that will be equal to 2 pi 4 times delta 4 this, and the pre-factor is what enters into here, into this, in this.

So, this is what I wanted to show. And then, we are almost there; I will make some connection; I will make a connection of this with the scattering amplitude. I will define it and then we will proceed from there.